Transformations

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Introduction

- It is often the case in statistics that one knows something about the convergence of $\mathbf{x}_n$, but then we want to know something about the convergence of some function of the random vector $g(\mathbf{x}_n)$
- Today, we'll go over three useful tools for drawing these kinds of conclusions
    - The continuous mapping theorem
    - Slutsky's theorem
    - The delta method

Continuous mapping theorem

- The continuous mapping theorem is a simple, but very useful result
- It says that if $\mathbf{x}_n \to \mathbf{x}$ (in any sense), then $g(\mathbf{x}_n) \to g(\mathbf{x})$ (in the same sense) if g is continuous
- **Theorem (continuous mapping):** Let $g : \mathbb{R}^d \to \mathbb{R}^k$ be continuous almost everywhere with respect to $\mathbf{x}$.
  (i) If $\mathbf{x}_n \xrightarrow{\text{d}} \mathbf{x}$, then $g(\mathbf{x}_n) \xrightarrow{\text{d}} g(\mathbf{x})$
  (ii) If $\mathbf{x}_n \xrightarrow{\text{P}} \mathbf{x}$, then $g(\mathbf{x}_n) \xrightarrow{\text{P}} g(\mathbf{x})$
  (iii) If $\mathbf{x}_n \xrightarrow{\text{as}} \mathbf{x}$, then $g(\mathbf{x}_n) \xrightarrow{\text{as}} g(\mathbf{x})$

Example #1

- The continuous mapping theorem is extremely useful and allows formal justification of all kinds of things seem obvious, such as: we should be able to "square both sides" of a convergence statement
- For example, suppose $Z_n \xrightarrow{\text{d}} \text{N}(0, 1)$; then we immediately have $Z_n^2 \xrightarrow{\text{d}} \chi_1^2$
- Again, the key requirement is continuity; if the function isn't continuous, all bets are off
- For example, suppose $X_n \xrightarrow{\text{P}} 0$, and

$$g(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases};$$

$g(0) = 0$, but $g(X_n) \xrightarrow{\text{P}} 1$ if X_n is continuous

Example #2

- Note that only continuity at 0 was relevant in that last example, since $P(X = 0) = 1$
- By contrast, if $X_n \xrightarrow{\text{P}} 1$ or $X_n \xrightarrow{\text{d}} \text{N}(0, 1)$, then the CMT would hold since $g(x)$ would be continuous almost everywhere
- As a multivariate example, we proved the central limit theorem

$$\sqrt{n}(\mathbf{x}_n - \boldsymbol{\mu}) \xrightarrow{\text{d}} \text{N}(\mathbf{0}, \boldsymbol{\Sigma});$$

by the continuous mapping theorem, we immediately have the corollary

$$\sqrt{n}\boldsymbol{\Sigma}^{-1/2}(\mathbf{x}_n - \boldsymbol{\mu}) \xrightarrow{\text{d}} \text{N}(\mathbf{0}, \mathbf{I})$$

Example #3

- The result even extends to matrices (you can imagine stacking the columns of the $d \times x$ matrix into a giant vector of dimension $d \cdot k$)
- Proving these facts is beyond the scope of this course, but:
    - Matrix inversion is a continuous function (unless the matrix is singular)
    - Taking the square root of a positive definite matrix is also continuous
- So for example, if $\mathbf{X}_n \xrightarrow{\text{P}} \mathbf{A}$, then $\mathbf{X}_n^{-1} \xrightarrow{\text{P}} \mathbf{A}^{-1}$ provided that $\mathbf{A}$ is not singular

Asymptotic equivalence

- Another very useful result is Slutsky's theorem, which we will present here in a rather general form
- Before we prove this result, we need to introduce the following lemma concerning asymptotically equivalent sequences
- Two sequences of random vectors $\mathbf{x}_n$ and $\mathbf{y}_n$ are said to be *asymptotically equivalent* if $\mathbf{x}_n - \mathbf{y}_n \xrightarrow{\text{P}} 0$.
- **Lemma (asymptotic equivalence):** If $\mathbf{x}_n \xrightarrow{\text{d}} \mathbf{x}$ and $\mathbf{x}_n - \mathbf{y}_n \xrightarrow{\text{P}} 0$, then $\mathbf{y}_n \xrightarrow{\text{d}} \mathbf{x}$.
- In words, the lemma is saying that asymptotically equivalent sequences have the same limiting distributions

Slutsky's theorem

- This lemma is necessary to prove Slutsky's theorem:
- **Theorem (Slutsky):** If $\mathbf{x}_n \xrightarrow{\text{d}} \mathbf{x}$ and $\mathbf{y}_n \xrightarrow{\text{P}} \mathbf{a}$, where $\mathbf{a}$ is a constant, then

$$\begin{bmatrix} \mathbf{x}_n \\ \mathbf{y}_n \end{bmatrix} \xrightarrow{\text{d}} \begin{bmatrix} \mathbf{x} \\ \mathbf{a} \end{bmatrix}.$$

- This is perhaps not the form in which you are used to seeing Slutsky's theorem; the name "Slutsky's theorem" is widely used in an inconsistent manner to mean a number of similar results

Remarks

- This result along with the continuous mapping theorem implies all of the results that people often call "Slutsky's theorem"
- For example, we have the familiar $X_n + Y_n \xrightarrow{\text{d}} X + a$ and $X_n Y_n \xrightarrow{\text{d}} aX$ since additional and multiplication are continuous functions
- But we also have much more complex statements; for example, if $\mathbf{x}_1, \mathbf{x}_2, \ldots$ is an iid sample with mean $\boldsymbol{\mu}$ and nonsingular variance, then

$$n(\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \mathbf{S}_n^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}) \xrightarrow{\text{d}} \chi^2_d$$

Delta method

- There is one last important type of transformation that Slutsky/CMT do not address: suppose we know the distribution of $\mathbf{x} - \boldsymbol{\mu}$ and want to know the distribution of $g(\mathbf{x}) - g(\boldsymbol{\mu})$
- This result is described by a theorem known as the delta method
- **Theorem (Delta method):** Let $g : \mathbb{R}^d \to \mathbb{R}^k$ such that ∇g is continuous in a neighborhood of $\boldsymbol{\mu} \in \mathbb{R}^d$ and suppose $\sqrt{n}(\mathbf{x}_n - \boldsymbol{\mu}) \xrightarrow{\text{d}} \mathbf{x}$. Then

$$\sqrt{n}(g(\mathbf{x}_n) - g(\boldsymbol{\mu})) \xrightarrow{\text{d}} \nabla g(\boldsymbol{\mu})^\top \mathbf{x}.$$

Normal distribution corollary

- This typically comes when dealing with functions of sample moments, which are multivariate normal by the CLT
- **Corollary (Delta method):** Let $g : \mathbb{R}^d \rightarrow \mathbb{R}^k$ such that ∇g is continuous in a neighborhood of $\boldsymbol{\mu} \in \mathbb{R}^d$ and suppose $\sqrt{n}(\mathbf{x}_n - \boldsymbol{\mu}) \xrightarrow{\text{d}} \text{N}(\mathbf{0}, \boldsymbol{\Sigma})$. Then

$$\sqrt{n}(g(\mathbf{x}_n) - g(\boldsymbol{\mu})) \xrightarrow{\text{d}} \text{N}(\mathbf{0}, \nabla g(\boldsymbol{\mu})^\top \boldsymbol{\Sigma} \nabla g(\boldsymbol{\mu})).$$

- Our proof of the delta method illustrates an important point worth noting for the future: applying a Taylor series expansion requires conditions, but these conditions only need to be met with probability tending to 1 in order to establish convergence in distribution (asymptotic equivalence lemma)

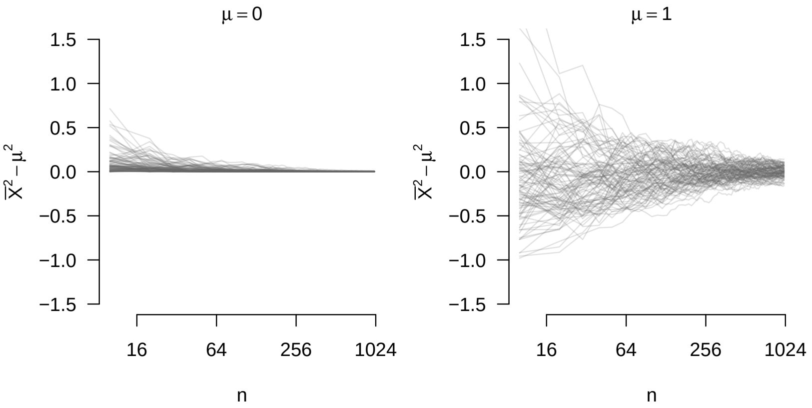
Example

- While the delta method is certainly a useful result, it should be noted that the rate of convergence can vary widely depending on both $\boldsymbol{\mu}$ and g
- For example, let's look at the function $g(\mu) = \mu^2$
- By the delta method and the CLT, we have

$$\sqrt{n}(\bar{x}^2 - \mu^2) \xrightarrow{\text{d}} \text{N}(0, 4\mu^2\sigma^2)$$

- Note that this isn't even the same rate of convergence for all μ: $\bar{x}^2 - \mu^2$ is $O_p(1/\sqrt{n})$ in general, but $O_p(1/n)$ when $\mu = 0$

Example (cont'd)



Remarks

- The relevance to statistical practice is that a common use of the delta method is to derive approximate confidence intervals for unknown parameters
- However, not all transformations and not all values of the unknown parameters converge to normality equally fast
- In practice, this means that some transformations produce much more accurate confidence intervals than others, and it is not always obvious which transformation is best
- Furthermore, a confidence interval procedure can be good for some values of $\boldsymbol{\theta}$ but poor at other values