An Introduction to the Generalized Factorials

Based on the Paper of Manjul Bhargava

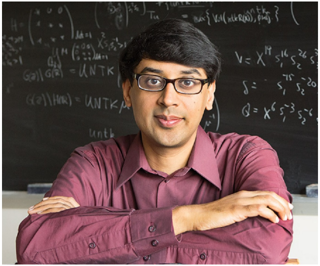
Nitesh Mathur

February 11, 2021

Table of Contents

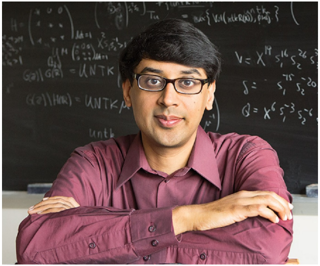
1. Introduction
2. The Generalized Factorial
3. Example
4. Research

About the Author



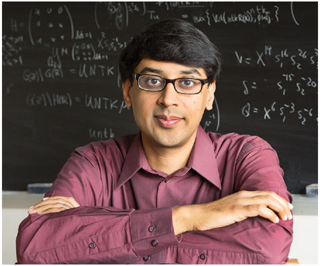
About the Author

- Paper Published in 2000



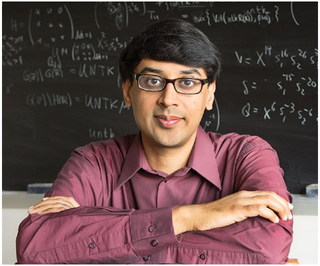
About the Author

- Paper Published in 2000
- Since worked on Higher Composition Laws, 15 and 290 Theorems, and Average Rank of Elliptic Curves



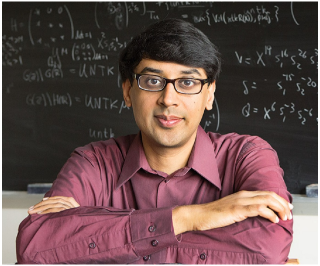
About the Author

- Paper Published in 2000
- Since worked on Higher Composition Laws, 15 and 290 Theorems, and Average Rank of Elliptic Curves
- Fields Medal Recipient in 2014



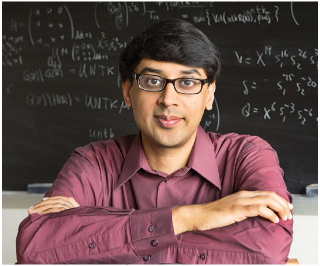
About the Author

- Paper Published in 2000
- Since worked on Higher Composition Laws, 15 and 290 Theorems, and Average Rank of Elliptic Curves
- Fields Medal Recipient in 2014
- Doctoral Advisor: Andrew Wiles



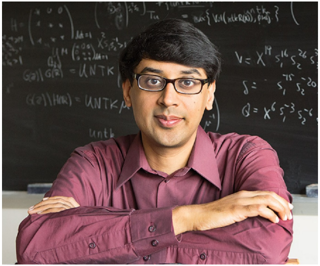
About the Author

- Paper Published in 2000
- Since worked on Higher Composition Laws, 15 and 290 Theorems, and Average Rank of Elliptic Curves
- Fields Medal Recipient in 2014
- Doctoral Advisor: Andrew Wiles
- Professor at Princeton, Leiden University, and adjunct professor in several others.



About the Author

- Paper Published in 2000
- Since worked on Higher Composition Laws, 15 and 290 Theorems, and Average Rank of Elliptic Curves
- Fields Medal Recipient in 2014
- Doctoral Advisor: Andrew Wiles
- Professor at Princeton, Leiden University, and adjunct professor in several others.
- Musician (Tabla Player)



Factorial Function in Number Theory

Factorial Function in Number Theory

- Definition: $n! = \prod_{k=1}^{n} k = n(n-1)(n-2)...(3)(2)(1)$

Factorial Function in Number Theory

- Definition: $n! = \prod_{k=1}^{n} k = n(n-1)(n-2)...(3)(2)(1)$
- Examples: $5! = (5)(4)(3)(2)(1) = 120$

Factorial Function in Number Theory

- Definition: $n! = \prod_{k=1}^{n} k = n(n-1)(n-2)...(3)(2)(1)$
- Examples: $5! = (5)(4)(3)(2)(1) = 120$
- The Gamma Function

Factorial Function in Number Theory

- Definition: $n! = \prod_{k=1}^{n} k = n(n-1)(n-2)...(3)(2)(1)$
- Examples: $5! = (5)(4)(3)(2)(1) = 120$
- The Gamma Function
- $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \; dt$

Factorial Function in Number Theory

- Definition: $n! = \prod_{k=1}^{n} k = n(n-1)(n-2)...(3)(2)(1)$
- Examples: $5! = (5)(4)(3)(2)(1) = 120$
- The Gamma Function
- $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\ dt$
- $\Gamma(n) = (n-1)!$

Factorial Function in Number Theory

- Definition: $n! = \prod_{k=1}^{n} k = n(n-1)(n-2)...(3)(2)(1)$
- Examples: $5! = (5)(4)(3)(2)(1) = 120$
- The Gamma Function
- $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\ dt$
- $\Gamma(n) = (n-1)!$
- $\Gamma(5) = 4! = 24, \Gamma(1/2) = \sqrt{\pi}$

Factorial Function in Number Theory

Factorial Function in Number Theory

- **Theorem 1**
  For any nonnegative integers, k and $l, (k+l)!$ is a multiple of $k!l!$.

Factorial Function in Number Theory

- **Theorem 1**
  For any nonnegative integers, k and $l, (k+l)!$ is a multiple of $k!l!$.
- **Theorem 2** Let f be a primitive polynomial of degree k and let $d(\mathbb{Z}, f) = \text{gcd}\{f(a) : a \in \mathbb{Z}\}$ Then, $d(\mathbb{Z}, f)$ divides $k!$.

Factorial Function in Number Theory

- **Theorem 1**
  For any nonnegative integers, k and l, $(k+l)!$ is a multiple of $k!l!$.
- **Theorem 2** Let f be a primitive polynomial of degree k and let $d(\mathbb{Z}, f) = \text{gcd}\{f(a) : a \in \mathbb{Z}\}$ Then, $d(\mathbb{Z}, f)$ divides $k!$.
- **Theorem 3**
  Let $a_0, a_1, ...a_n \in \mathbb{Z}$ be any $n+1$ integers. Then their product of their pairwise differences

$$\prod_{i<j}(a_i - a_j)$$

  is a multiple of $0!1!...n!$

- **Theorem 1**
  For any nonnegative integers, k and $l, (k+l)!$ is a multiple of $k!l!$.
- **Theorem 2** Let f be a primitive polynomial of degree k and let $d(\mathbb{Z}, f) = \text{gcd}\{f(a) : a \in \mathbb{Z}\}$ Then, $d(\mathbb{Z}, f)$ divides $k!$.
- **Theorem 3**
  Let $a_0, a_1, ...a_n \in \mathbb{Z}$ be any $n+1$ integers. Then their product of their pairwise differences
  $$\prod_{i<j}(a_i - a_j)$$
  is a multiple of $0!1!...n!$
- **Theorem 4** The number of polynomial functions from $\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$ is given by
  $$\prod_{k=0}^{n-1} \frac{n}{\text{gcd}(n, k!)}$$

Table of Contents

1. Introduction
2. The Generalized Factorial
3. Example
4. Research

Motivation

These theorems are true on $\mathbb{Z}$.

Is there a "Generalized Factorial Function" so that for any subset S of $\mathbb{Z}$, the theorems mentioned above still remain true?

p-Ordering

Let $S \subset \mathbb{Z}$ and fix a prime p.

p-Ordering

Let $S \subset \mathbb{Z}$ and fix a prime p.

- Choose $a_0 \in S$

p-Ordering

Let $S \subset \mathbb{Z}$ and fix a prime p.

- Choose $a_0 \in S$
- Choose $a_1 \in S$ that minimizes the highest power of p dividing $a_1 - a_0$

p-Ordering

Let $S \subset \mathbb{Z}$ and fix a prime p.

- Choose $a_0 \in S$
- Choose $a_1 \in S$ that minimizes the highest power of p dividing $a_1 - a_0$
- Choose an element $a_2 \in S$ that minimizes the highest power of p dividing $(a_2 - a_0)(a_2 - a_1)$

p-Ordering

Let $S \subset \mathbb{Z}$ and fix a prime p.

- Choose $a_0 \in S$
- Choose $a_1 \in S$ that minimizes the highest power of p dividing $a_1 - a_0$
- Choose an element $a_2 \in S$ that minimizes the highest power of p dividing $(a_2 - a_0)(a_2 - a_1)$
- For the k^{th} step, choose an element $a_k \in S$ that minimizes the highest power of p dividing $(a_k - a_0)(a_k - a_1) \cdot \ldots \cdot (a_k - a_{k-1})$

p-Ordering

Let $S \subset \mathbb{Z}$ and fix a prime p.

- Choose $a_0 \in S$
- Choose $a_1 \in S$ that minimizes the highest power of p dividing $a_1 - a_0$
- Choose an element $a_2 \in S$ that minimizes the highest power of p dividing $(a_2 - a_0)(a_2 - a_1)$
- For the k^{th} step, choose an element $a_k \in S$ that minimizes the highest power of p dividing $(a_k - a_0)(a_k - a_1) \cdot \ldots \cdot (a_k - a_{k-1})$
- Notation: For each $k, v_k(S, p)$ represents the highest power of p that fulfills the above expression $\{v_0(S, p), v_1(S, p), ..\}$

Table of Contents

1. Introduction
2. The Generalized Factorial
3. Example
4. Research

Example

Let S be the set of all primes. $S = \{2, 3, 5, 7...\}$ and fix prime $p = 2$

Example

Let S be the set of all primes. $S = \{2, 3, 5, 7...\}$ and fix prime $p = 2$

- Let $a_0 = 19$

Example

Let S be the set of all primes. $S = \{2, 3, 5, 7...\}$ and fix prime $p = 2$

- Let $a_0 = 19$
- We need to pick a_1.
  The highest power of p that divides $2 - a_0 = -17$ is $2^0 = 1$

Example

Let S be the set of all primes. $S = \{2, 3, 5, 7...\}$ and fix prime $p = 2$

- Let $a_0 = 19$
- We need to pick a_1.
  The highest power of p that divides $2 - a_0 = -17$ is $2^0 = 1$
- Let's pick a_2 $(a_2 - 19)(a_2 - 2)$. Pick
  $a_2 = 5 \Rightarrow (5 - 19)(5 - 2) = (-14)(3) = (2 \cdot -7)(3)$
  The highest power of p that divides $(a_2 - 19)(a_2 - 2)$ is $2^1 = 2$.

Example

Let S be the set of all primes. $S = \{2, 3, 5, 7...\}$ and fix prime $p = 2$

- Let $a_0 = 19$
- We need to pick a_1.
  The highest power of p that divides $2 - a_0 = -17$ is $2^0 = 1$
- Let's pick a_2 $(a_2 - 19)(a_2 - 2)$. Pick
  $a_2 = 5 \Rightarrow (5 - 19)(5 - 2) = (-14)(3) = (2 \cdot -7)(3)$
  The highest power of p that divides $(a_2 - 19)(a_2 - 2)$ is $2^1 = 2$.
- Similarly, for a_3, we need $(a_3 - 19)(a_3 - 2)(a_3 - 5)$.
  $(17 - 19)(17 - 2)(17 - 5) = (-2)(15)(2^2 \cdot 3)$ The corresponding power here is $2^3 = 8$.

Example

Let S be the set of all primes. $S = \{2, 3, 5, 7...\}$ and fix prime $p = 2$

- Let $a_0 = 19$
- We need to pick a_1.
  The highest power of p that divides $2 - a_0 = -17$ is $2^0 = 1$
- Let's pick a_2 $(a_2 - 19)(a_2 - 2)$. Pick
  $a_2 = 5 \Rightarrow (5 - 19)(5 - 2) = (-14)(3) = (2 \cdot -7)(3)$
  The highest power of p that divides $(a_2 - 19)(a_2 - 2)$ is $2^1 = 2$.
- Similarly, for a_3, we need $(a_3 - 19)(a_3 - 2)(a_3 - 5)$.
  $(17 - 19)(17 - 2)(17 - 5) = (-2)(15)(2^2 \cdot 3)$ The corresponding power here is $2^3 = 8$.
- Similarly for the rest a_k

Examples Continued

- The p-ordering for $p = 2$ is as follows: $\{19, 2, 5, 17, 23, 31, ...,\}$ and its corresponding p-sequence is as follows, $\{1, 1, 2, 8, 16, 128, ...\}$

Back to Theory

Back to Theory

- Construct such a p ordering for every p (Note: Not unique)

Back to Theory

- Construct such a p ordering for every p (Note: Not unique)
- **Punchline 1**: The associated p-sequence of S is independent of the choice of p-ordering.

Back to Theory

- Construct such a p ordering for every p (Note: Not unique)
- **Punchline 1**: The associated p-sequence of S is independent of the choice of p-ordering.
- **Punchline 2**: Let S be any subset of $\mathbb{Z}$. Then the *factorial function* of S, denoted by $k!_S$ is defined by

$$k!_s = \prod_p v_k(S, p)$$

Back to Example

Back to Example

- The p-ordering for the prime subset of $\mathbb{Z}$ is as follows:

Back to Example

- The p-ordering for the prime subset of $\mathbb{Z}$ is as follows:
- $p = 2$
  p-ordering: $\{19, 2, 5, 17, 23, 31, ...,\}$
  p-sequence is as follows, $\{1, 1, 2, 8, 16, 128, ...\}$

Back to Example

- The p-ordering for the prime subset of $\mathbb{Z}$ is as follows:
- $p = 2$
  p-ordering: $\{19, 2, 5, 17, 23, 31, ..., \}$
  p-sequence is as follows, $\{1, 1, 2, 8, 16, 128, ...\}$
- $p = 3$
  p-ordering: $\{2, 3, 7, 5, 13, 17, 19, ...\}$
  p-sequence: $\{1, 1, 1, 3, 3, 9, ...\}$

Examples

- $4!_P = 48, 6!_P = 11520, \ldots$
- Notice, one has to multiply across. Each k represents an index in each p-sequence.

Table of values of $v_k(P, p)$ and $k!_P$

	$p = 2$	$p = 3$	$p = 5$	$p = 7$	$p = 11$	...	$k!_P$
$k = 0$	1	1	1	1	1	...	1×1×1×1×1×... = 1
$k = 1$	1	1	1	1	1	...	1×1×1×1×1×... = 1
$k = 2$	2	1	1	1	1	...	2×1×1×1×1×... = 2
$k = 3$	8	3	1	1	1	...	8×3×1×1×1×... = 24
$k = 4$	16	3	1	1	1	...	16×3×1×1×1×... = 48
$k = 5$	128	9	5	1	1	...	128×9×5×1×1×... = 5760
$k = 6$	256	9	5	1	1	...	256×9×5×1×1×... = 11520

The Natural Numbers

The Natural Numbers

- Consider $\mathbb{N} \subset \mathbb{Z}$ The natural ordering of $\mathbb{N} = \{1, 2, 3, ...\}$ is a p-ordering of $\mathbb{N}$.
  The p-sequences of $\mathbb{N}$ are as follows:

The Natural Numbers

- Consider $\mathbb{N} \subset \mathbb{Z}$ The natural ordering of $\mathbb{N} = \{1, 2, 3, ...\}$ is a p-ordering of $\mathbb{N}$.
  The p-sequences of $\mathbb{N}$ are as follows:
- p = 2: $\{1, 1, 2, 2, 8, 8, 16, 16, ...\}$

The Natural Numbers

- Consider $\mathbb{N} \subset \mathbb{Z}$ The natural ordering of $\mathbb{N} = \{1, 2, 3, ...\}$ is a p-ordering of $\mathbb{N}$.
  The p-sequences of $\mathbb{N}$ are as follows:
- p = 2: $\{1, 1, 2, 2, 8, 8, 16, 16, ...\}$
- p = 3: $\{1, 1, 1, 3, 3, 3, 9, 9, 9, ...\}$

The Natural Numbers

- Consider $\mathbb{N} \subset \mathbb{Z}$ The natural ordering of $\mathbb{N} = \{1, 2, 3, ...\}$ is a p-ordering of $\mathbb{N}$.
  The p-sequences of $\mathbb{N}$ are as follows:
- p = 2: $\{1, 1, 2, 2, 8, 8, 16, 16, ...\}$
- p = 3: $\{1, 1, 1, 3, 3, 3, 9, 9, 9, ...\}$
- p = 5: $\{1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 25, ..\}$

The Natural Numbers

- Consider $\mathbb{N} \subset \mathbb{Z}$ The natural ordering of $\mathbb{N} = \{1, 2, 3, ...\}$ is a p-ordering of $\mathbb{N}$.
  The p-sequences of $\mathbb{N}$ are as follows:
- p $=$ 2: $\{1, 1, 2, 2, 8, 8, 16, 16, ...\}$
- p $=$ 3: $\{1, 1, 1, 3, 3, 3, 9, 9, 9, ...\}$
- p $=$ 5: $\{1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 25, ..\}$
- p $=$ 7: $\{1, 1, 1, 1, 1, 1, 1, 7, 7, 7, 7, 7, ..\}$
  Check Your Results:

The Natural Numbers

- Consider $\mathbb{N} \subset \mathbb{Z}$ The natural ordering of $\mathbb{N} = \{1, 2, 3, ...\}$ is a p-ordering of $\mathbb{N}$.
  The p-sequences of $\mathbb{N}$ are as follows:
- p $=$ 2: $\{1, 1, 2, 2, 8, 8, 16, 16, ...\}$
- p $=$ 3: $\{1, 1, 1, 3, 3, 3, 9, 9, 9, ...\}$
- p $=$ 5: $\{1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 25, ..\}$
- p $=$ 7: $\{1, 1, 1, 1, 1, 1, 1, 7, 7, 7, 7, 7, ..\}$
  Check Your Results:
- $0!_{\mathbb{N}} = 1 * 1 * 1 * 1 * 1... = 1$

The Natural Numbers

- Consider $\mathbb{N} \subset \mathbb{Z}$ The natural ordering of $\mathbb{N} = \{1, 2, 3, ...\}$ is a p-ordering of $\mathbb{N}$.
  The p-sequences of $\mathbb{N}$ are as follows:
- p $=$ 2: $\{1, 1, 2, 2, 8, 8, 16, 16, ...\}$
- p $=$ 3: $\{1, 1, 1, 3, 3, 3, 9, 9, 9, ...\}$
- p $=$ 5: $\{1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 25, ..\}$
- p $=$ 7: $\{1, 1, 1, 1, 1, 1, 1, 7, 7, 7, 7, 7, ..\}$
  Check Your Results:
- $0!_{\mathbb{N}} = 1 * 1 * 1 * 1 * 1... = 1$
- $2!_{\mathbb{N}} = 2 * 1 * 1 * 1 * 1... = 2$

The Natural Numbers

- Consider $\mathbb{N} \subset \mathbb{Z}$ The natural ordering of $\mathbb{N} = \{1, 2, 3, ...\}$ is a p-ordering of $\mathbb{N}$.
  The p-sequences of $\mathbb{N}$ are as follows:
- p = 2: $\{1, 1, 2, 2, 8, 8, 16, 16, ...\}$
- p = 3: $\{1, 1, 1, 3, 3, 3, 9, 9, 9, ...\}$
- p = 5: $\{1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 25, ..\}$
- p = 7: $\{1, 1, 1, 1, 1, 1, 1, 7, 7, 7, 7, 7, ..\}$
  Check Your Results:
- $0!_{\mathbb{N}} = 1 * 1 * 1 * 1 * 1... = 1$
- $2!_{\mathbb{N}} = 2 * 1 * 1 * 1 * 1... = 2$
- $3!_{\mathbb{N}} = 2 * 3 * 1 * 1 * 1... = 6$

The Natural Numbers

- Consider $\mathbb{N} \subset \mathbb{Z}$ The natural ordering of $\mathbb{N} = \{1, 2, 3, ...\}$ is a p-ordering of $\mathbb{N}$.
  The p-sequences of $\mathbb{N}$ are as follows:
- p = 2: $\{1, 1, 2, 2, 8, 8, 16, 16, ...\}$
- p = 3: $\{1, 1, 1, 3, 3, 3, 9, 9, 9, ...\}$
- p = 5: $\{1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 25, ..\}$
- p = 7: $\{1, 1, 1, 1, 1, 1, 1, 7, 7, 7, 7, 7, ..\}$
  Check Your Results:
- $0!_{\mathbb{N}} = 1 * 1 * 1 * 1 * 1... = 1$
- $2!_{\mathbb{N}} = 2 * 1 * 1 * 1 * 1... = 2$
- $3!_{\mathbb{N}} = 2 * 3 * 1 * 1 * 1... = 6$
- $6!_{\mathbb{N}} = 16 * 9 * 5 * 1.... = 720$

More Examples

Sl. No.	Set S	$k!_S$
1	Set of natural numbers	$k!$
2	Set of even integers	$2^k \times k!$
3	Set of integers of the form $an + b$	$a^k \times k!$
4	Set of integers of the form 2^n	$(2^k - 1)(2^k - 2) \ldots (2^k - 2^{k-1})$
5	Set of integers of the form q^n for some prime q	$(q^k - 1)(q^k - 2) \ldots (q^k - q^{k-1})$
6	Set of squares of integers	$(2k)!/2$

Revisit Theorems

Revisit Theorems

- **Theorem 1**
  For any nonnegative integers, k and $l, (k+l)!_S$ is a multiple of $k!_S l!_S$.

Revisit Theorems

- **Theorem 1**
  For any nonnegative integers, k and $l, (k+l)!_S$ is a multiple of $k!_S l!_S$.
- **Theorem 2** Let f be a primitive polynomial of degree k and let $d(S,f) =$gcd$\{f(a) : a \in S\}$ Then, $d(S,f)$ divides $k!_S$.

Revisit Theorems

- **Theorem 1**
  For any nonnegative integers, k and $l, (k+l)!_S$ is a multiple of $k!_S l!_S$.
- **Theorem 2** Let f be a primitive polynomial of degree k and let $d(S,f) = \text{gcd}\{f(a) : a \in S\}$ Then, $d(S,f)$ divides $k!_S$.
- **Theorem 3**
  Let $a_0, a_1, ...a_n \in S$ be any $n+1$ integers. Then their product of their pairwise differences
  $$\prod_{i<j}(a_i - a_j)$$
  is a multiple of $0!_S 1!_S ... n!_S$

Revisit Theorems

- **Theorem 1**
  For any nonnegative integers, k and $l, (k+l)!_S$ is a multiple of $k!_S l!_S$.
- **Theorem 2** Let f be a primitive polynomial of degree k and let $d(S,f) = \text{gcd}\{f(a) : a \in S\}$ Then, $d(S,f)$ divides $k!_S$.
- **Theorem 3**
  Let $a_0, a_1, ...a_n \in S$ be any $n+1$ integers. Then their product of their pairwise differences

  $$\prod_{i<j}(a_i - a_j)$$

  is a multiple of $0!_S 1!_S ... n!_S$
- **Theorem 4** The number of polynomial functions from S to $\mathbb{Z}/n\mathbb{Z}$ is given by

  $$\prod_{k=0}^{n-1} \frac{n}{\text{gcd}(n, k!_S)}$$

The Rest of the Paper

The Rest of the Paper

- A bunch of proofs.

The Rest of the Paper

- A bunch of proofs.
- Generalization to Dedekind Rings.

The Rest of the Paper

- A bunch of proofs.
- Generalization to Dedekind Rings.
- Generalization to Higher Dimensions.

The Rest of the Paper

- A bunch of proofs.
- Generalization to Dedekind Rings.
- Generalization to Higher Dimensions.
- Applications

Table of Contents

1. Introduction
2. The Generalized Factorial
3. Example
4. Research

Posed Questions

Posed Questions

- For a subset $S \subset \mathbb{Z}$, is there a natural combinatorial interpretation of $k!_S$.

Posed Questions

- For a subset $S \subset \mathbb{Z}$, is there a natural combinatorial interpretation of $k!_S$.
- What is the natural combinatorial interpretation for $\binom{n}{k}_S = \dfrac{n!_S}{k!_S(n-k)!_S}$ coefficients?

Posed Questions

- For a subset $S \subset \mathbb{Z}$, is there a natural combinatorial interpretation of $k!_S$.
- What is the natural combinatorial interpretation for $\binom{n}{k}_S = \dfrac{n!_S}{k!_S(n-k)!_S}$ coefficients?
- What is the "binomial theorem" for generalized binomial?

My Research

My Research

- *An Algorithm to Reverse the Generalized Factorials Process*

My Research

- *An Algorithm to Reverse the Generalized Factorials Process*
- Sequences of Generalized Factorials show up in the denominators of numerous series.

My Research

- *An Algorithm to Reverse the Generalized Factorials Process*
- Sequences of Generalized Factorials show up in the denominators of numerous series.
- **Research Question:** Given a sequence of numbers, presumably a sequence of generalized factorials for a particular set, can we figure out what set that is?

Sources

Bhargava, Manjul (2000). "The Factorial Function and Generalizations" (PDF). The American Mathematical Monthly. 107 (9): 783–799.

Thank You!

- GAUSS, University of Iowa
- Dr. O'Neil, Jon Bolin
- Oklahoma-Arkansas MAA Section Meeting
- University of Tulsa
- TU Journal Club

Questions?