Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 5
Due: Monday, October 6

1. *The Cramér-Wold device.* Let $\mathbf{x}_n \in \mathbb{R}^d$ be a sequence of random vectors. Prove that if $\mathbf{a}^\top \mathbf{x}_n \xrightarrow{\text{d}} \mathbf{a}^\top \mathbf{x}$ for all vectors $\mathbf{a} \in \mathbb{R}^d$, then $\mathbf{x}_n \xrightarrow{\text{d}} \mathbf{x}$. Hint: Lévy Continuity theorem.

2. *Simultaneous convergence in probability.* Suppose $X_{n1} \xrightarrow{\text{P}} X_1, \ldots, X_{nd} \xrightarrow{\text{P}} X_d$. Prove that the vector $\mathbf{x}_n$ converges to the vector $\mathbf{x}$ (i.e., that element-wise convergence in probability implies convergence in probability as defined in class). Note: do not assume that the elements of $\mathbf{x}_n$ are independent from each other, or anything else about their joint distribution. Hint: Union bound.

3. *Consistency of the OLS estimator.* Suppose that for a sequence of constant vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots,$ the random variables $Y_1, Y_2, \ldots$ are independent with mean $\mathbf{x}_i^\top \boldsymbol{\beta}$ and variance σ^2. Prove that $\widehat{\boldsymbol{\beta}} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}$ is a consistent estimator of $\boldsymbol{\beta}$. Do you need additional conditions on $\mathbf{X}$ in order to guarantee that $\widehat{\boldsymbol{\beta}}$ is consistent? If so, make any additional assumptions you wish, but clearly describe and label these additional conditions.

4. *A triangular array for binomial data.* Suppose $Y_n \sim \text{Binom}(n, \pi_n)$; note in particular that the binomial probability π here is changing with n. Use the Lindeberg-Feller central limit theorem to show that

$$\frac{Y_n - n\pi_n}{\sqrt{n\pi_n(1-\pi_n)}} \xrightarrow{\text{d}} \text{N}(0,1).$$

As above, are there any additional conditions you need to require on the sequence π_n in order for this result to be true? If so, tell me what they are. In your proof, you will need to set up a triangular array; clearly describe what random variables you are using for this array and prove that it satisfies the requirements to be a triangular array as defined in the notes.

5. *Slutsky constant.* In Slutsky's theorem, we require that one of the two random variables converges in probability to a constant. What happens if both $\mathbf{x}_n$ and $\mathbf{y}_n$ converge in distribution? Specifically, suppose $\mathbf{x}_n \xrightarrow{\text{d}} \mathbf{x}$ and $\mathbf{y}_n \xrightarrow{\text{d}} \mathbf{y}$; is it true that

$$\begin{bmatrix} \mathbf{x}_n \\ \mathbf{y}_n \end{bmatrix} \xrightarrow{\text{d}} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}?$$

For each of the following scenarios, either prove that this is true or prove that it is not by providing a counterexample.

   (a) Without any additional restrictions.
   (b) With the added restriction that $\mathbf{x}_n \perp\!\!\!\perp \mathbf{y}_n$ and $\mathbf{x} \perp\!\!\!\perp \mathbf{y}$.