Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 9

Due: Monday, November 9

1. *Score, Wald, and likelihood ratio tests for the rate parameter of the Gamma distribution.* This is a continuation of the problem, "Quadratic approximation for the Gamma distribution". In that problem, you derived the score and information, and simulated a random sample from the Gamma distribution. Now, implement score, Wald, and likelihood ratio approaches for carrying out inference regarding the rate parameter β. We discussed these approaches conceptually in class; this problem ensures that you understand all the specifics well enough to program them. Specifically, turn in a separate `.R` file that provides the following answers (please comment your code well enough that I can find each answer easily):
   (a) 99% Wald confidence interval for β.
   (b) Wald test of $H_0 : \beta = 1$.
   (c) 99% Score confidence interval for β.
   (d) Score test of $H_0 : \beta = 1$.
   (e) 99% likelihood ratio confidence interval for β.
   (f) Likelihood ratio test of $H_0 : \beta = 1$.

2. *Simulation study comparing likelihood ratio, Wald, and Score tests.* As in the above problem, simulate from the gamma distribution, only this time with a sample size of 20:

```
x <- rgamma(20, shape=2, rate = 1)
```

   In this problem, you will carry out simulations in which you generate $N = 10,000$ samples in this manner.
   (a) Carry out Wald, likelihood ratio, and score tests of $H_0 : \beta = 1$. Report the observed type I error using $p < 0.05$ as a rejection threshold; note that this is 1 - the coverage of the 95% confidence interval, but the tests are easier to calculate.
   (b) Now, suppose we reparameterize the problem in terms of the scale parameter:
   $$f(x|\alpha, \theta) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}.$$
   Re-derive the score and information matrix in terms of this new parameterization.
   (c) Carry out Wald, likelihood ratio, and score tests of $H_0 : \theta = 1$. Report the observed type I error rate; note in this particular case, $\beta^* = \theta^* = 1$. Note that you can re-use much of the code you have already written; for example, $\hat{\alpha}$ is the same in both parameterizations.

3. *Newton's method and iteratively weighted least squares.* This problem explores the relationship between Newton's method and weighted least squares.
   (a) Suppose $\mathbf{y} \sim \mathrm{N}(\mathbf{X}\boldsymbol{\beta}, \mathbf{V})$, where $\mathbf{V}$ is a known variance-covariance matrix. Derive the MLE of $\boldsymbol{\beta}$.

(b) In class, we derived the Newton update for Poisson regression

$$\widehat{\boldsymbol{\beta}}^{(m+1)} = \widehat{\boldsymbol{\beta}}^{(m)} + (\mathbf{X}^\top \mathbf{W}^{(m)} \mathbf{X})^{-1} \mathbf{X}^\top (\mathbf{y} - \boldsymbol{\mu}^{(m)}),$$

by taking a first-order approximation to the score function about $\widetilde{\boldsymbol{\beta}}$. Suppose we instead take a second-order Taylor series expansion of the log-likelihood about $\widetilde{\boldsymbol{\beta}}$. As we have seen in previous assignments, this approximate likelihood can be written as the likelihood of a multivariate normal distribution as in part (a). Show that the MLE of this approximate likelihood is the Newton update.