Applications of Analysis In PDE

An Overview and Applications

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Examples of PDEs

1. Laplace's equation:

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0 \tag{1}$$

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4. Wave Equation:

$$u_{tt} - \Delta u = 0 \tag{4}$$

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Main Idea

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3. Instead of attempting to find explicit solutions, **energy estimates** are used to prove 'weak solutions' to PDE.

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1. For some of the nonlinear PDEs mentioned above, we can find fundamental solutions and explicit formulas using particular techniques.
2. The other way to look at solutions is to view it from a 'functional analysis' point of view.
3. Instead of attempting to find explicit solutions, **energy estimates** are used to prove 'weak solutions' to PDE.
4. The ideal setting to study many of these energy method techniques for PDEs occur in **Sobolev spaces**.

Introduction

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2. In particular, we say that a complex vector space X is a **normed linear space** if for each $x \in X$, we have a norm $||x||$ defined as follows:
   (a) $||x+y|| \leq ||x|| + ||y|| \quad$ for all $x, y \in X$
   (b) $||\alpha x|| = |\alpha|||x||$ for $x \in X, \alpha$ scalar.
   (c) $||x|| = 0 \Rightarrow x = 0$.

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2. The Holder space $C^{k,\gamma}(\bar{U})$ consists of all functions $u \in C^k(\bar{U})$ for which the norm:

$$||u||_{C^{k,\gamma}(\bar{U})} = \sum_{|\alpha| \leq k} ||D^{\alpha}u||_{C(\bar{U})} + \sum_{|\alpha|=k} |D^{\alpha}u|_{C^{0,\gamma}(\bar{U})} \tag{6}$$

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3. In particular, the space of functions $C^{k,\gamma}(\bar{U})$ is a Banach space.

Motivation To Sobolev Spaces

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1. Holder spaces are not often suitable for PDE theory since it is hard to make good analytic estimates
2. We need to find spaces that have less smoothness properties than Holder spaces but are still smooth enough

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Weak Derivatives

1. Let $C_c^{\infty}(U)$ be the space of infinitely differentiable functions $\phi : U \to \mathbb{R}$ with compact support in U.

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2. Let $\phi \in C_c^\infty$ be denoted as a **test function**.
3. Suppose $u, v \in L^1_{\text{loc}}(U)$ and α is a multiindex. We say that v is the α^{th}-weak partial derivative of u, written as $D^\alpha u = v$ if

$$\int_U u D^\alpha \phi \; dx = (-1)^{|\alpha|} \int_U v \phi \; dx \tag{7}$$

for **all** test functions $\phi \in C_c^\infty(U)$.

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   for **all** test functions $\phi \in C_c^\infty(U)$.
4. Basically, if there exists such a v, we say that $D^\alpha u = v$ **in the weak sense**. Otherwise, u does not possess a weak α-th partial derivative.

Sobolev Space Definition

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3. For $u \in W^{k,p}(U)$, define the norm as follows:

$$||u||_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p \, dx\right)^{1/p} & (1 \leq p < \infty) \\ \sum_{|\alpha| \leq k} \text{ess sup}_U |D^\alpha u| & (p = \infty). \end{cases}$$

Sobolev Spaces(Continued)

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1. For example, for $p = 2$,
$H^k(U) = W^{k,2}(U), \qquad (k = 0, 1, 2, ...)$ is a Hilbert space.
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2. In particular, a Sobolev space is a Banach space.
3. Lots of properties, inequalities, and embeddings related to Sobolev spaces

Sobolev Inequalities/Embeddings

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2. **Gagliardo-Nirenberg-Sobolev inequality:** Assume $1 \leq p < n$. There exists a constant C, depending only on p and n such that

$$||u||_{L^{P*}(\mathbb{R}^n)} \leq C||Du||_{L^p(\mathbb{R}^n)} \tag{8}$$

for all $u \in C_c^1(\mathbb{R}^n)$.

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3. **Morrey's inequality**: Assume $n < p \leq \infty$. Then there exists a constant C, depending only on p and n, such that

$$||u||_{C^{0,\gamma}(\mathbb{R}^n)} \leq C||u||_{W^{1,p}(\mathbb{R}^n)} \tag{9}$$

   for all $u \in C^1(\mathbb{R}^n)$, where $\gamma = 1 - n/p$.

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Applications to Elliptic PDEs

1. Notation: $U \subset \mathbb{R}^n, u : \bar{U} \to \mathbb{R}$ unknown function, $f : U \to \mathbb{R}$ given, and L a second-order partial differential operator

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3. This is called the Dirchlet's boundary condition.

Differential Operator

1. The differential operator have either the form:

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \qquad (11)$$

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4. This is known as the nondivergence form.

Continued

1. We say that the partial differential operator L is (uniformly) elliptic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \tag{13}$$

for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$.

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2. (Notation) Let H be a real Hilbert space, $||$ be the norm, $(,)$ the inner product, and $<,>$ denote the pairing of H with its dual space.

Theorems

1. **Lax-Milgram Theorem:** Assume that $B : H \times H \to \mathbb{R}$ is a bilinear mapping, for which there exist constants $\alpha, \beta > 0$ such that

$$|B[u, v]| \leq \alpha ||u|| ||v|| \ (u, v \in H) \text{ and } \beta ||u||^2 \leq B[u, u] \ (u \in H).$$

Finally, let $f : H \to \mathbb{R}$ be a bounded linear functional on H. Then there exists a unique element $u \in H$ such that

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2. **Energy Estimates:** There exists $\alpha, \beta > 0 \& \gamma \geq 0$ such that

   $$|B[u, v]| \leq \alpha ||u||_{H_0^1(U)} ||v||_{H_0^1(U)} \ \& \ \beta ||u||^2_{H_0^1(U)} \leq B[u, u] + \gamma ||u||^2_{L^2(U)}$$

   for all $u, v \in H_0^1(U)$.

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The Navier-Stokes

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4. Physical Applications in weather models, ocean currents, water flow, video game systems

Other Interesting PDEs

1. Black-Scholes Equation - Applications in Stock Market

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{14}$$

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4. Stochastic Processes and BVP (Kakutani's solution to Dirichlet problem using Brownian motion)

The End

- Thank You!
- Questions?