Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 1
Due: Wednesday, September 4

1. *Exchange paradox.* Consider the following seemingly paradoxical situation. There are two identical envelopes, each containing money. One contains twice as much as the other. You may pick one envelope and keep the money it contains. You randomly choose an envelope, but before opening it, you are told that the envelope contains \$20, and you are given the opportunity to switch envelopes. Should you? You reason as follows: "The other envelope contains either \$10 or \$40 dollars, with equal probability. Therefore, if I switch envelopes, my expected value is

$$\tfrac{1}{2} \cdot 10 + \tfrac{1}{2} \cdot 40 = 25,$$

   which is larger than the \$20 dollars I can expect if I keep the original envelope." So you switch.

   This seems unreasonable, however. In particular, the argument would be exactly the same regardless of the value of your envelope – if you are told that your envelope contains x dollars, the reasoning above concludes that the other envelope contains $5x/4$ dollars. If we should always switch envelopes, regardless of how much money our envelope contains, it would seem that something has gone wrong.

   What is wrong with the above argument? Explain, and provide an approach to the envelope switching problem that does not lead to a paradox. There are many possible solutions, although in keeping with the spirit of the course, I would encourage you to work the concept of likelihood into your answer.

2. *Combining likelihoods, part 1.* Suppose we have two independent samples drawn from $N(\theta, 1)$. All we know about the first sample is that $n_1 = 5$ and the maximum value of these five observations is 3.5. For the second sample, we know that $n_2 = 3$ and $\bar{x}_2 = 4$.

   (a) Derive the combined likelihood.
   (b) Plot the following three likelihood curves (within a single graph):
      - The likelihood from sample 1
      - The likelihood from sample 2
      - The combined likelihood

      All three likelihoods should be scaled such that their maximum value is 1. Comment on how the combined likelihood compares to the likelihoods from samples 1 and 2.

3. *Combining likelihoods, part 2.* In the previous exercise, we combined density and probability terms ... maybe this causes problems? Modify the combined likelihood from the previous exercise so that all density terms are replaced by probability terms such that an observation that $X = x$ means $x - \epsilon < X < x + \epsilon$.

   Plot the following three likelihood curves within a single graph:

   - The original combined likelihood (with density terms)
   - The "discretized" likelihood with $\epsilon = 0.01$
   - The "discretized" likelihood with $\epsilon = 0.5$

Provide two separate plots: one in which the likelihoods are on their "raw" scale and the other in which they are normalized such that their maximum value is 1.

In addition to the plots themselves:

- Compare the raw and normalized versions of the plots. What do you learn from each plot and which one is more meaningful?
- Discuss whether combining density and probability is an issue for likelihood-based inference.