Math 6000, Fall 2020 (Prof. Kinser), Homework 6

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Source Discussed problem/solutions with Zach Bryhtan.

Problem 1. *Skills developed: Using short exact sequences in conjunction with length and chains.*

In this problem we generalize some familiar facts about dimensions of vector spaces to length of modules. Recall that the *length* of a module M is the length of a composition series of M, if one exists, and ∞ or undefined if M does not have a composition series. This problem only deals with finite length modules.

(a) Prove that "length is additive on short exact sequence": Let $0 \to A \to B \to C \to 0$ be a short exact sequence of left $R-$modules, and suppose that B is of finite length. Prove that $\ell(B) = \ell(A) + \ell(C)$.

(b) Use (a) to prove the "sum-intersection formula' for modules of finite length: If $K, N \subset M$ are submodules and have finite length, then

$$\ell(K+N) + \ell(K \cap N) = \ell(K) + \ell(N).$$

The point of the exercise is use (a) to prove this – do not try t do it explicitly considering chains in all these spaces.

Thms/Defs 1. A **composition series** of a module M is a finite chain in M such that each factor is a simple module.

2. (i) $I = \{1, ..., n\}, \ \ M_1 \subset M_2 \subset ... \subset M_n \subset M$.

(ii) (Ascending Filtrations) $0 \subset M_1 \subset M_2 \subset M_3 \subset ....$

(iii) (Descending Filtrations) $M \supset M_1 \supset M_2 \supset M_3 \supset ...$

2b. The **factors** of a chain are the quotients $\{\frac{M_{i+1}/M_i}{M_i}\}$ (in (i) and (ii)) and $\{\frac{M_i}{M_{i+1}}\}$ in (iii).

2c. Suppose $M_0 = 0 \subset M_1 \subset M_2 ... \subset M_n = M$. The **length** of this chain is n.

2d. A **refinement** of a chain is one which contains the first as a subsequence.

3. (Rotman) **Jordan-Holder Theorem:** Any two composition series of a module M are equivalent. In particular, the length of a composition series, if one exists, is an invariant of M, called the **length** of M.

3b. Corollary - If module M has length n, then every chain of submodules of M has length $\leq n$.

Proof (a) Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of left $R-$modules.

Suppose $\ell(B) = n < \infty$. Then, it has a composition series:

$$B_0 = 0 \leq B_1 \leq B_2 .... \leq B_n = B,$$

where each factor is a simple module.

Since B has finite length, A and C also have finite length. This follows since:

(i) $\alpha : A \to B$ is injective and length of B is finite, $\ell(A) \leq \ell(B) = n$. Denote $\ell(A) = a < n$.

(ii) $\beta : B \to C$ is surjective, then $C \cong B/A$ by the First Isomorphism theorem and exactness of the sequence. Since $\ell(B), \ell(A) < \infty$, we then also have that $\ell(C)$ is finite. Denote $\ell(C) = c < n$.

Because A is a module of finite length a, we can create the following composition series:

$$A_0 = 0 \leq A_1 \leq A_2 .... \leq A_a = A$$

Since im $(\alpha) \subset B$, we can embed this composition series in the composition series of B:

$$A_0 = 0 \leq A_1 \leq A_2 .... \leq A_a = A = B_a \leq ... \leq B_n = B$$

We will quotient every submodule in the composition series by A. (The motivation is that we want to look at the composition series of $C \cong B/A$). Then, we get the following series:

$$\underbrace{A_0/A = 0 \leq A_1/A \leq ...}_{0} \leq A_a/A = A/A \leq B_a/A \leq B_{a+1}/A \leq ... \leq B_n/A = B/A \cong C$$

Note, each factor is a simple module, so this is a composition series:

$$A/A \leq B_a/A \leq B_{a+1}/A \leq ... \leq B_n/A = B/A \cong C$$

This follows since $\dfrac{B_a/A}{A/A} = B_a/A$ and $\dfrac{B_{i+1}/A}{B_i/A} = B_{i+1}/B_i$ (by the third isomorphism theorem).

Since B/A and B_{i+1}/B_i were simple, $\dfrac{B_a/A}{A/A}$ and $\dfrac{B_{i+1}/A}{B_i/A}$ are simple for each $a \leq i \leq n$ and this is a composition series for $C \cong B/A$.

Hence, $\ell(c) = c = n - a$, where $n = \ell(B), a = \ell(a)$.

$\therefore, \ell(B) = \ell(A) + \ell(C)$.

(b) Let $K, N \subset M$ are submodules and have finite length.

Suppose the short exact sequence is split. Consider the following sequence:

$$0 \to K \xrightarrow{i} K \oplus N \xrightarrow{\pi_2} N \to 0$$

Since K, N of finite length $\iff$ $K \oplus N$ is of finite length. Now, we can apply part (a) to conclude that $\ell(K \oplus N) = \ell(K) + \ell(N)$.

Similarly, consider the chain:

$$0 \to K \cap N \to K \oplus N \to K + N \to 0$$

By part (a), we have $\ell(K \oplus N) = \ell(K \cap N) + \ell(K + N)$.

Putting these two together, we get the following:

$$\begin{aligned} \ell(K \cap N) + \ell(K + N) &= \ell(K \oplus N) \\ &= \ell(K) + \ell(N) \\ \Rightarrow \ell(K \cap N) + \ell(K + N) &= \ell(K) + \ell(N) \end{aligned}$$

Problem 2. *Skills developed: using Noetherian and Artinian properties together with homomorphisms.*

This problem generalizes the fact that a linear operator on a finite dimensional vector space is injective iff it is surjective iff it is an isomorphism. Prove (a) carefully. For (b), just briefly indicate the main ideas of a proof (a few sentences)

(a) Let M be a Noetherian left $R-$modules, and $f : M \to M$ a surjective homomorphims. Show that f must be an isomorphism.

(b) Let N be an Artinian left $R-$module, and $g : N \to N$ an injective homomorphism. Show that g must be an isomorphism.

Defs/Thms 1a. A ring is a (left/right/2-sided) **Noetherian ring** if it has ACC on (left/right/2-sided ideals) , i.e. R has ACC as $R-$module.

1b. A ring is (left/right/2-sided) **Artinian** if it has DCC on (left/right/2-sided ideals), i.e. R has DCC as $R-$module.

2a. A left R-module M has an **ascending chain condition** (ACC) if every ascending chain of submodule

$0 \subset M_1 \subset M_2 \subset M_3 ... \subset M$ **stabilizes**, meaning $\exists N$ such that $M_i = M_{i+1}$ $\forall i \geq N$.

2b. (Just like ACC above) If every descending chain of submodules stabilizes.

3. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $R-$modules. Then, B is Noetherian. $\iff$ A, C both Noetherian (Similarly for Artinian).

3b. If R is a left Noetherian ring, every finitely generated left $R-$ module is Noetherian.

4. (Pg. 458 D and F) **Theorem 1** Let R be a ring and let M be a left R- module. Then, the following are equivalent:

(1) M is a Noetherian R-module.

(2) Every nonempty set of submodules of M contains a maximal element under inclusion.

(3) Every submodule of M is finitely generated.

Proof Suppose M is a Noetherian left $R-$module, and $f : M \to M$ is a surjective homomorphism.

We need to show that f is also injective, to conclude that f is an isomorphism. We will do this by showing that the kernel is trivial.

Since M is Noetherian, it is finitely generated (since every submodule of M is finitely generated and M is a submodule of itself) i.e. there exists $a_1, a_2, ..., a_n \in M$ such that for any $x \in M$, there exist $r_1, r_2, ..., r_n \in R$ with $x = r_1a_1 + r_2a_2 + ... + r_na_n$.

Here $\{a_1, a_2, ..., a_n\}$ is the minimal set of generators of M.

Since $f : M \to M$ is a surjective, for each $m_2 \in M$, $\exists\, m_1 \in M$ such that $f(m_1) = m_2$.

Let $m_1 = r_1a_1 + r_2a_2 + ... + r_na_n$. Then, we have the following:

$$
\begin{aligned}
\ker(f) &= \{m_1 \in M | m_2 = 0\} \\
&= \{m_1 \in M | f(m_1) = 0\} \\
&= \{m_1 \in M | f(r_1a_1 + r_2a_2 + ... + r_na_n) = 0\} \\
&= \{m_1 \in M | f(r_1a_1) + f(r_2a_2) + ...f(r_na_n) = 0\} \qquad \text{(since } f \text{ is a R-module homomorphism)} \\
&= \{m_1 \in M | r_1f(a_1) + r_2f(a_2) + ...r_nf(a_n) = 0\}
\end{aligned}
$$

Since $\{a_1, ..., a_n\}$ is the minimal set of generators and f is surjective, it is necessarily the case that each $f(a_i) \neq 0$. Then it must be the case each r_i is 0, $0 < i \leq n$.

Hence, $\{m_1 \in M | f(m_1) = 0\} \iff \{m_1 \in M | m_1 = 0\}$. Therefore, the kernel is trivial.

(b) Since N is an Artinian left R-module, by definition, it has a descending chain condition as R-module. Consider the following descending chain:

$$N \supset g(N) \supset g(g(N)) \supset ... \supset \underbrace{g(g(....g(N))}_{n \text{ times}} \supset .... \supset 0$$

Denote $\underbrace{g(g(....g(N))}_{n \text{ times}} = g^n$.

Since it stabilizes, $\exists\, N$ such that $g^i(N) = g^{i+1}(N) = 0 \;\; \forall i \geq N$.

Let n_2 be an arbitrary element of N. Then, $g^i(n) \subset g^{i+1}(n)$. Because $\forall i \geq N$, the chain stabilizes, $\exists\, n_1 \in N$ such that $g^i(g(n_1)) = g^i(n_2)$.

Since g is injective, this means that $g(n_1) = n_2$. $\therefore$, g is surjective and thus, an isomorphism.

(b) (**Alternatively** - not to be graded. I think this works but not sure if I need to elaborate more on the choice of n_1).

Let N be an Artinian left $R-$module. It is given that $g : N \to N$ is an injective homomorphism. We need to show that g is an surjective.

Fact: An Artinian Ring is a Noetherian Ring.

Similar to part (a), a Noetherian ring is finitely generated. So, N can be finitely generated with $\{b_1, b_2, ..., b_n\}$ as the minimal set of generators of M.

Let n_2 be an arbitrary element of N. Show $\exists\, n_1 \in N$ such that $f(n_1) = n_2$.

We can write n_2 as follows: $n_2 = r_{21}b_1 + r_{22}b_2 + ... + r_{2n}b_n$.

Then, we can find $n_1 \in N$ such that:

$$
\begin{aligned}
f(n_1) &= f(r_{11}b_1 + r_{12}b_2 + ...r_{1n}b_n) \\
&= r_{11}f(b_1) + r_{12}f(b_2) + ... + r_{1n}f(b_n)
\end{aligned}
$$

In particular, since f is injective, for $b_i \neq b_j$ for $0 \leq i, j \leq n$, $f(b_i) \neq f(b_j)$.

Hence,since $\{b_1, ..., b_n\}$ are the minimal set of generators of N, $f(b_1), ...f(b_n)$ will map to $b_1, ..., b_n$.

Therefore, we have found n_1 such that $f(n_1) = n_2$. Since n_2 was arbitrary, we have that f is surjective and thus, an isomorphism.

Problem 3. *Skills developed: computation of Jacobson radical in the familiar setting of quotients of polynomial rings*

Let K be a field, $R - K[x]$ the polynomial ring in one variable over K, and $I \subset R$ an ideal. Describe the Jacobson radical $J(R)$.

Thms/Defs 1. For a ring R, the **Jacobson radical** $J(R) \subset R$ is the intersection of all maximal left ideals of R.

1b. Recall that a **maximal (left) ideal** of R is a proper ideal of R not contained inside any ideal besides itself and R.

2. For $x \in R$, the following are equivalent:

(i) $x \in J(R)$

(ii) $\forall r \in R, 1 - rx$ has left inverse

(iii) $x \in \text{ann}_R(R/I)$ $\forall$ maximal left ideals $I \subset R$.

(iv) $x \in \text{ann}_S$ $\forall$ simple $R-$modules S .

3. (pg.457 Corollary 2) If R is a P.I.D., then every nonempty set of ideals of R has a maximal element and R is a Noetherian ring.

4. Every nonzero prime ideal in a PID is a maximal ideal.

5. Assume R is commutative. The ideal M is a maximal ideal iff R/M is a field.

6. In a PID, a nonzero element is a prime iff it is irreducible.

Proof Let $R = K[x]/I$, where $I \subset K[x]$ is an ideal.

Let $\overline{r(x)} \in K[x]/I$ be arbitrary, and suppose $\overline{p(x)} \in J(R)$ for $\overline{p(x)} \in R$.

Then, by the theorem stated above, $1 - \overline{r(x)} \cdot \overline{p(x)}$ has a left inverse.

In particular, the inverse needs to be an element of $K^\times/(I \cap K)$. We have two cases here:

(i) Suppose $I \cap K = 0$. Then, $K^\times/(I \cap K) = K^\times$. The inverse here is 1. So, $1 - \overline{r(x)} \cdot \overline{p(x)} = 1 \Rightarrow \overline{p(x)} = 0$.

Since $\overline{p(x)} \in K[x]/I$ and $\overline{p(x)} = 0$, we have that $p(x) \in I$.

Hence, $J(R) = J(K[x]/I) = J(K[x])/I$.

(ii) Now suppose that $I \cap K \neq 0 \Rightarrow I \cap K = K$. Then, $K^\times/(I \cap K) = 0 \Rightarrow J(R) = \{0\}$.

(**Source:** Zach for inspiration on this approach)

Attempt 2 (Not for a grade)

Since K is a field, the polynomial ring $K[x]$ is a Principal Ideal Domain (PID).

(i) Suppose K is finite.

Fact There exists irreducible polynomial over F of arbitrarily high degree.

Let $p(x) \in K[x]$ be such an irreducible polynomial with $p(x) \neq 0$. In a PID, a nonzero element is prime if and only if it is irreducible. Hence $p(x)$ is prime.

In addition, every nonzero prime ideal in a PID is a maximal ideal. Hence, $(p(x)) = I$ is a maximal ideal.

Since $p(x)$ was arbitrary, there are infinitely many such maximal ideals.

Now, we can take the intersection of all maximal (left) ideals of $K[x]/I$ to get the Jacobson radical $J(K[x]/I)$.

(ii) The maximal ideals are exactly $(x - c)$ for $c \in K[x]/I$.

Then, $\cap_{c \in K[x]/I}(x - c)$ consists of polynomials divisble by $x - c$ for all $c \in K[x]/I$.

Hence, $\cap_{c \in K[x]/I}(x - c) = 0$.

$\therefore J(K[x]/I) = \{0\}$.

Problem 4. Let $R = M_2(K[[t]])$ be the ring of 2×2 matrices with entries in $K[[t]]$, the ring of formal power series over a field K (defined in class and on the last homework). Let $\mathfrak{m} = (t)$ be the unique maximal ideal of $K[[t]]$, consisting of power series with 0 constant term. Let $S \subset R$ be the subring of matrices of the form:

$$\begin{bmatrix} K[[t]] & K[[t]] \\ \mathfrak{m} & K[[t]] \end{bmatrix}$$

These are matrices that are "upper-triangular modulo t". Prove that the Jacobson radical of S is

$$J(S) = \begin{bmatrix} \mathfrak{m} & K[[t]] \\ \mathfrak{m} & \mathfrak{m} \end{bmatrix}$$

Hint: You can show $J(S)$ is contained in the right hand side by finding two simple $S-$modules (both $1-$dimensional over K) and showing that the intersection of their annihilators is the right hand side. These may take some work to find, they are not exactly the same as modules appearing in other examples in class. Then you can show the right hand side is contained in the left imitating techniques demonstrated in class for upper triangular matrices.

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1. Let K be a field and $R = K[[t]]$, the ring of formal power with coefficients in K.

An element is an expression:

$f = a_0 + a_1 t + a_2 t^2 + ... = \sum_{i=0}^{\infty} a_i t^i$ with addition and multiplication defined as usual.

2a. $u \in R$ is a **unit** if $\exists\, v \in R$ such that $uv = vu = 1$.

2b. (Rotman 8.36) If R is a ring, then

(i)

$$J(R) = \{x \in R : 1 + rxs \text{ is a unit in } R \text{ for all } r, s \in R\}$$

(ii) If R is a ring and $J'(R)$ is the intersection of all the maximal right ideals of R, then $J'(R) = J(R)$.

3a. Ann $_R(M) = \{r \in R | rm = 0\ \forall m \in M\}$

(i) Ann $_R(M)$ is a 2 sided ideal.

(ii) For any left ideal $I \subset R,$ Ann $_R(R/I) \subset I.$

3b. $J(R) = \cap_{I \subset R \text{ maximal}} \text{ann}_R(R/I) = \cap_{S \text{ simple}} \text{ann}_R S$

(a-i) We will show that $J(S) \subset Z$, where $Z = \begin{bmatrix} \mathfrak{m} & K[[t]] \\ \mathfrak{m} & \mathfrak{m} \end{bmatrix}$.

Consider simple module $K_1 = \{\begin{bmatrix} x \\ 0 \end{bmatrix} | x \text{ scalars } \in K\}$ with R acting by restriction of scalars along $S \subset R = M_2(K[[t]]) \xrightarrow{\text{mod } (t)} M_2(K)$.
Since K_1 is a simple $M_2(K)$ module, it is also a simple $S-$module.
We see that $\begin{bmatrix} \mathfrak{m} & K[[t]] \\ \mathfrak{m} & K[[t]] \end{bmatrix} \cdot \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} \mathfrak{m}x \\ \mathfrak{m}x \end{bmatrix} \xrightarrow{\text{mod } t} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence, $\text{Ann}_R(K_1) = \begin{bmatrix} \mathfrak{m} & K[[t]] \\ \mathfrak{m} & K[[t]] \end{bmatrix}$.
Similarly, let $K_2 = \{\begin{bmatrix} 0 \\ y \end{bmatrix} | y \in K\}$ with R acting as above. We get that $\text{Ann}_R(K_2) = \begin{bmatrix} K[[t]] & \mathfrak{m} \\ K[[t]] & \mathfrak{m} \end{bmatrix}$.
Then, since the Jacobson radical can be defined as intersection of annihilators over simple modules, we have:

$J(S) \subset \text{ann}_S K_1 \cap \text{ann}_S K_2 = Z$, where S is a simple module.

$\therefore J(S) \subset Z$.

(The choice of K_2 does not seem to be correct).

(a-ii) Now, show that $Z \subset J(S)$.

We will use the computational criterion for the Jacobson radical here (8.36).

Let $X \in Z$, i.e. $X = \begin{bmatrix} tf_{11} & f_{12} \\ tf_{21} & tf_{22} \end{bmatrix}$

Let $A, B \in S$ be as follows:

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ ta_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ tb_{21} & b_{22} \end{bmatrix}$

Note, $AXB \in J(S)$. The computation is given below.

$$\begin{aligned}
AX &= \begin{bmatrix} a_{11} & a_{12} \\ ta_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} tf_{11} & f_{12} \\ tf_{21} & tf_{22} \end{bmatrix} \\
&= \begin{bmatrix} ta_{11}f_{11} + ta_{12}f_{21} & a_{11}f_{12} + ta_{21}f_{22} \\ t^2a_{21}f_{11} + ta_{22}f_{21} & ta_{21}f_{12} + ta_{22}f_{22} \end{bmatrix} \\
&= \begin{bmatrix} t(a_{11}f_{11} + a_{12}f_{21}) & a_{11}f_{12} + ta_{21} \\ t(ta_{21}f_{11} + a_{22}f_{21}) & t(a_{21}f_{12} + a_{22}) \end{bmatrix} \\
\Rightarrow AXB &= \begin{bmatrix} t(a_{11}f_{11} + a_{12}f_{21}) & a_{11}f_{12} + ta_{21} \\ t(ta_{21}f_{11} + a_{22}f_{21}) & t(a_{21}f_{12} + a_{22}) \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ tb_{21} & b_{22} \end{bmatrix} \\
&= \begin{bmatrix} t(a_{11}f_{11} + a_{12}f_{21})b_{11} + t(a_{11}f_{12} + ta_{21})b_{21} & t(ta_{21}f_{11} + a_{22}f_{21})b_{12} + (a_{11}f_{12} + ta_{21})b_{22} \\ t(ta_{21}f_{11} + a_{22}f_{21})b_{11} + t^2(a_{21}f_{12} + a_{22})b_{21} & t(ta_{21}f_{11} + a_{22}f_{21})b_{12} + t(a_{21}f_{12} + a_{22})b_{22} \end{bmatrix} \\
&= \begin{bmatrix} t[(a_{11}f_{11} + a_{12}f_{21})b_{11} + (a_{11}f_{12} + ta_{21})b_{21}] & a_{11}f_{12}b_{22} + t[ta_{21}f_{11} + a_{22}f_{21})b_{12} + ta_{21}b_{22}] \\ t[(ta_{21}f_{11} + a_{22}f_{21})b_{11} + t(ta_{21}f_{12} + ta_{22})b_{21}] & t[(ta_{21}f_{11} + a_{22}f_{21})b_{12} + (a_{21}f_{12} + a_{22})b_{22}] \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} tg_{11} & g_{12} \\ tg_{21} & tg_{22} \end{bmatrix}$$

$$\therefore AXB \in \begin{bmatrix} \mathfrak{m} & K[[t]] \\ \mathfrak{m} & \mathfrak{m} \end{bmatrix}$$

Here

(i) $g_{11} = (a_{11}f_{11} + a_{12}f_{21})b_{11} + (a_{11}f_{12} + ta_{21})b_{21}$,

(ii) $g_{12} = a_{11}f_{12}b_{22} + t[ta_{21}f_{11} + a_{22}f_{21})b_{12} + ta_{21}b_{22}$,

(iii) $g_{21} = (ta_{21}f_{11} + a_{22}f_{21})b_{11} + t(ta_{21}f_{12} + ta_{22})b_{21}$,

(iv) $g_{22} = (ta_{21}f_{11} + a_{22}f_{21})b_{12} + (a_{21}f_{12} + a_{22})b_{22}$.

Next, we use the proposition as follows:

$$\begin{aligned} 1 + AXB &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} tg_{11} & g_{12} \\ tg_{21} & tg_{22} \end{bmatrix} \\ &= \begin{bmatrix} 1 + tg_{11} & g_{12} \\ tg_{21} & 1 + tg_{22} \end{bmatrix} \end{aligned}$$

Now, we compute the determinant of $1 + AXB$.

$$\begin{aligned} \det(1 + AXB) &= \det \begin{bmatrix} 1 + tg_{11} & g_{12} \\ tg_{21} & 1 + tg_{22} \end{bmatrix} \\ &= (1 + tg_{11})(1 + tg_{22}) - g_{12} \cdot (tg_{21}) \\ &= [1 + t(g_{11} + g_{22}) + t^2 g_{11}g_{22}] - tg_{12}g_{21} \\ &= 1 + t(g_{11} + g_{22} - g_{12}g_{21}] + t^2 g_{11}g_{12} \\ &= 1 + t[(g_{11} + g_{22} - g_{12}g_{21}) + (tg_{11}g_{12})] \\ &= 1 + th \end{aligned}$$

where $h = (g_{11} + g_{22} - g_{12}g_{21}) + (tg_{11}g_{12})$.

Note, that a matrix is invertible over a commutative ring $\iff \det(S) = \pm 1$.

Since $\mathfrak{m} = (t), th = 0$. Hence, $\det(1 + AXB) = 1 \iff 1 + AXB$ is invertible.

Therefore, $1 + AXB$ is a unit in $R = M_2(K[[t]])$ for all $A, B \in R$.

$\therefore, Z \subset J(S)$.

Conclude Since we have shown that $J(S) \subset Z$ and $Z \subset J(S)$, we have shown that $J(S) = Z$.