Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 8
Due: Monday, October 24

1. *Neyman-Scott problem.* Suppose Y_{i1} and Y_{i2} are an iid sample from $\mathrm{N}(\mu_i, \sigma^2)$, for $i = 1, \ldots, n$.
   (a) Find the maximum likelihood estimate of σ^2.
   (b) Show that the MLE of σ^2 is not consistent.
   (c) Why does our theorem about the consistency of maximum likelihood estimates not hold in this situation?
   (d) Find a consistent estimator of σ^2.

2. Suppose $X_1, \ldots, X_n$ are an iid sample with density
$$p(x|\theta) = \theta x^{\theta-1} 1\{x \in (0,1)\}, \quad \Theta = (0, \infty).$$
   (a) Check whether this problem satisfies regularity conditions (A)-(D) as defined in the notes. For the purposes of the problem, I would like you to explicitly check each condition (as opposed to, for example, just saying that the problem is in the exponential family). However,
      - You can skip condition C(i); checking this condition is a bit tedious and we'll discuss it in class instead
      - Make sure that $\boldsymbol{\Theta}^*$ is the same in condition (B) as it is in (C)
   (b) Find the MLE and its limiting distribution (i.e., a convergence in distribution statement involving some function of the MLE; the MLE itself probably just converges to θ^*).

3. *Combination of Poisson variables.* Suppose in one month, a public health department records X cases of a new disease. At the end of the month, it is discovered that this disease is really two different diseases with similar symptoms. In month two, data recording practices are changed and the number of cases of each disease, Y_1 and Y_2, is recorded separately. For the purposes of the problem, assume that the number of cases of disease i in a month follows a Poisson distribution with rate λ_i, that cases of the two diseases arise independently, and that the number of cases in one month is independent of the number of cases in the next month.
   (a) Find the MLE of λ_1 and λ_2.
   (b) Suppose λ_1 and λ_2 are large enough that we feel comfortable applying the central limit theorem to (X, Y_1, Y_2). In other words, you may apply the approximation given by
$$\frac{Y_1 - \lambda_1}{\sqrt{\lambda_1}} \xrightarrow{\mathrm{d}} \mathrm{N}(0,1);$$
   and so on for X and Y_2 (this holds as $\lambda_1 \to \infty$). What is the approximate variance of $\hat{\lambda}_1$ as a function of λ_1 and λ_2?