Riemann Problem For A Non-Strictly Hyperbolic System In Chemotaxis

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Table of Contents

1. Introduction
2. Crash Lesson
3. Set Up
4. Shock Waves, Rarefaction Curves, and Contact Discontinuities
5. Theorem and Showcase
6. Conclusion

System

Equation

$$\begin{aligned} u_t + (uv)_x &= 0 \\ v_t - u_x &= 0 \end{aligned} \quad (1)$$

where $t > 0, x \in \mathbb{R}$ with initial data:

Riemann Problem

$$(u(x,0), v(x,0)) = \begin{cases} (u_-, v_-) \text{ for } x < 0 \\ (u_+, v_+) \text{ for } x > 0, \end{cases} \quad (2)$$

where (u_-, v_-) and (u_+, v_+) are constant states.

Features

1. The system is physically motivated in chemotaxis, which is the movement of organisms due to chemical response.
2. u - represents the concentration of bacteria, v - function of chemical concentration.
3. The system has a strictly hyperbolic and an elliptic region, with a nonstrictly hyperbolic boundary (parabola shaped).
4. We solved the Riemann problem **up to the non-strictly hyperbolic boundary** as well as a linearly degenerate region for $u \geq 0$ (physically relevant).
5. We will see **shock waves**, **rarefaction waves**, and **contact discontinuities** in our solution.

History

1. Rascle (1985): Solved problem that was reflected along the y axis.
2. Hillen and Wang (2008): Describe shock wave solutions in parameterized form for our equation.
3. Tong Li, L. Wang, and Liu, H. (2016): Derived (1) from the physical point of view via the Keller-Segel model.

Table of Contents

1 Introduction
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3 Set Up
4 Shock Waves, Rarefaction Curves, and Contact Discontinuities
5 Theorem and Showcase
6 Conclusion

Riemann Problem

- We want to solve the Riemann Problem with n equations.
- Consider $U \in \mathbb{R}^n$.
- $U = (u_1, u_2, ..., u_n), F(U) = (f_1(u), f_2(u), ..., f_n(u))$
- The system

$$U_t + F(U)_x = 0$$

with initial data

$$U(x,0) = U_0(x) = \begin{cases} U_-, & x < 0 \\ U_+, & x > 0 \end{cases} \tag{3}$$

is the Riemann problem with u_l, u_r constant vectors.

System: Shock Waves

- By Rankine-Hugoniot conditions, if U has a discontinuity across $x = st$, the jump conditions need to be satisfied:

$$s[U] = [F(U)],$$

where $[U] = U_- - U_+$ and $F(U) = F(U_-) - F(U_+)$

- For systems, the *entropy inequalities* are as follows:

$$\lambda_k(U_+) < s < \lambda_{k+1}(U_+)$$

$$\lambda_{k-1}(U_-) < s < \lambda_k(U_-)$$

for $1 \leq k \leq n$.

- Such a discontinuity is called a *k-shock wave*.

Contact Discontinuities

- They occur when $\nabla\lambda_i \cdot r_i = 0$ (in linearly degenerate regions).
- If two nearby states U_- and U_+ have the same $k-$ Riemann invariants with respect to a linearly degenerate field, then they are connected to each other by a contact discontinuity of speed $s = \lambda_k(U_-) = \lambda_k(U_+)$.

Theory - Rarefaction Waves

- A *rarefaction wave* is a continuous solution of the above system in the form $U = U(x/t)$.
- The kth family is genuinely nonlinear, $\nabla\lambda_k \cdot r_k \neq 0$, where r_k is the right eigenvalue.

Rarefaction - Continued

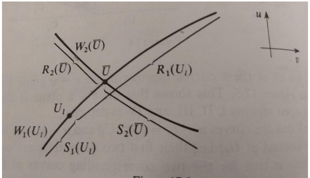
- Let $\xi = x/t$
- $$-\xi U_\xi + F(U)_\xi = 0$$
  or
  $$(dF - \xi I)U_\xi = 0$$
  .
- 

Figure 17.6

Table of Contents

1 Introduction

2 Crash Lesson

3 Set Up

4 Shock Waves, Rarefaction Curves, and Contact Discontinuities

5 Theorem and Showcase

6 Conclusion

Jacobian

Initial Setup

$$U_t + (F(U))_x = 0 \tag{4}$$

Jacobian

$U = (u, v)$ and $F(U) = (uv, -u)$, The Jacobian of F is as follows:

$$A(U) = \begin{pmatrix} v & u \\ -1 & 0 \end{pmatrix}$$

Characteristic Equations

Characteristic

The characteristic equation in $H \cup \Sigma$ is

$$\lambda^2 - v\lambda + u = 0 \tag{5}$$

$$\Rightarrow \lambda_1 = \frac{v - \sqrt{v^2 - 4u}}{2} \leq \frac{v + \sqrt{v^2 - 4u}}{2} = \lambda_2$$

Eigenvectors

$$r_{1,2} = \frac{1}{2}(-v \mp \sqrt{v^2 - 4u}, 1)^T \tag{6}$$

Notation

Regions

$$\begin{aligned} H &= \{(u,v) \in \mathbb{R}^2 | v^2 - 4u > 0\} \\ E &= \{(u,v) \in \mathbb{R}^2 | v^2 - 4u < 0\} \\ \Sigma &= \{(u,v) \in \mathbb{R}^2 | v^2 - 4u = 0\} \end{aligned} \tag{7}$$

- $D_- = \{u = 0, v \leq 0\}$
- $D_+ = \{u = 0, v \geq 0\}$
- $GNL \iff$ genuinely nonlinear
- $LD \iff$ linearly degenerate

Nonstrictly Hyperbolic

Feature of System

(i) (1) is strictly hyperbolic in H when $v^2 - 4u > 0$.
(ii) (1) is non-strictly hyperbolic on Σ, $v^2 - 4u = 0$, i.e. $\lambda_1 = \lambda_2$.

Lemma

(i) On D_+, we have $\lambda_2 = v$ is GNL, $\lambda_1 = 0$ is LD.
(ii) On D_-, we have $\lambda_1 = v$ is GNL, $\lambda_2 = 0$ is LD.

Regions

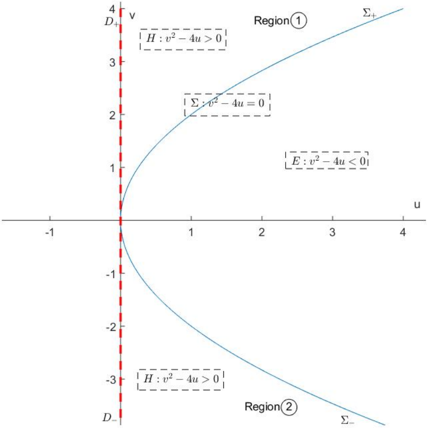

v
u
D_+
D_-
Σ_+
Σ_-
Region 1
Region 2
$H : v^2 - 4u > 0$
$\Sigma : v^2 - 4u = 0$
$E : v^2 - 4u < 0$
$H : v^2 - 4u > 0$

Table of Contents

1. Introduction
2. Crash Lesson
3. Set Up
4. Shock Waves, Rarefaction Curves, and Contact Discontinuities
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Hugoniot Locus

- We now study shock waves.
- The Rankine-Hugoniot conditions applied to (1) yields:

$$\begin{aligned} s(u-u_-) &= uv - u_-v_- \\ s(v-v_-) &= -(u-u_-) \end{aligned} \tag{8}$$

Hugionot Locus

$$\begin{aligned} S_1(U_-) : u - u_- &= -(v-v_-)\frac{v-\sqrt{v^2-4u_-}}{2}, v > v_- \\ S_2(U_-) : u - u_- &= -(v-v_-)\frac{v+\sqrt{v^2-4u_-}}{2}, v < v_-. \end{aligned} \tag{9}$$

Rarefaction Waves

- We now study rarefaction waves.
- Rarefaction waves can be derived by finding the integral curves of the right eigenvectors. For (1), we have:

$$\begin{pmatrix} v-\lambda & u \\ -1 & -\lambda \end{pmatrix} \begin{pmatrix} u_\xi \\ v_\xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Rarefaction

$$\begin{aligned} R_1(U_-) &: \frac{du}{dv} = \frac{-v+\sqrt{v^2-4u}}{2}, v < v_- \\ R_2(U_-) &: \frac{du}{dv} = \frac{-v-\sqrt{v^2-4u}}{2}, v > v_-. \end{aligned} \tag{10}$$

Riemann Invariants I

- Riemann invariants are functions w_i such that $\nabla w_i \cdot r_i = 0$.
- Across rarefaction waves, Riemann invariant does not change.
- $R_i(U_-) = \{U \in \mathbb{R}^2 | w_i(U) = w_i(U_-)\}, i = 1, 2$
-

$$w_1(U) = \begin{cases} \sqrt{\lambda_1}(\lambda_1 + 3\lambda_2), & v > 0 \\ \sqrt{-\lambda_1}(\lambda_1 + 3\lambda_2), & v < 0. \end{cases}$$
$$w_2(U) = \begin{cases} \sqrt{\lambda_2}(3\lambda_1 + \lambda_2), & v > 0 \\ \sqrt{-\lambda_2}(3\lambda_1 + \lambda_2), & v < 0. \end{cases} \tag{11}$$

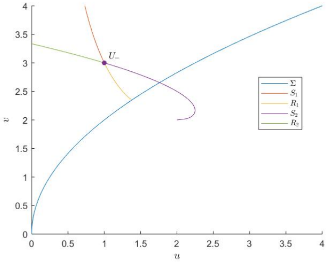
Riemann Invariants - II

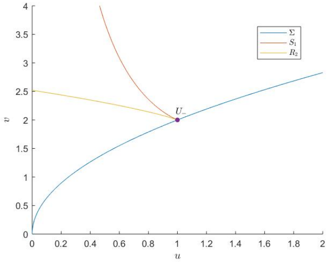
- Computationally, it was easier to write the Riemann invariants in terms of u and v.

$$w_1(U) = \begin{cases} \dfrac{\sqrt{v - \sqrt{v^2 - 4u}}\left(\frac{1}{2}(v - \sqrt{v^2 - 4u}) + \frac{3}{2}(v + \sqrt{v^2 - 4u})\right)}{\sqrt{2}}, \\ \dfrac{\sqrt{-v + \sqrt{v^2 - 4u}}\left(\frac{1}{2}(v - \sqrt{v^2 - 4u}) + \frac{3}{2}(v + \sqrt{v^2 - 4u})\right)}{\sqrt{2}}, \end{cases}$$

$$w_2(U) = \begin{cases} \dfrac{\sqrt{v + \sqrt{v^2 - 4u}}\left(\frac{3}{2}(v - \sqrt{v^2 - 4u}) + \frac{1}{2}(v + \sqrt{v^2 - 4u})\right)}{\sqrt{2}}, \\ \dfrac{\sqrt{-v - \sqrt{v^2 - 4u}}\left(\frac{3}{2}(v - \sqrt{v^2 - 4u}) + \frac{1}{2}(v + \sqrt{v^2 - 4u})\right)}{\sqrt{2}}, \end{cases} \tag{12}$$

Shock and Rarefaction Curves





Contact Discontinuity

- Let $U_-, U_+ \in D_- \cup D_+$. Then,
- U_- and U_+ can be connected a contact discontinuity of speed $s = 0$.

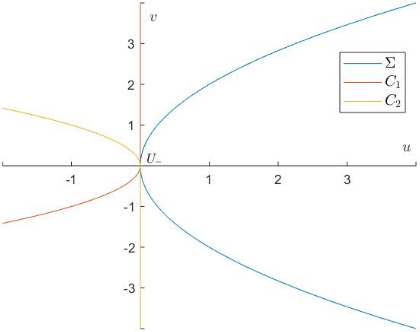


Table of Contents

1 Introduction

2 Crash Lesson

3 Set Up

4 Shock Waves, Rarefaction Curves, and Contact Discontinuities

5 Theorem and Showcase

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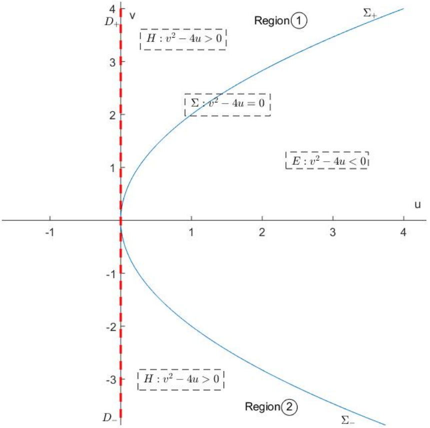
Theorem

Theorem 4

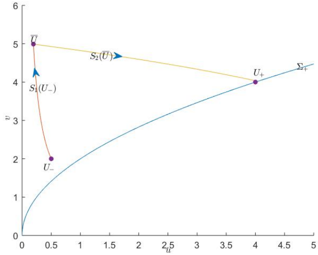
Suppose U_- and $U_+ \in (H \cup \Sigma) \cap \{u \geq 0\}$. Then the Riemann solutions can be constructed if an intermediate state $\overline{U}$, which connects U_- and U_+, exists in the same region.

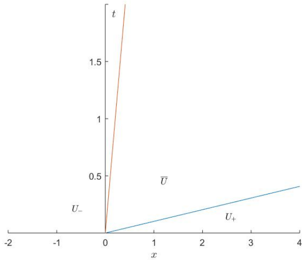
- We will show this via showcases.

Regions

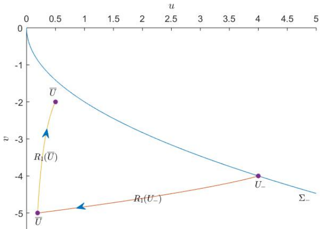

v
u
D_+
D_-
Region 1
Region 2
Σ_+
Σ_-
$H : v^2 - 4u > 0$
$\Sigma : v^2 - 4u = 0$
$E : v^2 - 4u < 0$
$H : v^2 - 4u > 0$

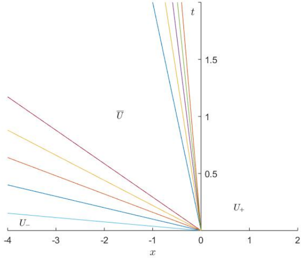
$U_{-} = (0.5, 2), U_{+} = (4, 4) \in \Sigma_{+}$ and $\overline{U} \approx (0.194, 4.99)$.



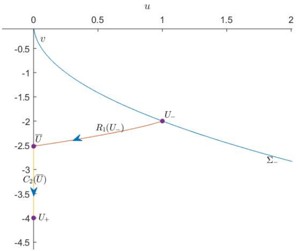


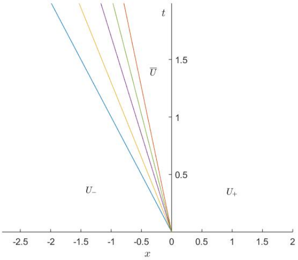
$U_- = (4, -4) \in \Sigma_-$, $U_+ = (0.5, -2)$ and $\overline{U} \approx (0.19, -5)$.



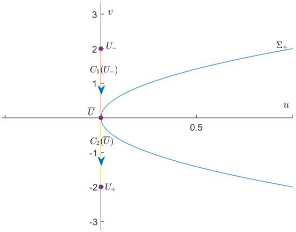


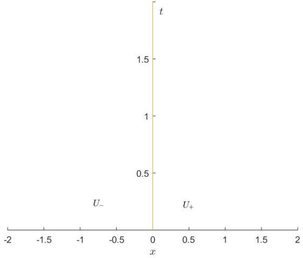
$U_- = (0, 2), U_+ = (1, 2) \in \Sigma_+$ and $\overline{U} \approx (0, -2.52)$.



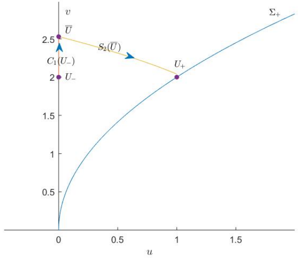


$U_- = (1, -2), U_+ = (0, -4) \in \Sigma_+$ and $\overline{U} \approx (0, -2.52)$.





$U_- = (0, 2), U_+ = (0, -2) \in \Sigma_+$ and $\overline{U} \approx (0, 0)$.



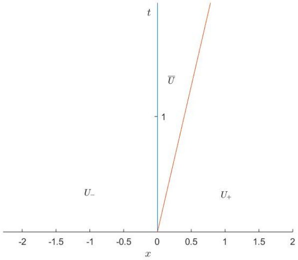


Table of Contents

1 Introduction

2 Crash Lesson

3 Set Up

4 Shock Waves, Rarefaction Curves, and Contact Discontinuities

5 Theorem and Showcase

6 Conclusion

Conclusion

- Graphing capabilities of Matlab were utilized to present the plots.
- We are currently looking at nonclassical transitional waves .
- In the future, we may look at solving other Riemann Problems for 2×2 up to the boundary.

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The End

- Thank You!
- Questions?