Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 4

Due: Monday, September 28

1. *Convergence in quadratic mean and probability.* This problem involves proving the theorems from the slide titled "Convergence in mean vs convergence in probability" from the "Modes of convergence" lecture. Hint: You may wish to review Markov's inequality.

   (a) Prove that if $\mathbf{x}_n \xrightarrow{r} \mathbf{x}$ for some $r > 0$, then $\mathbf{x}_n \xrightarrow{\mathrm{P}} \mathbf{x}$.

   (b) Prove that if $\mathbf{a} \in \mathbb{R}^d$, then $\mathbf{x}_n \xrightarrow{\mathrm{qm}} \mathbf{a}$ if and only if $\mathbb{E}\mathbf{x}_n \to \mathbf{a}$ and $\mathbb{V}\mathbf{x}_n \to \mathbf{0}_{d\times d}$.

2. *Bounded in probability vs convergence in distribution.* Prove that if a sequence of random vectors $\mathbf{x}_n \in \mathbb{R}^d$ converges in distribution, then $\mathbf{x}_n$ is bounded in probability.

3. *The Cramér-Wold device.* Let $\mathbf{x}_n \in \mathbb{R}^d$ be a sequence of random vectors. Prove that if $\mathbf{a}^\top\mathbf{x}_n \xrightarrow{\mathrm{d}} \mathbf{a}^\top\mathbf{x}$ for all vectors $\mathbf{a} \in \mathbb{R}^d$, then $\mathbf{x}_n \xrightarrow{\mathrm{d}} \mathbf{x}$. Hint: Continuity theorem.