Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 8

Due: Monday, November 4

1. *Slutsky's extension.* Suppose $\mathbf{y}_n \xrightarrow{\text{d}} \mathbf{y}$, where $\mathbf{y}$ is a $d \times 1$ random vector, $\mathbf{A}_n \xrightarrow{\text{P}} \mathbf{A}$, where $\mathbf{A}$ is a positive definite matrix, and that $\mathbf{y}_n = \mathbf{A}_n \mathbf{x}_n$. Note: This problem would be trivial if we knew that $\mathbf{A}_n$ were positive definite; the point of this problem is that you do *not* know this about $\mathbf{A}_n$, only that its limit $\mathbf{A}$ is positive definite. In other words, if at any point you write $\mathbf{A}_n^{-1}$ in your solution, then it is incorrect.

   (a) Find a lower bound $L > 0$ such that

   $$\mathbb{P}\left\{\frac{\mathbf{x}_n^\top \mathbf{A}_n^\top \mathbf{A}_n \mathbf{x}_n}{\|\mathbf{x}_n\|^2} > L\right\} \to 1.$$

   Hint: If $\mathbf{A}$ is positive definite, $\mathbf{A}^\top \mathbf{A}$ is positive definite.

   (b) Prove that $\mathbf{x}_n$ is bounded in probability. Hint: Use the result from (a).

   (c) Prove that $\mathbf{x}_n \xrightarrow{\text{d}} \mathbf{A}^{-1}\mathbf{y}$. Hint: Use the result from (b).

2. *Bernstein-von Mises theorem.* Suppose the prior $p(\boldsymbol{\theta})$ is continuous with $p(\boldsymbol{\theta}) > 0$ for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, and that regularity conditions (A)-(C) as defined in the "Likelihood: Consistency" lecture are satisfied. Below, $\hat{\boldsymbol{\theta}}$ is the MLE and $\boldsymbol{\delta}$ is an arbitrary $d \times 1$ vector.

   (a) Consider the log-likelihood ratio

   $$\log\left(\frac{L(\hat{\boldsymbol{\theta}} + \boldsymbol{\delta}/\sqrt{n})}{L(\hat{\boldsymbol{\theta}})}\right);$$

   what does this quantity converge to as $n \to \infty$? Hint: Taylor series.

   (b) Consider the prior ratio

   $$\frac{p(\hat{\boldsymbol{\theta}} + \boldsymbol{\delta}/\sqrt{n})}{p(\hat{\boldsymbol{\theta}})};$$

   what does this quantity converge to as $n \to \infty$?

   (c) Prove that

   $$p(\hat{\boldsymbol{\theta}} + \boldsymbol{\delta}/\sqrt{n}|\mathbf{x})/p(\hat{\boldsymbol{\theta}}|\mathbf{x}) \xrightarrow{\text{P}} \exp\{-\tfrac{1}{2}\boldsymbol{\delta}^\top \boldsymbol{\mathscr{I}}(\boldsymbol{\theta}^*)\boldsymbol{\delta}\}.$$

   Note: this follows fairly directly from (a) and (b), but note that "prove" here mean that rigorous justifications of each step are required. If you provided such justifications in (a) and (b), you do not have to repeat the arguments, but I do need to see those justifications somewhere.

Note: I realize that the version we presented in class involved almost sure convergence; you are only asked to show convergence in probability here.