Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 8

Due: Monday, November 2

1. *Completing the square part 2.* Do part (b) of problem 1 from assignment 6.

2. *Score test with nuisance parameters.* Prove the following theorem, which was stated in our lecture on score, Wald, and likelihood ratio approaches:

   Theorem: If (A)-(C) hold and $\boldsymbol{\theta}_0 = \boldsymbol{\theta}_1^*$, then

   $$\mathbf{u}_1(\hat{\boldsymbol{\theta}}_0)^\top \mathscr{V}_{11}(\hat{\boldsymbol{\theta}}_0)\mathbf{u}_1(\hat{\boldsymbol{\theta}}_0) \xrightarrow{\text{d}} \chi_r^2.$$

   Hint: Take a Taylor series expansion of $\mathbf{u}(\boldsymbol{\theta})$ about $\boldsymbol{\theta}^*$, evaluated at $\hat{\boldsymbol{\theta}}_0$.

   In this theorem, you may make use of the following lemma, which you do not need to prove (its proof is essentially identical to the case for the unrestricted MLE $\hat{\boldsymbol{\theta}}$).

   Lemma: If (A)-(C) hold and $\boldsymbol{\theta}_0 = \boldsymbol{\theta}_1^*$, then

   $$\sqrt{n}(\hat{\boldsymbol{\theta}}_2(\boldsymbol{\theta}_0) - \boldsymbol{\theta}_2^*) = \mathscr{I}_{22}^{-1}(\boldsymbol{\theta}^*)\frac{1}{\sqrt{n}}\mathbf{u}_2(\boldsymbol{\theta}^*) + o_p(1),$$

   where $\mathbf{u} = (\mathbf{u}_1^\top\ \mathbf{u}_2^\top)^\top$ describes the partitioned score vector, $\mathscr{I}_{22}$ is the lower right block of the Fisher information for a single observation, and here the final $o_p(1)$ term is a $(d-r)\times 1$ vector, all of whose elements are converging in probability to zero.

3. *Projection matrices and the χ^2 distribution.* A matrix $\mathbf{P}$ satisfying $\mathbf{P} = \mathbf{PP}$ is known as an *idempotent matrix*, or *projection matrix*. Below, suppose that $\mathbf{P}$ is a symmetric projection matrix.

   (a) Show that every eigenvalue of $\mathbf{P}$ must be either 1 or 0.

   (b) Show that the rank of $\mathbf{P}$ equals the trace of $\mathbf{P}$.

   (c) Show that if $\mathbf{z} \sim \text{N}(\mathbf{0}, \mathbf{I})$, then $\mathbf{z}^\top\mathbf{P}\mathbf{z} \sim \chi_r^2$, where r is the rank of $\mathbf{P}$.