The Riemann Problem

Systems of Conservation Laws

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Introduction to the Riemann Problem - Scalar Case

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- For $u \in \mathbb{R}^1$

$$u_t + f(u)_x = 0, \tag{1}$$

$$u(x,0) = u_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$$

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where u_l and u_r are constants.

- This is in the scalar case of the Riemann problem.
- f must be genuinely nonlinear so WLOG, let $f'' > 0$.

Weak Solutions of Conservation Laws

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- A bounded measurable function $u(x,t)$ is called a *weak solution* of this IVP for any $\phi \in C_0^1(\mathbb{R}^1 \times [0,\infty))$

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- ϕ is called a test function.
- ϕ has compact support in $\mathbb{R}^1 \times [0,\infty)$.

Solution to the Riemann Problem - Shock Wave

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- The shock wave solution is:

$$u(x,t) = \begin{cases} u_l, & x < st \\ u_r, & x > st, \end{cases} \tag{3}$$

where

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} \tag{4}$$

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- This is known as the jump condition (Rankine-Hugoniot condition)

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- The rarefaction wave solution is:

$$u(x,t) = \begin{cases} u_l, & x < f'(u_l)t \\ (f')^{-1}(\frac{x}{t}) & f'(u_l)t < t < f'(u_r)t \\ u_r, & x > f'(u_r)t, \end{cases} \tag{5}$$

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- Since we assumed $f'' > 0$ for all u, $u_l < u_r \Rightarrow f'(u_l) < f'(u_r)$.

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- If $f'(u_l) > s > f'(u_r)$, then we say that the entropy conditions are satisfied.
- If not, $f'(u_l) \not> s \not> f'(u_r)$, we have a rarefaction wave.
- Entropy condition **guarantees** uniqueness of weak solution.

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- Consider $\mathbf{u} \in \mathbb{R}^n$.
- $\mathbf{u} = (u_1, u_2, ..., u_n), \mathbf{f(u)} = (f_1(\mathbf{u}), f_2(\mathbf{u}), ..., f_n(\mathbf{u}))$
- The system

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}$$

with initial data

$$\mathbf{u(x,0)} = \mathbf{u_0}(\mathbf{x}) = \begin{cases} \mathbf{u}_l, & x < 0 \\ \mathbf{u}_r, & x > 0 \end{cases} \tag{6}$$

is the Riemann problem with $\mathbf{u}_l, \mathbf{u}_r$ constant vectors.

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- By (4), if $\mathbf{u}$ has a discontinuity across $x = st$, the jump conditions need to be satisfied:

$$s[\mathbf{u}] = [\mathbf{f(u)}],$$

where $[\mathbf{u}] = \mathbf{u}_l - \mathbf{u}_r$ and $\mathbf{f(u)} = \mathbf{f}(\mathbf{u}_l) - \mathbf{f}(\mathbf{u}_r)$

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- For systems, the *entropy inequalities* are as follows:

$$\lambda_k(\mathbf{u}_r) < s < \lambda_{k+1}(\mathbf{u}_r)$$

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for $1 \leq k \leq n$.

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- Such a discontinuity is called a *k-shock wave*.

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where $p' < 0, p'' > 0$.

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where $p' < 0, p'' > 0$.

- Rewrite the p-system as follows:

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}$$

where $\mathbf{u} = (v, u)$ and $\mathbf{f(u)} = (-u, p(v))$

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- $p' < 0 \Rightarrow$ we have real and distinct eigenvalues $\Rightarrow$ **strict hyperbolic system.**

Example (Continued)

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- Now, we can write the Riemann problem for the following initial conditions:

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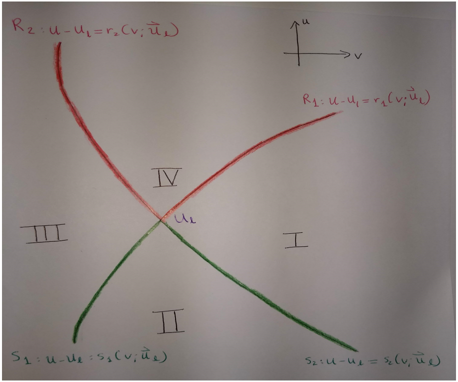
- Entropy condition is satisfied.
- Given left state, what kind of right state can be connected to it?

Graphics


R2 : u - ul = r2(v; ul)
u
v
R1 : u - ul = r1(v; ul)
IV
III
ul
I
II
S1 : u - ul = s1(v; ul)
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- Hence, we have

$$-\sqrt{-p'(v_r)} < s < -\sqrt{-p'(v_l)}$$

and

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- $$-\frac{(u - u_l)}{(v - v_l)} = \frac{p(v) - p(v_l)}{u - u_l}$$
  $$u - u_l = \pm\sqrt{(p(v_l) - p(v))(v - v_l)}$$

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- Similarly,
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- We have 2 families of rarefaction waves, corresponding to either λ_1 or λ_2.
- The kth family is genuinely nonlinear, $\nabla \lambda_k \cdot \mathbf{r_k} \neq 0$, where $\mathbf{r_k}$ is the right eigenvalue.

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- $\lambda_1 v_\xi + u_\xi = 0, \Rightarrow v_\xi, u_\xi \neq 0$
- Since $v_\xi \neq 0, u_\xi / v_\xi = -\lambda_1$

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- Similarly 2$-$ rarefaction wave curve is given by
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- Represents the conservation of mass and momentum
- v denotes the specific volume, i.e. $v = \rho^{-1}$, where ρ is the density, u denotes the velocity, and γ is the adiabatic gas constant.
- Note, in the p-system, if we choose $p(v) = kv^{-\gamma}$, we retrieve this isentropic gas dynamics equations.

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Definition

A *centered simple wave*, centered at (x_0, t_0) is a simple wave depending on $\dfrac{(x - x_0)}{(t - t_0)}$

Definition

The kth characteristic family is said to be *genuinely nonlinear* in a region $D \subset \mathbb{R}^n$ provided that $\nabla\lambda_k \cdot r_k \neq 0$ in D. If this is the case, normalize r_k by $\nabla\lambda_k \cdot r_k = 1$.

Solution of Riemann Problem for general hyperbolic systems

Theorem (Lax (1957))

Let $\mathbf{u}_l \in N \subset \mathbb{R}^n$.
Consider the system of n equations

$$\mathbf{u}_t + \mathbf{f(u)}_x = 0, x \in \mathbb{R}, t > 0,$$

where $\mathbf{u} = (u_1, ..., u_n), \mathbf{f(u)} = (f_1(\mathbf{u}), f_2(\mathbf{u}), ...f_n(\mathbf{u}))$, *the system is* **hyperbolic.**, *and each characteristic field is either genuinely nonlinear or linear degenerate in N.*
Then, there is a neighborhood $\hat{N} \subset N$ *of* $\mathbf{u}_l$ *such that if* $\mathbf{u}_r \in \hat{N}$, *the Riemann problem has precisely one solution, consisting of at most* $(n+1)$ *constant states.*

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- $$\begin{aligned} v_t - u_x &= 0 \\ u_t + p_x &= 0 \\ (e + 1/2u^2)_t + (pu)_x &= 0 \end{aligned}$$

Application To Gas Dynamics

- $$v_t - u_x = 0$$
  $$u_t + p_x = 0$$
  $$(e + 1/2u^2)_t + (pu)_x = 0$$

- $$\begin{pmatrix} 0 & -1 & 0 \\ p_v & 0 & p_s \\ 0 & 0 & 0 \end{pmatrix}$$

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- The eigenvalues are $\lambda_1 = -\sqrt{-p_v}, \lambda_2 = 0, \lambda_3 = \sqrt{-p_v}$
- We have two *genuinely nonlinear* characteristic families, and one linearly degenerate. So, we only have 2 families of shock waves and rarefaction waves.

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References

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The End

- Thank You!
- Questions?