Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 5
Due: Monday, September 26

1. *Convergence in quadratic mean and probability.* This problem involves proving the theorems from the slide titled "Convergence in mean vs convergence in probability" from the "Modes of convergence" lecture. Hint: Markov's inequality.
   (a) Prove that if $\mathbf{x}_n \xrightarrow{r} \mathbf{x}$ for some $r > 0$, then $\mathbf{x}_n \xrightarrow{\mathrm{P}} \mathbf{x}$.
   (b) Prove that if $\mathbf{a} \in \mathbb{R}^d$, then $\mathbf{x}_n \xrightarrow{\mathrm{qm}} \mathbf{a}$ if and only if $\mathbb{E}\mathbf{x}_n \to \mathbf{a}$ and $\mathbb{V}\mathbf{x}_n \to \mathbf{0}_{d\times d}$.
2. *The Cramér-Wold device.* Let $\mathbf{x}_n \in \mathbb{R}^d$ be a sequence of random vectors. Prove that if $\mathbf{a}^\top\mathbf{x}_n \xrightarrow{\mathrm{d}} \mathbf{a}^\top\mathbf{x}$ for all vectors $\mathbf{a} \in \mathbb{R}^d$, then $\mathbf{x}_n \xrightarrow{\mathrm{d}} \mathbf{x}$. Hint: Continuity theorem.
3. *Simultaneous convergence in probability.* Suppose $X_{n1} \xrightarrow{\mathrm{P}} X_1, \ldots, X_{nd} \xrightarrow{\mathrm{P}} X_d$. Prove that the vector $\mathbf{x}_n$ converges to the vector $\mathbf{x}$ (i.e., that element-wise convergence in probability implies convergence in probability as defined in class). Note: do not assume that the elements of $\mathbf{x}_n$ are independent from each other, or anything else about their joint distribution. Hint: Union bound.