Global BV solution to a system of balance laws from traffic flow

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Traffic System

- Constructing global solutions and finding zero relaxation limits of traffic flow
- Roadways, Vehicles, Drivers
- Microscopic Vs Macroscopic
- We will be focusing on a specific macroscopic model

History of Traffic Flow

- Lighthill-Whitham-Richards (LWR) model [1955, 1956]
- Payne-Whitham (PW) model [1971, 1974]
- Viscous models studied by Kerner-Konhauser, Kühne, Beckshulte, and Li [1984-1994, 2008]
- Aw-Rascle and Zhang's higher continuum (ARZ) models [2000, 2001]
- There are more references not mentioned above

Nonlinear Balance Laws

- Let $U \in \mathbb{R}^n$.
- $U = (u_1, u_2, ..., u_n), F(U) = (f_1(u), f_2(u), ..., f_n(u))$
- Consider the general conservation form

$$U_t + F(U)_x + P(U) = 0 \tag{1}$$

with initial data

$$U(x, 0) = U_0(x), \tag{2}$$

where $x \in \mathbb{R}, t > 0$.

The Model

- To analyze the 2×2 traffic flow model:

$$\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ v_t + (\frac{1}{2}v^2 + g(\rho))_x + \frac{v - v_e(\rho)}{\tau} &= 0, \end{aligned} \tag{3}$$

with initial data

$$(\rho(x,0), v(x,0)) = (\rho_0(x), v_0(x)) \tag{4}$$

where $x \in \mathbb{R}, t > 0, \tau > 0$.

- ρ - density, v - velocity, $v_e(\rho)$ - equilibrium velocity.
- $g(\rho)$ - anticipation factor and satisfies

$$g'(\rho) = \rho(v_e'(\rho)/\theta)^2, \tag{5}$$

where $g'(\rho) \geq 0$, $0 < \theta < 1$.

LWR Model

- The equilibrium flow is described by Lighthill-Whitham-Richards (LWR) model [8, 9]

$$\rho_t + (\rho v_e(\rho))_x = 0, \quad x \in \mathbb{R}, t > 0, \tag{6}$$

  with initial data $\rho(x, 0) = \rho_0(x) > 0$.
- $q(\rho) = \rho v_e(\rho)$ is known as the fundamental diagram
- For our work, we let

$$v_e(\rho) = -a\rho + b, \tag{7}$$

  where $a > 0, b > 0$.
- In our study, the equilibrium flux $q(\rho) = \rho(-a\rho + b)$ is a concave function of ρ.

Work Overview

- We showed in [1] the existence of a global BV solution for a system of balance laws arising in traffic flow in the framework of Dafermos [2]
- Computed entropy-entropy flux pair, Kawashima condition, sub-characteristic condition, and the partial dissipative inequality
- With these conditions we show the existence of BV solutions for the Cauchy problem

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First Transformation

- We want $U \equiv 0$ to be an equilibrium solution we need to do a change of variables $v = u + b$.
- Now we can rewrite (3) as follows

$$\begin{aligned} \rho_t + (\rho(u+b))_x &= 0 \\ u_t + (\frac{1}{2}(u+b)^2 + g(\rho))_x + \frac{u+b-v_e(\rho)}{\tau} &= 0 \end{aligned} \tag{8}$$

- In terms of the general form, we have

$$\begin{aligned} U &= (\rho, v-b) = (\rho, u)^T \\ F(U) &= (\rho(u+b), \frac{1}{2}u^2 + ub + g(\rho)))^T \\ P(U) &= (0, \frac{u+b-v_e(\rho)}{\tau})^T \end{aligned} \tag{9}$$

Preliminaries

- The Jacobian is

$$\begin{bmatrix} u+b & \rho \\ g'(\rho) & u+b \end{bmatrix} \tag{10}$$

- Using (5) and (7), the eigenvalues are

$$\lambda_{1,2} = u + b \mp \frac{a}{\theta}\rho \tag{11}$$

- The corresponding right eigenvectors are

$$r_{1,2} = (\mp\frac{\theta}{a}, 1)^T. \tag{12}$$

- The system (8) is genuinely nonlinear since

$$\nabla\lambda_i \cdot r_i = \frac{q''(\rho)}{v_e'(\rho)} = 2 \neq 0, \qquad i = 1, 2. \tag{13}$$

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Obstacles

In order to apply Dafermos' theory [2], we had to

- Search for a convex entropy-entropy flux pair
- Verify conditions
- Transform system (3) once again into equivalent form

Entropy-Entropy Flux Pair

- We need to find smooth entropy flux pair $(\eta, q)(U)$ where η is convex and has been normalized by $\eta(0), D\eta(0) = 0$.
- This is important since admissible solutions U must satisfy the entropy inequality

$$\partial_t \eta(U(x,t)) + \partial_x q(U(x,t)) + D\eta(U(x,t))P(U(x,t)) \leq 0 \quad (14)$$

- We also want our system to be a **symmetrizable**, which means it needs to be endowed with nontrivial **companion balance laws**.
- So we also need to solve

$$\begin{aligned} DQ_1(U,X) &= B(U,X)^T DG_1(U,X) \\ DQ_2(U,x) &= B(U,X)^T DG_2(U,X), \end{aligned} \quad (15)$$

where $G_1 = U, G_2 = F(U), DQ_i = [\frac{\partial Q_i}{\partial \rho}, \frac{\partial Q_i}{\partial u}], i = 1, 2.$

Continued

- Solving (15), we then constructed an explicit solution of a convex entropy-entropy flux pair

$$\eta(\rho, u) = Q_1(\rho, u) = (u - s\rho)^2 + \Gamma(u + s\rho)^2, \tag{16}$$

$$\begin{aligned} q(\rho, u) = Q_2(\rho, u) &= ((u - s\rho)^2 + \Gamma(u + s\rho)^2)(u + b) \\ &+ (1 + \Gamma)(u(s\rho)^2) \\ &+ 2(\Gamma - 1)\frac{(s\rho)^3}{3} - \frac{1 + \Gamma}{3}u^3, \end{aligned} \tag{17}$$

where $s = \dfrac{a}{\theta}, \Gamma = \dfrac{1 + \theta}{1 - \theta} > 1$.

- With this entropy-entropy flux pair, the convexity conditions are satisfied

Partial Dissipative Inequality

- We assume that P is dissipative semidefinite relative to η, i.e.

$$D\eta(U) \cdot P(U) \geq \alpha |P(U)|^2, \tag{18}$$

with $\alpha > 0$.

- For our system (8), we needed to find a condition such that

$$\begin{bmatrix} \frac{\partial \eta}{\partial \rho} & \frac{\partial \eta}{\partial u} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \frac{u + b - v_e(\rho)}{\tau} \end{bmatrix} \geq \alpha \left(\frac{u + b - v_e(\rho)}{\tau}\right)^2 \tag{19}$$

- After simplification, we require

$$0 < \alpha \leq \tau(2\Gamma + 1), \tag{20}$$

where $\Gamma > 1$.

Kawashima Condition

- The Kawashima condition is given by

$$DP(0)r_i(0) \neq 0, \qquad i = 1, 2. \tag{21}$$

- For our system, we have

$$DP(0)r_i(0) = \begin{bmatrix} 0 \\ \frac{\mp\theta + 1}{\tau} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{22}$$

since $0 < \theta < 1$.

Sub-characteristic condition

- The sub-characteristic is satisfied when

$$\lambda_1 < \lambda_* < \lambda_2. \tag{23}$$

- For $v = v_e(\rho)$,

$$\lambda * (\rho) = -2a\rho + b.$$

- The sub-characteristic condition is satisfied for (8) since we have

$$v_e(\rho) - \frac{a}{\theta}\rho < -2a\rho + b < v_e(\rho) + \frac{a}{\theta}\rho \tag{24}$$

for $0 < \theta < 1$.

Equivalent Form

- In order to apply Dafermos' theory, we needed to convert (8) into an equivalent form

$$\begin{aligned} \partial_t V + \partial_x G(V, W) + X(V, W) &= 0 \\ \partial_t W + \partial_x H(V, W) + CW + Y(V, W) &= 0, \end{aligned} \tag{25}$$

where $x \in \mathbb{R}$, $t > 0$,and $\eta_{WW} C(0, 0) > 0$.

- We followed Dafermos [2] and found the following change of variables

$$Z = (V, W) = (\rho, a\rho + u), \tag{26}$$

which transforms (8) to

$$\begin{aligned} V_t + [V(W - aV + b)]_x &= 0 \\ W_t + [\frac{1}{2}(W^2 - a^2V^2) + bW + g(V)]_x + \frac{1}{\tau} W &= 0 \end{aligned} \tag{27}$$

with initial conditions

$$Z_0 = (V_0, W_0) = (\rho_0, a\rho_0 + u_0). \tag{28}$$

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Theorem

Theorem (Admissible *BV* solution to the Cauchy problem)

Consider the Cauchy problem (27), (28) *with genuinely nonlinear characteristic fields* (13), *endowed with a convex entropy* η (16) (17). *The source is dissipative semidefinite* (18) *relative to the entropy* η, *and the Kawashima condition* (21) *holds. Then there are positive constants* $\delta_1, \sigma_0, c_0, c_1, \nu, b$ *so that the Cauchy problem* (27) (28) *under initial data* Z_0 *with*

$$\int_{-\infty}^{\infty} (1 + x^2)|Z_0(x)|^2 \, dx = \sigma^2 < \sigma_0^2, \tag{29}$$

$$TV_{(-\infty,\infty)} Z_0(\cdot) = \delta < \delta_1, \tag{30}$$

$$\int_{-\infty}^{\infty} V_0(x) \, dx = 0, \tag{31}$$

possesses an admissible BV solution Z *on* $(-\infty, \infty) \times [0, \infty)$ *and*

Theorem (Continued)

$$\int_{-\infty}^{\infty} |Z(x,t)| \, dx \leq b\sigma, \qquad 0 \leq t < \infty, \tag{32}$$

$$TV_{(-\infty,\infty)} Z(\cdot, t) \leq c_0\sigma + c_1\delta e^{-\nu t}, \qquad 0 \leq t < \infty, \tag{33}$$

$$\int_{-\infty}^{\infty} |Z(x,t)| \, dx \to 0, \qquad \text{as } t \to \infty, \tag{34}$$

$$TV_{(-\infty,\infty)} Z(\cdot, t) \to 0, \qquad \text{as } t \to \infty, \tag{35}$$

where $\delta > 0$.

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