Relative efficiency

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Relative efficiency

- Suppose test 1 requires n_1 observations to obtain a certain power β, and that test 2 required n_2 observations to reach the same power
- Then the *relative efficiency* of test 1 with respect to test 2 is n_2/n_1
- For example, if test 2 needs 100 observations to achieve 80% power and test 1 needs only 50, then the relative efficiency is $n_2/n_1 = 100/50 = 2$; in other words, test 1 is twice as efficient

Pitman efficiency

- In all but the simplest situations, this is impossible to calculate
- Instead, we must rely on either simulations or asymptotic arguments
- Asymptotic arguments are not entirely straightforward, either, because the power of any consistent test goes to 1 as $n \to \infty$
- To avoid the power of both tests going to 1, we must have $\theta \to \theta_0$ as $n \to \infty$
- Specifically, E.J.G. Pitman proposed the sequence of alternatives

$$\theta_n = \theta_0 + n^{-1/2}\delta + o(n^{-1/2})$$

Noether's theorem

For this sequence of alternatives, Gottfried Noether proposed the following theorem:

Theorem: Suppose the test statistic T satisfies the following (i)-(iv):

(i) There exist functions $\mu_n(\theta)$, $\sigma_n(\theta)$ such that

$$\frac{T - \mu_n(\theta)}{\sigma_n(\theta)} \xrightarrow{\text{d}} N(0, 1)$$

for all $\theta \in [\theta_0, \theta_0 + \epsilon]$

(ii) $\mu_n(\theta)$ is differentiable at θ_0

Noether's Theorem (cont'd)

(iii)

$$\lim_{n \to \infty} \frac{\mu_n'(\theta_n)}{\mu_n'(\theta_0)} = \lim_{n \to \infty} \frac{\sigma_n(\theta_n)}{\sigma_n(\theta_0)} = 1$$

(iv)

$$\lim_{n \to \infty} n^{-1/2} \frac{\mu_n'(\theta_0)}{\sigma_n(\theta_0)} = c$$

Then the asymptotic power of the test is $\Phi(c\delta + z_\alpha)$, where z_α is the αth quantile of the standard normal distribution

Two-sided vs. one-sided tests

- The preceding result assumes a one-sided test
- **Corollary:** For a two-sided test, the asymptotic power is

$$\Phi(c\delta + z_{\alpha/2}) + 1 - \Phi(c\delta - z_{\alpha/2})$$

- Either way, the asymptotic power is determined by $c\delta$

Asymptotic relative efficiency

- The factor c is called the *asymptotic efficiency* of the test, and the *asymptotic relative efficiency* of test 1 with respect to test 2 is

$$e_{12} = \left(\frac{c_1}{c_2}\right)^2$$

- This is a very compelling way to compare two tests; its usefulness, however, is limited by the difficulty of finding c

AE of the Wilcoxon rank sum test

- It is possible to work out the asymptotic relative efficiency of the Wilcoxon rank sum test, although it is much easier to work with the algebraically equivalent Mann-Whitney test statistic:

$$T_{mw} = \frac{1}{mn} \sum_i \sum_j \mathbf{1}(x_i < y_j)$$
$$= \frac{1}{mn} \left(T_w - \frac{1}{2}n(n+1) \right)$$

  where T_w is the Wilcoxon rank sum test statistic $\sum_j r_j g_j$, and $\{x_i\}$ and $\{y_j\}$ are the values in the two groups, with m observations in group 1 and n observations in group 2

- Let's also introduce the notation $N = m + n$ and $\lambda = m/N$

AE of the Wilcoxon rank sum test (cont'd)

- Now, for the shift-in-location problem where $X_i \sim F(x + g_i\Delta)$,

$$\sigma_N(\theta_0) = \sqrt{\frac{N+1}{12mn}}$$

$$\mu_N(\theta) = \int F(y + \theta)dF(y)$$

$$\mu'_N(\theta_0) = \int f^2(y)dy$$

- Thus,

$$c_w = \lim_{N \to \infty} \frac{\mu'_N(\theta_0)}{\sqrt{(N)}\sigma_N(\theta_0)}$$

$$= \sqrt{12\lambda(1-\lambda)} \int f^2(y)dy$$

Homework

Homework: Show that

$$\sigma_N(\theta_0) = \sqrt{\frac{N+1}{12mn}}$$

AE of the t-test

- Meanwhile, the t-test statistic is

$$T_t = \frac{\bar{y} - \bar{x}}{s},$$

where s^2 is the pooled sample variance and $s^2 \xrightarrow{\mathsf{P}} \sigma_f^2$

- Thus,

$$\sigma_N(\theta_0) = \sqrt{\frac{N}{mn}}$$

$$\mu_N'(\theta) = \frac{1}{\sigma_f}$$

and

$$c_t = \frac{\sqrt{\lambda(1-\lambda)}}{\sigma_f}$$

Relative efficiency under the exponential distribution

- Suppose that the true distribution of the data is exponential with mean β

- Then

$$c_w = \sqrt{3\lambda(1-\lambda)}\beta^{-1}$$
$$c_t = \sqrt{\lambda(1-\lambda)}\beta^{-1}$$
$$e_{wt} = 3$$

- In other words, when the true distribution of the data is exponential, the Wilcoxon rank sum test is 3 times more efficient than the t-test at detecting a shift in location

Other distributions

One can repeat this exercise for other distributions:

Distribution	e_{wt}
Exponential	3
Double exponential	1.5
Normal	0.96
Uniform	1
Cauchy	∞

Homework

- Of course, these are asymptotic arguments
- What happens for finite n?
- **Homework:** Conduct a simulation comparing the relative power of the Wilcoxon and t-tests for $n = 6, \ldots, 100$ (you can choose the intervals) with an equal number of observations in each group (*i.e.*, 3 in each group, $\ldots$, 50 in each group). Use the following progression of Δ values: $\Delta = \sqrt{2/n}$. Conduct two simulations, one in which the true distribution of the data is normal, the other in which it is double exponential. Plot the relative power of the Wilcoxon test with respect to the t-test versus sample size. Comment on how well the asymptotic results seem to agree with your finite-n results.

Homework (cont'd)

Homework: Slide 13 claims that the Wilcoxon rank sum test is (asymptotically) 1.5 times as efficient as a two-sample t-test when the data follows a double exponential distribution. Derive this result.