Lax Oleinik Formula

Integral Solution, Entropy Condition, and Uniqueness

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Introduction

1. We are studying the IVP for scalar conservation laws:

$$\begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{1}$$

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2. Recall that $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is an integral solution of (**??**) if

$$\int_0^\infty \int_{-\infty}^\infty uv_t + F(u)v_x \; dx \; dt + gv \; dx|_{t=0} = 0 \tag{2}$$

holds for all test functions v.

Theorem 2

1. The Lax-Oleinik formula is defined by

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2. **Theorem**
3. Assume $F : \mathbb{R} \to \mathbb{R}$ is smooth, uniformly convex, and $g \in L^{\infty}(\mathbb{R})$.
4. Then, $u(x,t)$ is an **integral solution** for the initial value problem (**??**).

Notation

1. Define

$$w(x,t) = \min_{y \in \mathbb{R}} \{ tL(\frac{x-y}{t}) + h(y) \}, \qquad (x \in \mathbb{R}, t > 0), \qquad (4)$$

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2. Then w is a solution of the IVP for the Hamilton-Jacobi:

$$\begin{aligned} w_t + F(w_x) &= 0 \quad \text{a.e. in } \mathbb{R}\times(0,\infty) \\ w &= h \quad \text{on } \mathbb{R}\times\{t=0\}. \end{aligned} \qquad (5)$$

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3. Recall that $w(x,0) = h(x) = \int_0^x g(y)\ dy$.
4. Now, we choose any test function v satsifying $v : \mathbb{R}\times[0,\infty)\to\mathbb{R}$ smooth with compact support.

Proof

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4. $$0 = \int_0^\infty \int_{-\infty}^\infty \underbrace{w_t v_x}_{I_1} + \underbrace{F(w_x)v_x}_{I_2} \; dx \; dt$$

Proof (Continued)

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4. $$\int_0^\infty \int_{-\infty}^\infty w_x v_t \; dx \; dt + F(w_x)v_x \; dxdt + \int_{-\infty}^\infty gv \; dx|_{t=0}$$
5. Let $u = w_x$, we get that u is an **integral solution** (**??**) for (**??**).

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3. Since $F'' > 0$, we conclude that $u_l > u_r$.

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2. $$u(x+z,t) - u(x,t) \leq \frac{C}{t} z \qquad (6)$$
3. (**??**) is known as the entropy condition.
4. Observe that for $t > 0$, $u(x,t) - \dfrac{C}{t} x$ is nonincreasing. So, $u_l(x,t) \geq u_r(x,t)$.

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3. $$u(x,t) = G(\frac{x - y(x,t)}{t})$$
4. $$\begin{aligned} u(x,t) &\geq G(\frac{x - y(x+z,t)}{t}) \quad \text{for } z > 0 \\ &\geq G(\frac{x + z - y(x+z,t)}{t}) - \frac{\text{Lip}(G)z}{t} \\ &= u(x+z,t) - \frac{\text{Lip}(G)z}{t} \end{aligned}$$

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4. For test functions $v : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ w/ compact support, &
5. $$u(x+z,t) - u(x,t) \leq C(1 + \frac{1}{t})z$$
constant $C \geq 0$ and a.e. $x, z \in \mathbb{R}, t > 0, z > 0$.

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2. Assume F is convex and smooth.
3. Then, there exists (up to a set of measure zero) at most one entropy solution of the initial value problem (**??**).

Proof Outline

1. **Step 1**: Assume $u, \tilde{u}$ are two entropy solutions of (**??**) and $w = u - \tilde{u}$. Then,

$$
\begin{aligned}
F(u(x,t)) - F(\tilde{u}(x,t)) &= \int_0^1 \frac{d}{dr} F(ru(x,t) + (1-r)\tilde{u}(x,t))\, dr \\
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4. $u_x^\epsilon(x,t), \tilde{u}_x^\epsilon(x,t) \leq C(1 + \frac{1}{t})$ and $u^\epsilon \to u, \tilde{u}^\epsilon \to \tilde{u}$

Proof Outline-2

1. **Step 3:** Then,

$$b_\epsilon(x,t) = \int_0^1 \frac{d}{dr} F(ru^\epsilon(x,t) + (1-r)\tilde{u}^\epsilon(x,t))\, dr$$

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3. **Step 4:** Let $T > 0$ and for any smooth $\Psi : \mathbb{R} \times (0,T) \to \mathbb{R}$, choose v that is the solution for a linear transport equation:

$$\begin{aligned} v_t^\epsilon + b_\epsilon v_x^\epsilon = \Psi \qquad & \text{in } \mathbb{R} \times (0,T) \\ v = 0 \qquad & \text{on } \mathbb{R} \times \{t = T\} \end{aligned} \tag{8}$$

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4. Then $v^\epsilon(x,t)$ is the unique solution of (**??**) via method of characteristics.

Proof Outline - 3

1. **Step 5:** Then show that for each $s > 0$, there exists a constant C_s such that

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for all $0 \leq t \leq T$ and some constant D.

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4. We need to choose partitions, define variations and take the supremum over all partitions.

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4. Hence, $w = u - \tilde{u} = 0 \Rightarrow u = \tilde{u}$ a.e.

The End

- Thank You!
- Questions?