Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 2

Due: Monday, September 14

1. *Vector norms.*

   (a) Show that $\|\cdot\|_2$ is a norm.

   (b) Show that $\|\cdot\|_\infty$ is a norm.

   (c) Let $\|\cdot\|_{1/2}$ denote the function of $\mathbf{x}$ you would obtain by using $p = 1/2$ in the definition of an L_p norm. Is $\|\cdot\|_{1/2}$ a norm? Why or why not?

2. *Logistic regression.* The logistic regression model states that Y_i is equal to 1 with probability π_i and 0 otherwise, with π_i related to a set of linear predictors $\{\eta_i\}$ by the following model:

$$\log \frac{\pi_i}{1-\pi_i} = \eta_i \qquad \text{for } i = 1, 2, \ldots, n$$
$$\boldsymbol{\eta} = \mathbf{X}\boldsymbol{\beta}$$

   where $\boldsymbol{\eta} \in \mathbb{R}^n$, $\boldsymbol{\beta} \in \mathbb{R}^d$, and $\mathbf{X}$ is an $n \times d$ matrix.

   (a) Let ℓ_i denote the contribution to the log-likelihood from observation i. Find the partial derivative of ℓ_i with respect to η_i. Simplify your answer as much as possible.

   (b) Let $\ell : \mathbb{R}^n \to \mathbb{R}$ denote the log-likelihood as a function of the linear predictors $\boldsymbol{\eta}$. Find $\nabla_{\boldsymbol{\eta}} \ell$.

   (c) Find $\nabla_{\boldsymbol{\beta}} \boldsymbol{\eta}$.

   (d) Find $\nabla_{\boldsymbol{\beta}} \ell$.

3. *O-notation proofs.* Prove the following results:

   (a) $O(1)o(1) = o(1)$.

   (b) $\{1 + o(1)\}^{-1} = O(1)$.

   (c) $o\{O(1)\} = o(1)$.

4. *Exponential Taylor series.*

   (a) Show that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

   (b) What is the infinite series for e^{ax}?

   (c) What is the infinite series for b^x?

   (d) Let $f : \mathbb{R}^d \to \mathbb{R}$. What is the second-order Taylor series for $f(\mathbf{x}) = \exp(\mathbf{a}^\top \mathbf{x})$ about $\mathbf{x} = \mathbf{0}$? Give both the o-notation and Lagrange forms.

(e) Suppose $\mathbf{a} = [2 \quad -1]^\top$ and $\mathbf{x} = [1 \quad 1]^\top$. Find the point $\mathbf{x}^*$ on the line segment connecting $\mathbf{x}$ and $\mathbf{0}$ that satisfies the Lagrange form of Taylor's theorem.

5. *Uniform convergence.* For each of the following sequences, determine the pointwise limit of $\{f_n\}$ and decide whether $f_n \to f$ uniformly on the set given or not.

   (a) $f_n(x) = \sqrt[n]{x}$ on $[0, 1]$.

   (b) $f_n(x) = e^x/x^n$ on $(1, \infty)$.

   (c) $f_n(\mathbf{x}) = n^{-1}\exp\{-\|\mathbf{x}\|^2\}$ on $\mathbb{R}^d$.