Lindeberg-Feller central limit theorem

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Introduction

- Last time, we proved the central limit theorem for the iid case
- Obviously, this is very useful, but at the same time, has clear limitations – the majority of practical applications of statistics involve modeling the relationship between some outcome Y and a collection of potential predictors $\{X_j\}_{j=1}^d$
- Those predictors are not the same for each observation; hence, Y is not iid and the ordinary CLT does not apply

Introduction (cont'd)

- Nevertheless, we'd certainly hope it to be the case that $\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ converges to a normal distribution even if the errors are not normally distributed
- Does it? If so, under what circumstances?
- Before getting to this question, let's first introduce the concept of a "triangular array" of variables

Triangular array

- A triangular array of random variables is of the form

$$\begin{array}{lll} X_{11} & & \\ X_{21} & X_{22} & \\ X_{31} & X_{32} & X_{33} \\ \ldots, & & \end{array}$$

  where the random variables in each row (i) are independent of each other, (ii) have zero mean and (iii) have finite variance.
- The requirement that the variables have zero mean is only for convenience; we can always construct zero-mean variables by considering $X_{ni} = Y_{ni} - \mu_{ni}$
- I've stated the definition here in terms of scalar variables, but the entries in this triangle can also be random vectors $\mathbf{x}_{ni}$

Triangular array (cont'd)

- We are going to be concerned with $Z_n = \sum_{i=1}^n X_{ni}$, the row-wise sum of the array
- Since the elements of each row are independent, we have

$$s_n^2 = \mathbb{V} Z_n = \sum_{i=1}^n \mathbb{V} X_{ni} = \sum_{i=1}^n \sigma_{ni}^2$$

or, if the elements in the array are random vectors,

$$\mathbf{V}_n = \mathbb{V}\mathbf{z}_n = \sum_{i=1}^n \mathbb{V}\mathbf{x}_{ni} = \sum_{i=1}^n \boldsymbol{\Sigma}_{ni}$$

Non-IID laws of large numbers

- Before moving on to central limit theorems, it's worth mentioning how the law of large numbers extends to the non-iid case
- **Theorem (Law of Large Numbers, non-IID)**: Suppose $\mathbf{x}_1, \mathbf{x}_2, \ldots$ are independent random variables with $\frac{1}{n}\sum_i \boldsymbol{\mu}_i \to \boldsymbol{\mu}$ and $\frac{1}{n}\sum \mathbb{V}\mathbf{x}_i$ is bounded. Then $\bar{\mathbf{x}} \xrightarrow{\text{P}} \boldsymbol{\mu}$.
- Note that if there is a uniform bound on the individual variances, meaning that $(\mathbb{V}\mathbf{x}_i)_{jk} < M$ for all i, j, k, then $\frac{1}{n}\sum \mathbb{V}\mathbf{x}_i$ is bounded as well

Lindeberg condition

- There are a few different ways of extending the central limit theorem to non-iid random variables; the most general of these is the Lindeberg-Feller theorem
- This version of the CLT involves a new condition known as the *Lindeberg condition*: for every $\epsilon > 0$,

$$\frac{1}{s_n^2}\sum_{i=1}^{n}\mathbb{E}\{X_{ni}^2 1(|X_{ni}| \geq \epsilon s_n)\} \to 0$$

  as $n \to \infty$
- We'll discuss the multivariate version of this condition a bit later

Example

- The Lindeberg condition is a bit abstract at first, so let's see how it works, starting with the simplest case: iid random variables
- **Theorem:** Suppose $X_1, X_2, \ldots$ are iid with mean zero and finite variance. Then the Lindeberg condition is satisfied.
- There are three key steps in this proof:
    - Replacing the infinite sum with a single quantity $\propto \mathbb{E}T_n$
    - $T_n \xrightarrow{\text{P}} 0$ (which happens if $s_n \to \infty$)
    - $\mathbb{E}T_n \to 0$ by the Dominated Convergence Theorem (requires finite variance)

Non-iid case

- The last two steps work out essentially the same way in non-iid settings
- The first step, however, requires some resourcefulness
- Typically, the proof proceeds along the lines of bounding the elements of the sum by their "worst-case scenario"; this eliminates the sum, but requires a condition requiring that the worst-case scenario can't be too extreme
- We'll see a specific example of this later as it pertains to regression

Lindeberg's theorem

- We are now ready to present the Lindeberg-Feller theorem, although we won't be proving it in this course
- **Theorem (Lindeberg)**: Suppose $\{X_{ni}\}$ is a triangular array with $Z_n = \sum_{i=1}^n X_{ni}$ and $s_n^2 = \mathbb{V} Z_n$. If the Lindeberg condition holds: for every $\epsilon > 0$,

$$\frac{1}{s_n^2} \sum_{i=1}^n \mathbb{E}\{X_{ni}^2 1(|X_{ni}| \geq \epsilon s_n)\} \to 0,$$

then $Z_n / s_n \xrightarrow{\text{d}} \text{N}(0, 1)$.

Lindeberg's theorem, alternate statement

- The preceding theorem is expressed in terms of sums; it is often more natural to think about Lindeberg's theorem in terms of means
- **Theorem (Lindeberg)**: Suppose $\{X_{ni}\}$ is a triangular array such that $Z_n = \frac{1}{n}\sum_{i=1}^n X_{ni}$, $s_n^2 = \frac{1}{n}\sum_{i=1}^n \mathbb{V}X_{ni}$, and $s_n^2 \to s^2 \neq 0$. If the Lindeberg condition holds: for every $\epsilon > 0$,

$$\frac{1}{n}\sum_{i=1}^n \mathbb{E}\{X_{ni}^2 1(|X_{ni}| \geq \epsilon\sqrt{n})\} \to 0,$$

  then $\sqrt{n}Z_n \xrightarrow{\text{d}} \text{N}(0, s^2)$.
- Note: we've added an assumption that $s_n^2 \to s^2$, but made the Lindeberg condition easier to handle (s_n no longer appears)

Feller's Theorem

- The preceding theorem(s) show that the Lindeberg condition is sufficient for asymptotic normality
- Feller showed that it was also a necessary condition, if we introduce another requirement:

$$\max_i \frac{\sigma_{ni}^2}{\sum_{j=1}^n \sigma_{nj}^2} \to 0$$

  as $n \to \infty$; i.e., no one term dominates the sum
- **Theorem (Feller)**: Suppose $\{X_{ni}\}$ is a triangular array with $Z_n = \sum_{i=1}^n X_{ni}$ and $s_n^2 = \mathbb{V} Z_n$. If $Z_n/s_n \xrightarrow{\text{d}} \text{N}(0,1)$ and $\max_i \sigma_{ni}^2/s_n^2 \to 0$, then the Lindeberg condition holds.

Lindeberg-Feller theorem

- Putting these two theorems together, the Lindeberg-Feller Central Limit Theorem says that if no one term dominates the variance, then we have asymptotic normality if and only if the Lindeberg condition holds
- The forward (Lindeberg) part of the theorem is the most important part in practice, as our goal is typically to prove asymptotic normality
- However, it is worth noting that the Lindeberg condition is the minimal condition that must be met to ensure this
- For example, there is another CLT for non-iid variables called the Lyapunov CLT, which requires a "Lyapunov condition"; not surprisingly, this implies the Lindeberg condition, as it is a stronger condition than necessary for asymptotic normality

Multivariate CLT

- Now let's look at the multivariate form of the Lindeberg-Feller CLT, which I'll give in the "mean" form
- **Theorem (Lindeberg-Feller CLT):** Suppose $\{\mathbf{x}_{ni}\}$ is a triangular array of $d \times 1$ random vectors such that $\mathbf{z}_n = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_{ni}$ and $\mathbf{V}_n = \frac{1}{n}\sum_{i=1}^n \mathbb{V}\mathbf{x}_{ni} \to \mathbf{V}$, where $\mathbf{V}$ is positive definite. If for every $\epsilon > 0$,

$$\frac{1}{n}\sum_{i=1}^n \mathbb{E}\{\|\mathbf{x}_{ni}\|^2 1(\|\mathbf{x}_{ni}\| \geq \epsilon\sqrt{n})\} \to 0,$$

  then $\sqrt{n}\mathbf{z}_n \xrightarrow{\text{d}} \text{N}(\mathbf{0}, \mathbf{V})$.
- Or equivalently, $\sqrt{n}\mathbf{V}_n^{-1/2}\mathbf{z}_n \xrightarrow{\text{d}} \text{N}(\mathbf{0}, \mathbf{I})$

Multivariate Feller condition

- Similar to the univariate case, the Lindeberg condition is both necessary and sufficient if we add the condition that no one term dominates the variance
- In the multivariate setting, this means that

$$\frac{\mathbb{V}\mathbf{x}_i}{\sum_{j=1}^{n}\mathbb{V}\mathbf{x}_j} \to \mathbf{0}_{d\times d}$$

for all i; the division here is element-wise

CLT for linear regression

- OK, now let's take what we've learned and put it into practice, answering our question from the beginning of lecture: do we have a central limit theorem for linear regression?
- **Theorem:** Suppose $\mathbf{y} = \mathbf{X}\beta^* + \mathbf{w}$, where $w_i \overset{\text{iid}}{\sim} (0, \sigma^2)$. Suppose $\frac{1}{n}\mathbf{X}^\top\mathbf{X} \to \mathbf{\Sigma}$, where $\mathbf{\Sigma}$ is positive definite, and let $\mathbf{x}_i$ denote the $d \times 1$ vector of covariates for subject i (taken to be fixed, not random). If $\|\mathbf{x}_i\|$ is uniformly bounded, then

$$\frac{1}{\sigma}(\mathbf{X}^\top\mathbf{X})^{1/2}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*) \xrightarrow{\text{d}} \text{N}(\mathbf{0}, \mathbf{I}).$$

- In other words, $\widehat{\boldsymbol{\beta}} \mathrel{\dot\sim} \text{N}(\boldsymbol{\beta}^*, \sigma^2(\mathbf{X}^\top\mathbf{X})^{-1})$

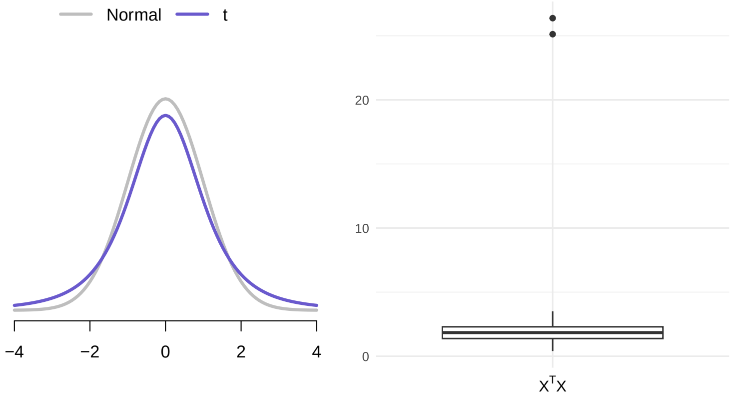
Remarks

- Note that in proving this result, we needed two key conditions
  - $\frac{1}{n}\mathbf{X}^\top\mathbf{X}$ converging to a p.d. matrix; this seems obvious since if $\mathbf{X}^\top\mathbf{X}$ was not invertible, $\widehat{\beta}$ isn't even well-defined
  - $\|\mathbf{x}_i\|$ bounded; this is less obvious, but is connected to the idea of influence in regression
- In iid data, all observations essentially carry the same weight for the purposes of estimation and inference
- In regression, however, observations far from the mean of the covariate have much greater influence over the model fit
- This is essentially what $\|\mathbf{x}_i\|$ is measuring: in words, we are requiring that no one observation can exhibit too great an influence

Simulation

- This is one of those situations where theory helps to guide intuition and practice
- Let's carry out a simulation to illustrate
- We will challenge the central limit theorem in two ways:
  - $\mathbf{w}$ will follow a t distribution with ν degrees of freedom
  - The elements of $\mathbf{X}$ will be uniformly distributed (from -1 to 1) except for the first two elements of column 1, which will be set to $\pm a$
- In what follows, $n = 100$ unless otherwise noted; 1000 simulations were run for each example

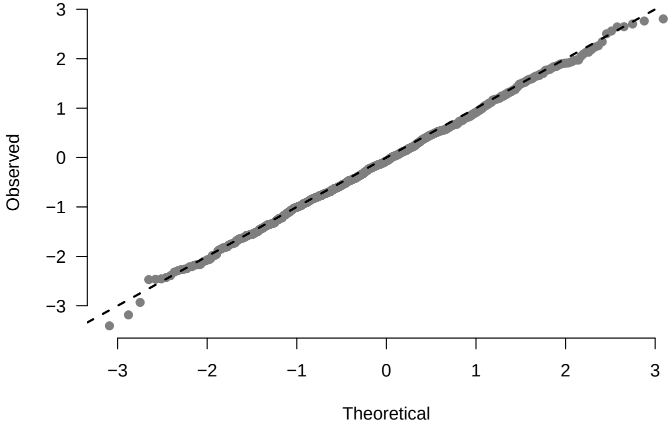
Illustration of the two conditions ($\nu = 3, a = 5$)



As we will see, the more comfortably the Lindeberg condition holds, the faster the rate of convergence to normality

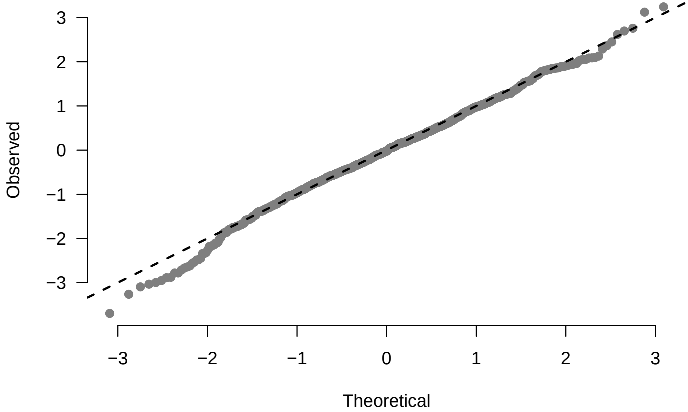
Results: $\nu = 50, a = 5$

Influential observations, but ε close to normal



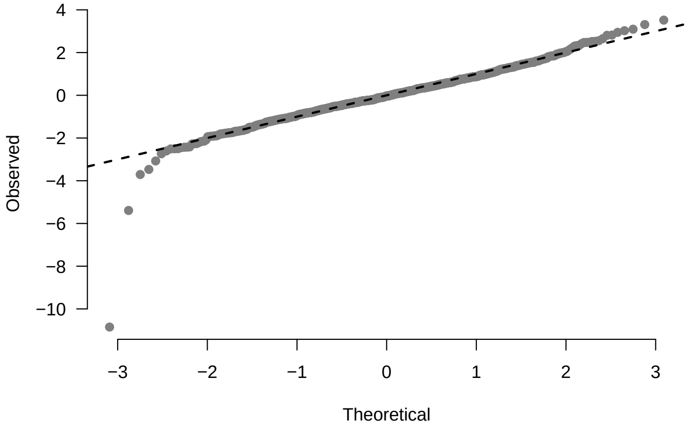
Results: $\nu = 3, a = 1$

Heavy tails, but no terribly influential observations



Results: $\nu = 3, a = 5$

Heavy tails and influential observations



Results: $\nu = 3, a = 5$

Heavy tails and influential observations, but $n = 1000$