Introduction to analytic proofs: Solutions

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Convergence

Theorem. *Suppose $x_n \to x$ and $y_n \to y$. Then $x_n + y_n \to x + y$.*

Proof. Let $\epsilon > 0$.

$$\text{①} \qquad \exists N_x : n > N_a \implies |x_n - x| < \tfrac{\epsilon}{2} \qquad x_n \to x$$
$$\text{②} \qquad \exists N_y : n > N_y \implies |y_n - y| < \tfrac{\epsilon}{2} \qquad y_n \to y$$

Thus, for all $n > N = \max(N_x, N_y)$, we have

$$\begin{aligned} |x_n + y_n - (x+y)| &\le |x_n - x| + |y_n - y| && \text{Triangle inequality} \\ &< \epsilon && \text{①, ②} \end{aligned}$$

□

Theorem. *Suppose $x_n \to x$ and $y_n \to y$. Then $x_n y_n \to xy$.*

Proof. First, let's establish an identity:

$$\begin{aligned} x_n y_n - xy &= x_n y_n - x_n y + x_n y - xy \\ &= x_n(y_n - y) + y(x_n - x) \\ &= (x_n - x + x)(y_n - y) + y(x_n - x) \\ &= (x_n - x)(y_n - y) + x(y_n - y) + y(x_n - x) \end{aligned}$$

Now, let $\epsilon > 0$.

$$\text{①} \qquad \exists N_x : n > N_x \implies |x_n - x| < \frac{\sqrt{\epsilon}}{3} + \frac{\epsilon}{3\,|y|} \qquad x_n \to x$$
$$\text{②} \qquad \exists N_y : n > N_y \implies |y_n - y| < \frac{\sqrt{\epsilon}}{3} + \frac{\epsilon}{3\,|x|} \qquad y_n \to y$$

Thus, for all $n > N = \max(N_x, N_y)$, we have

$$\begin{aligned} |x_n y_n - xy| &= |(x_n - x)(y_n - y) + x(y_n - y) + y(x_n - x)| && \text{Identity above} \\ &\le |x_n - x|\,|y_n - y| + |x|\,|y_n - y| + |y|\,|x_n - x| && \text{Triangle inequality} \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} && \text{①, ②} \\ &= \epsilon \end{aligned}$$

In the construction of N above, note that we are assuming $x, y \neq 0$. If either is zero, the second term in the sum can simply be omitted, as the corresponding term below is zero. □

Theorem. *Suppose $x_n \to x$, with $x_n \neq 0$ for all n and $x \neq 0$. Then $1/x_n \to 1/x$.*

Proof. First, let us note that $|a-b| < \frac{1}{2}|b| \implies |a| > \frac{1}{2}b$. This is fairly obvious when you think about it; to prove it, we can break the claim up into cases:

- $b > 0$ and $a > b$: $a > b > b/2$
- $b > 0$ and $b > a$: $b - a < \frac{1}{2}b$, so $a > \frac{1}{2}b$

The cases where $b < 0$ follow the same reasoning. Now, let $\epsilon > 0$.

①	$\exists N_1 : n > N_1 \implies |x_n - x| < \frac{1}{2}|x|^2 \epsilon$	$x_n \to x$
②	$\exists N_2 : n > N_2 \implies |x_n - x| < \frac{1}{2}|x|$	$x_n \to x$
③	so that $|x_n| > \frac{1}{2}|x|$	②, see above

Thus, for all $n > N = \max(N_1, N_2)$, we have

$$\begin{aligned} \left|\frac{1}{x_n} - \frac{1}{x}\right| &= \left|\frac{x - x_n}{x_n x}\right| \\ &\leq \frac{2}{|x|^2}|x_n - x| && ③ \\ &< \epsilon && ① \end{aligned}$$

Note that in this third theorem, the requirement that $x_n \neq 0$ is unnecessary. As we see from ③, if $x_n \to x$ and $x \neq 0$, then there is an N such that $x_n \neq 0$ for all $n > N$. □

Continuity

The first two theorems are essentially the same as their sequence counterparts, but the differences are worth paying attention to.

Theorem. *Let the functions f and g be continuous at x_0. Then $h = f + g$ is continuous at x_0.*

Proof. Let $\epsilon > 0$.

①	$\exists \delta_f : |x - x_0| < \delta_f \implies |f(x) - f(x_0)| < \frac{\epsilon}{2}$	f continuous at x_0
②	$\exists \delta_g : |x - x_0| < \delta_g \implies |g(x) - g(x_0)| < \frac{\epsilon}{2}$	g continuous at x_0

Thus, for all $x : |x - x_0| < \delta = \min(\delta_f, \delta_g)$, we have

$$\begin{aligned} |h(x) - h(x_0)| &= |f(x) + g(x) - f(x_0) - g(x_0)| && \text{Def } h \\ &\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| && \text{Triangle inequality} \\ &\leq \epsilon && ①, ② \end{aligned}$$

□

Theorem. *Let the functions f and g be continuous at x_0. Then $h = f \cdot g$ is continuous at x_0.*

Proof. Let $\epsilon > 0$.

$$\text{①} \quad \exists \delta_f : |x - x_0| < \delta_f \implies |f(x) - f(x_0)| < \frac{\sqrt{\epsilon}}{3} + \frac{\epsilon}{3\,|g(x_0)|} \qquad f \text{ continuous at } x_0$$

$$\text{②} \quad \exists \delta_g : |x - x_0| < \delta_g \implies |g(x) - g(x_0)| < \frac{\sqrt{\epsilon}}{3} + \frac{\epsilon}{3\,|f(x_0)|} \qquad g \text{ continuous at } x_0$$

Thus, for all $x : |x - x_0| < \delta = \min(\delta_f, \delta_g)$, we have

$$\begin{aligned}
|h(x) - h(x_0)| &= |f(x)g(x) - f(x_0)g(x_0)| && \text{Def } h \\
&\leq |\{f(x) - f(x_0)\}\{g(x) - g(x_0)\}| \\
&\quad + |f(x_0)\{g(x) - g(x_0)\}| \\
&\quad + |g(x_0)\{f(x) - f(x_0)\}| && \text{See earlier proof} \\
&< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} && \text{①, ②} \\
&= \epsilon && \square
\end{aligned}$$

Theorem. *Let the function f be continuous at x_0 and the function g be continuous at $f(x_0)$. Then $h(x) = g(f(x))$ is continuous at x_0.*

Proof. Let $\epsilon > 0$.

$$\begin{aligned}
&\text{①} \quad \exists \eta : |y - f(x_0)| < \eta \implies |g(y) - g(f(x_0))| < \epsilon && g \text{ continuous at } f(x_0) \\
&\text{②} \quad \exists \delta : |x - x_0| < \delta \implies |f(x) - f(x_0)| < \eta && f \text{ continuous at } x_0
\end{aligned}$$

Thus, for all $x : |x - x_0| < \delta$, we have

$$\begin{aligned}
|h(x) - h(x_0)| &= |g(f(x)) - g(f(x_0))| && \text{Def } h \\
&< \epsilon && \text{②} \implies \text{①} \quad \square
\end{aligned}$$

Exercise: Write an R function `n(eps)` that returns the smallest N for which $n > N \implies |f(x_n) - f(x_0)| < \epsilon$ for $x_n = 2^{1/n}$ and $f(x) = e^x$.

Conceptually, this is a three-part process:

1. Determine what x_n is converging to. Here, $x_n \to 1$.
2. Determine the largest value of delta that satisfies $e^{1+\delta} - e^1 < \epsilon$.
3. Determine the smallest value of N such that $2^{1/n} - 1 < \delta$.

```
n <- function(eps) {
  delta <- log(eps + exp(1)) - 1
  ceiling(1/log2(1+delta))
}
n(0.01)  ## 190

# Check solution
exp(2^(1/189)) - exp(1)  ## 189 not good enough
exp(2^(1/190)) - exp(1)  ## 190 within 0.01
```