Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 1

Due: Monday, August 31

1. *Exchange paradox.* Consider the following seemingly paradoxical situation. There are two identical envelopes, each containing money. One contains twice as much as the other. You may pick one envelope and keep the money it contains. You randomly choose an envelope, but before opening it, you are told that the envelope contains x dollars, and you are given the opportunity to switch envelopes. Should you? You reason as follows: "The other envelope contains either $x/2$ or $2x$ dollars, with equal probability. Therefore, if I switch envelopes, my expected value is

   $$\tfrac{1}{2} \cdot \tfrac{x}{2} + \tfrac{1}{2} \cdot 2x = \tfrac{5}{4}x,$$

   which is larger than the x dollars I can expect if I keep the original envelope." So you decide to switch.

   This seems unreasonable, however, for many reasons. Among them, the argument proceeds exactly the same way regardless of the value of x. So, even if you aren't told what x is, you should switch. Then having switched, you can repeat the above argument and decide to switch again . . . and again . . . and so on forever.

   What is wrong with the above argument? Explain, and provide an approach to the envelope switching problem that does not lead to a paradox. There are many possible answers to this problem.

2. *Combining likelihoods, part 1.* Suppose we have two independent samples drawn from $N(\theta, 1)$. From the first sample, we know only that $n_1 = 5$ and the maximum value of the samples is 3.5. From the second sample, we know that $n_2 = 3$ and $\bar{x}_2 = 4$.

   (a) Derive the combined likelihood.

   (b) Plot the following three likelihood curves (within a single graph):

   - The likelihood from sample 1
   - The likelihood from sample 2
   - The combined likelihood

   All three likelihoods should be scaled such that their maximum value is 1.

3. *Combining likelihoods, part 2.* In the previous exercise, we combined density and probability terms . . . maybe this causes problems? Modify the combined likelihood from the previous exercise so that all density terms are replaced by probability terms such that an observation that $X = x$ means $x - \epsilon < X < x + \epsilon$.

   Plot the following three likelihood curves (within a single graph):

   - The original combined likelihood (with density terms)
   - The "discretized" likelihood with $\epsilon = 0.01$
   - The "discretized" likelihood with $\epsilon = 0.1$

   Again, all three likelihoods should be scaled such that their maximum value is 1.