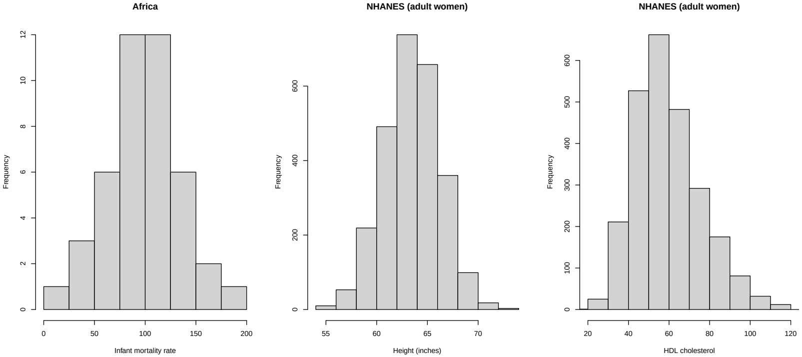
The normal distribution

Patrick Breheny

March 4

A common histogram shape

Histograms of infant mortality rates, heights, and cholesterol levels:



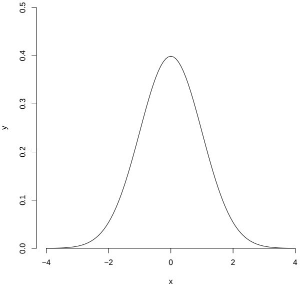
What do these histograms have in common?

The normal curve

Mathematicians discovered long ago that the equation

$$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

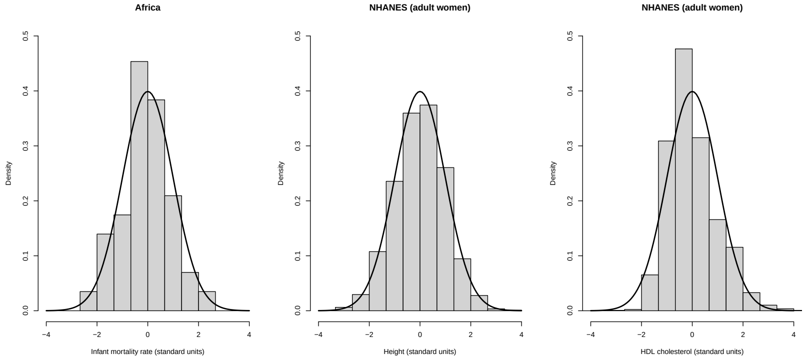
described the histograms of many random variables



Features of the normal curve

- The normal curve is symmetric around $x = 0$
- The normal curve is always positive
- The normal curve drops rapidly down near zero as x moves away from 0

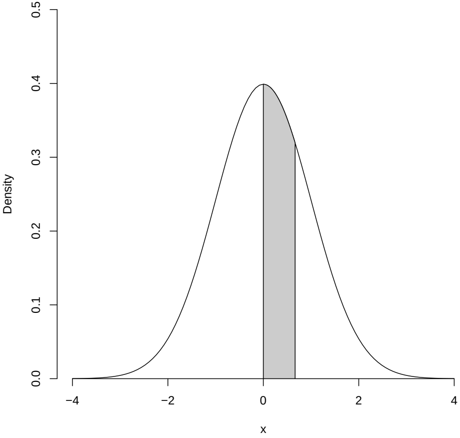
The normal curve in action



- Note that the data has been standardized and that the vertical axis is now called "density"
- Data whose histogram looks like the normal curve are said to be *normally distributed* or to *follow a normal distribution*

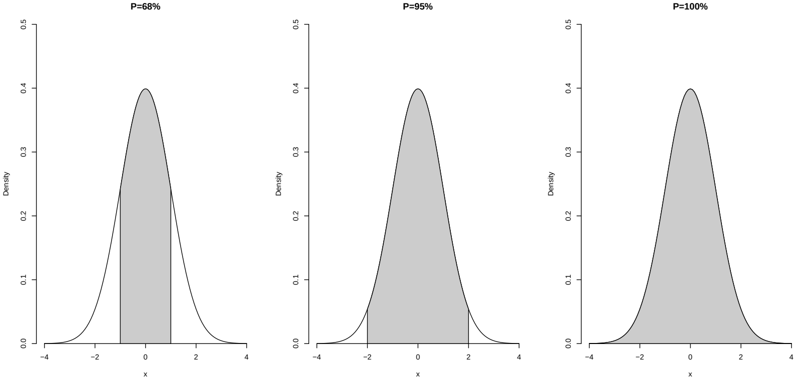
Probabilities from the normal curve

Probabilities are given by the area under the normal curve:



The 68%/95% rule

This is where the 68%/95% rule of thumb that we discussed earlier comes from:

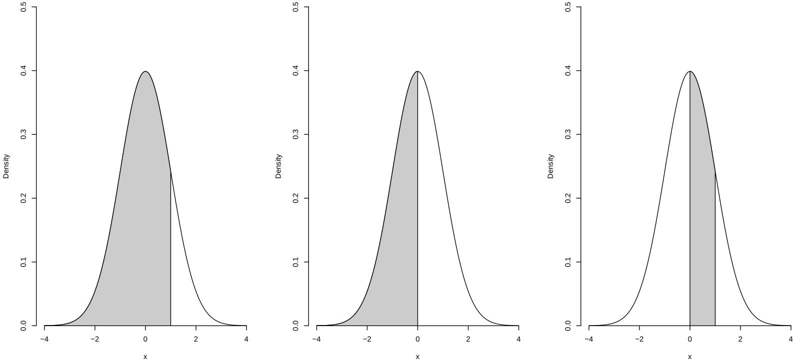


Calculating probabilities

- By knowing that the total area under the normal curve is 1, we can get a rough idea of the area under a curve by looking at a plot
- However, to get exact numbers, we will need a computer
- "How much area is under this normal curve?" is an extremely common question in statistics, and programmers have developed algorithms to answer this question very quickly
- The output from these algorithms is commonly collected into tables, which is what you will have to use for exams

Calculating the area under a normal curve, example 1

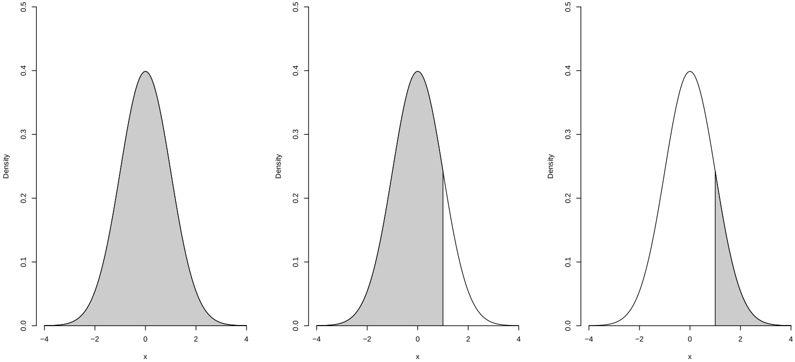
Find the area under the normal curve between 0 and 1



$$.84 - .5 = .34$$

Calculating the area under a normal curve, example 2

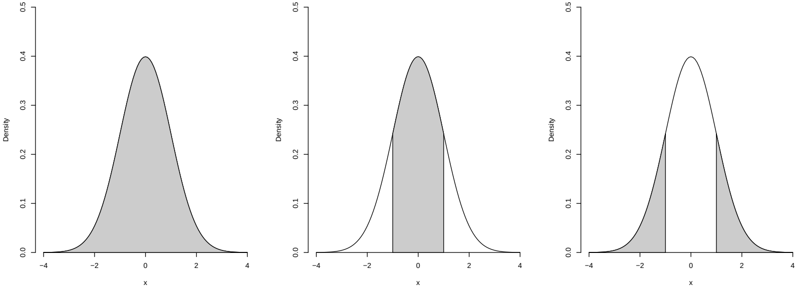
Find the area under the normal curve above 1



$$1 - .84 = .16$$

Calculating the area under a normal curve, example 3

Find the area under the normal curve that lies outside -1 and 1



- 1 - (.84-.16) = .32
- Alternatively, we could have used symmetry: 2(.16)=.32

Calculating percentiles

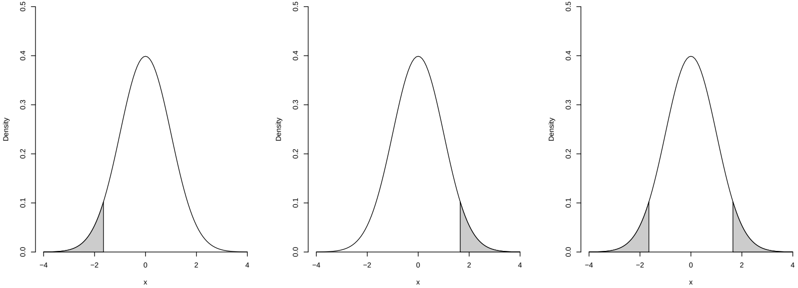
- A related question of interest is, "What is the xth percentile of the normal curve?"
- This is the opposite of the earlier question: instead of being given a value and asked to find the area to the left of the value, now we are told the area to the left and asked to find the value
- With a table, we can perform this inverse search by finding the probability in the body of the table, then looking to the margins to find the percentile associated with it

Calculating percentiles (cont'd)

- What is the 60th percentile of the normal curve?
- There is no ".600" in the table, but there is a ".599", which corresponds to 0.25
- The real 60th percentile must lie between 0.25 and 0.26 (it's actually 0.2533)
- For this class, 0.25, 0.26, or anything in between is an acceptable answer
- How about the 10th percentile?
- The 10th percentile is -1.28

Calculating values such that a certain area lies within/outside them

Find the number x such that the area outside $-x$ and x is equal to 10%



Our answer is therefore ± 1.645 (the 5th/95th percentile)

Reconstructing a histogram

- In week 2, we said that the mean and standard deviation provide a two-number summary of a histogram
- We can now make this observation a little more concrete
- Anything we could have learned from a histogram, we will now determine by approximating the real distribution of the data by the normal distribution
- This approach is called the *normal approximation*

NHANES adult women

- The data set we will work with on these examples is the NHANES sample of the heights of 2,649 adult women
- The mean height is 63.5 inches
- The standard deviation of height is 2.75 inches

Procedure: Probabilities using the normal curve

The procedure for calculating probabilities with the normal approximation is as follows:

#1 Draw a picture of the normal curve and shade in the appropriate probability

#2 Convert to standard units: letting x denote a number in the original units and z a number in standard units,

$$z = \frac{x - \bar{x}}{SD}$$

where $\bar{x}$ is the mean and SD is the standard deviation

#3 Determine the area under the normal curve using a table or computer

Estimating probabilities: Example # 1

- Suppose we want to estimate the percent of women who are under 5 feet tall
- 5 feet, or 60 inches, is 3.5/2.75=1.27 standard deviations below the mean
- Using the normal distribution, the probability of more than 1.27 standard deviations below the mean is $P(x < -1.27) = 10.2\%$
- In the actual sample, 282 out of 2,649 women were under 5 feet tall, which comes out to 10.6%

Estimating probabilities: Example # 2

- Another example: suppose we want to estimate the percent of women who are between 5'3 and 5'6 (63 and 66 inches)
- These heights are 0.18 standard deviations below the mean and 0.91 standard deviations above the mean, respectively
- Using the normal distribution, the probability of falling in this region is 39.0%
- In the actual data set, 1,029 out of 2,649 women were between 5'3 and 5'6: 38.8%

Procedure: Percentiles using the normal curve

- We can also use the normal distribution to approximate percentiles
- The procedure for calculating percentiles with the normal approximation is as follows:
  - #1 Draw a picture of the normal curve and shade in the appropriate area under the curve
  - #2 Determine the percentiles of the normal curve corresponding to the shaded region using a table or computer
  - #3 Convert from standard units back to the original units:

$$x = \bar{x} + z(SD)$$

where, again, x is in original units, z is in standard units, $\bar{x}$ is the mean, and SD is the standard deviation

Approximating percentiles: Example

- Suppose instead that we wished to find the 75th percentile of these women's heights
- For the normal distribution, 0.67 is the 75th percentile
- The mean plus 0.67 standard deviations in height is 65.35 inches
- For the actual data, the 75th percentile is 65.39 inches

The broad applicability of the normal approximation

- These examples are by no means special: the distribution of many random variables are very closely approximated by the normal distribution
- Indeed, this is why statisticians call it the "normal" distribution
- Other names for the normal distribution include the Gaussian distribution (after its inventor) and the bell curve (after its shape)
- For variables with approximately normal distributions, the mean and standard deviation essentially tell us everything about the data – other summary statistics and graphics are redundant

Caution

- Other variables, however, are not approximated by the normal distribution well, and give misleading or nonsensical results when you apply the normal approximation to them
- For example, the value 0 lies 1.63 standard deviations below the mean infant mortality rate for Europe
- The normal approximation therefore predicts a probability that 5.1% of the countries in Europe will have negative infant mortality rates

Caution (cont'd)

- As another example, the normal distribution will always predict the median to lie 0 standard deviations above the mean
- *i.e.*, it will always predict that the median equals the mean
- As we have seen, however, the mean and median can differ greatly when distributions are skewed
- For example, according to the U.S. census bureau, the mean income in the United States is $66,570, while the median income is $48,201

Summary

- The distribution of many random variables are very closely approximated by the normal distribution
- Know how to calculate the area under the normal curve
- Know how to determine percentiles of the normal curve
- Know how to approximate probabilities for real data using the normal approximation
- Know how to approximate quantiles for real data using the normal approximation