Applications of Algebraic Geometry

With A Hint of Robotics

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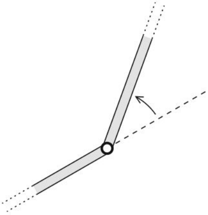
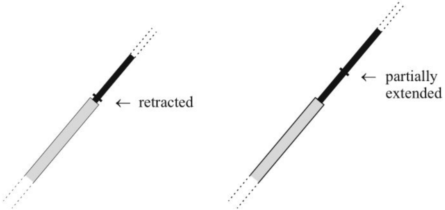
The Setup

- Highly Idealized Robots
- Consider robots constructed from rigid links or segments, connected by joints
- Possible motions are constructed using (1) planar revolute joints, and (2) prismatic joints
- A planar revolute joint permits a **rotation** of one segment relative to another.
- A prismatic joint permits one segment of one segment of a robot to move by **sliding or translation around an axis**

Visuals

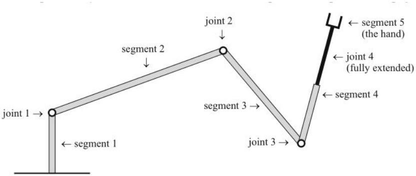
- 

  a revolute joint

- 

  a prismatic joint

Example

- Joints are connected by segments, and we write them in increasing order
- Example with 3 revolute joints, one prismatic joint, and 5 segments.
- 

The Space

- Revolute joint can be described by measuring angle θ counterclockwise and can be parameterized **by a circle** S^1.
- Prismatic joint is given by the distance the joint is extended to and can be parameterized by a **finite interval of real numbers.**
- For a planar robot with r revolute joints and p prismatic joints, the parameterization for the*joint space* $\mathcal{J}$ looks like the Cartesian product: $\mathcal{J} = \underbrace{S^1 \times \ldots \times S^1}_{r \text{ times}} \times I_1 \times \ldots \times I_p$
- Fixing a Cartesian coordinate system in the plane, represent the "hand" of a planar robots by (a, b) in a region $U \subset \mathbb{R}^2$, where the **possible hand orientations** are parameterized by vectors $\mathbf{u}$ in $V = S^1$.
- $\mathcal{C} = U \times V$ is called the **configuration or operational space** of the robot's hand.

Problems

- The mapping $f : \mathcal{J} \to \mathcal{C}$ encodes how **different possible joint settings** yields **different hand configurations**.
- **Forward Kinematic Problem**
- Can we give an explicit description or formula for f in terms of the joint settings and the dimensions of the segments of the robots arm?
- **Inverse Kinematic Problem**
- Given $c \in \mathcal{C}$, can we determine one or all the $j \in \mathcal{J}$ such that $f(j) = c$.

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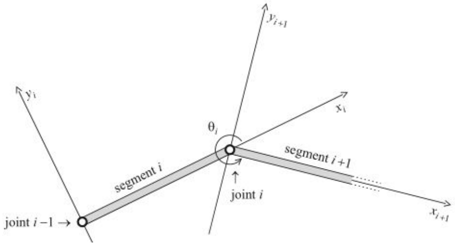
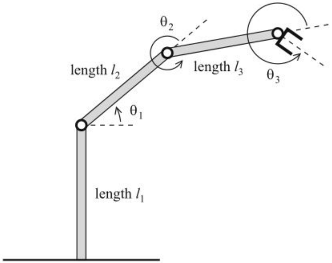
Setup

- Introduce local rectangular coordinate system at each of the revolute joints
- The origin is place at joint i; x_{i+1}-axis lies along segment $i+1$; y_{i+1} axis forms a normal; l_i is the length of segment i.
- 

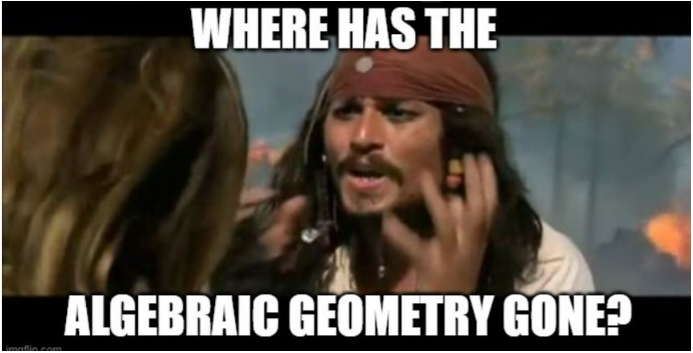
The Math

- If q has (x_{i+1}, y_{i+1}) coordinates, i.e. $q = (a_{i+1}, b_{i+1})$, the then to obtain $q = (a_i, b_i)$, we do the following:
- Rotate by angle θ_i to align the x_i and x_{i+1}-axes
- Then, translate by vector $(l_i, 0)$ (to make the origins of the coordinate systems coincide)
- $\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} \cos\theta_i & \sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} + \begin{pmatrix} l_i \\ 0 \end{pmatrix}$
- Commonly written as:
$\begin{pmatrix} a_i \\ b_i \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_i & \sin\theta_i & 1 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix} = A_i \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix}$

Example

- 

- The map $f : \mathcal{J} \to \mathcal{C}$ can be given by:
- $f(\theta_1 + \theta_2 + \theta_3) = \begin{pmatrix} l_3 \cos(\theta_1 + \theta_2) + l_2 \cos\theta_1 \\ l_3 \sin(\theta_1 + \theta_2) + l_2 \sin\theta_1 \\ \theta_1 + \theta_2 + \theta_3 \end{pmatrix}$
- We leave the computation as an exercise to the reader...


WHERE HAS THE
ALGEBRAIC GEOMETRY GONE?

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Gröbner Basis is Back

- Let $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$, then all the possible ways to place the hand at a given point $(x_1, y_1) = (a, b)$ are described by the following polynomial equations:
-

$$\begin{aligned} a &= l_3(c_1c_2 - s_1s_2) + l_2c_1 \\ b &= l_3(c_1s_2 + c_2s_1) + l_2s_1 \\ 0 &= c_1^2 + s_1^2 - 1 \\ 0 &= c_2^2 + s_2^2 - 1 \end{aligned}$$

- Compute a grevlex Gröbner basis with $c_1 > s_1 > c_2 > s_2$.
- This is the reduced Gröbner basis for ideal I generated by polynomials in the ring $\mathbb{R}(a, b, l_2, l_3)[c_1, s_1, c_2, s_2]$.

Specialization of Gröbner Basis

Propositions

- Assume $\bar{I} =< f_1, ..., f_s >\subset k[\mathbf{x}, \mathbf{t}]$ satisfies $\bar{I} \cap k[\mathbf{t}] = \{0\}$ and fix a monomial order as above. If $G = \{g_1, ..., g_t\}$ is a Gr ö bner basis for $\bar{I}$, Then:
- (i) G is a Gr ö bner basis for the ideal of $k(\mathbf{t}[\mathbf{x}]$ generated by the f_i with respect to the induced monomial order.
- (ii) For $i = 1, ...,$ write $g_i \in G$ in the form: $g_i = h_i(\mathbf{t})\mathbf{x}^{\alpha_i} +$ terms $< \mathbf{x}^{\alpha_i}$,
- where $h_i(\mathbf{t}) \in k[\mathbf{t}]$ is nonzero. If we set $W = \mathbf{V}(h_, ..., h_t) \subset k^m$, then for any specialization $\mathbf{t} \mapsto \mathbf{a} \in k^m - W$, the $g_i(\mathbf{x}, \mathbf{a})$ form a Gröbner basis with respect to the induced monomial order for the ideal generated by the $f_i(\mathbf{x}, \mathbf{a})$ in $k[\mathbf{x}]$.

Propositions - Continued

- **Definition** A **kinematic singularity** for a robot is a point $j \in \mathcal{J}$ such that $J_f(j)$ has rank strictly less than $\min(\dim(\mathcal{J}, \dim(\mathcal{C}))$.
- **Proposition** Let $f : \mathcal{J} \to \mathcal{C}$ be the configuration mapping for a planar robot with $n \geq 3$ revolute points. Then there exist kinematic singularities $j \in \mathcal{J}$.

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Direct Applications

- Automated Theorem Proving
- Wu's Method

Other Applications of Algebraic Geometry

- Statistics, Control Theory, Error-Correcting Codes, phylogenetics, Geometric Modeling
- Also connections to string theory, game theory, graph matching, integer programming
- Under the umbrella of 'computational algebraic geometry' and 'numerical algebraic geometry'

The End

- Thank You!
- Questions?