Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 2
Due: Monday, September 16

1. *Vector norms.*
   (a) Show that $\|\cdot\|_2$ is a norm.
   (b) Show that $\|\cdot\|_\infty$ is a norm.
   (c) Let $\|\cdot\|_{1/2}$ denote the function of $\mathbf{x}$ you would obtain by using $p = 1/2$ in the definition of an L_p norm. Is $\|\cdot\|_{1/2}$ a norm? Why or why not?

2. *Uniform convergence.* For each of the following sequences, determine the pointwise limit of $\{f_n\}$ and decide whether $f_n \to f$ uniformly on the set given or not. You must justify your conclusion – don't just say uniform/not uniform.
   (a) $f_n(x) = \sqrt[n]{x}$ on $[0, 1]$.
   (b) $f_n(x) = e^x/x^n$ on $(1, \infty)$.
   (c) $f_n(\mathbf{x}) = n^{-1} \exp\{-\|\mathbf{x}\|^2\}$ on $\mathbb{R}^d$.

3. *Equivalence of norms and continuity.* Suppose that $f : \mathbb{R}^d \mapsto \mathbb{R}$ is continuous with respect to the L_1 norm at a point $\mathbf{p}$. Show that f is also continuous with respect to the L_2 norm at point $\mathbf{p}$.

4. *O-notation proofs.* Prove the following results:
   (a) $O(1)o(1) = o(1)$.
   (b) $\{1 + o(1)\}^{-1} = O(1)$.
   (c) $o\{O(1)\} = o(1)$.

   Remarks:
   - Part (b) is trivial if you use properties of limits. For the purposes of this problem, however, prove the result using only the definition of o and O. I realize that a simpler proof exists, but I consider the longer proof quite instructive.
   - In part (c), you cannot use the result $o(r_n) = r_n o(1)$. This result is a consequence of the results in part (a) and part (c); using it would be circular logic.

5. *The Riemann-Stieltjes Integral.* Suppose μ is a bounded, nondecreasing function on $[a, b]$ and that μ is continuous at a point $x_0 \in [a, b]$. Further suppose that $g(x_0) = 1$ and $g(x) = 0$ if $x \neq x_0$. Prove that $\int g d\mu = 0$. Hint: Show that for any $\epsilon > 0$, we can find a partition P such that $0 \leq U(P, g, \mu) < \epsilon$.