# Math 6000, Fall 2020 (Prof. Kinser), Homework 4

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**Source** Discussed solutions to these problems with Zach Bryhtan, and we read each others' writing as well. Also contacted office hour for clarification.

**Problem 1.** *Skills developed: practice with universal property and other features of tensor products.*

Let $\phi : R \to S$ be a ring homomorphism. Recall from a previous homework that $N \in S-\text{Mod}$ can be considered an $R-\text{Mod}$ by restriction of scalars. We will denote this by ${}_RN$ here. Let $M$ be an $R-$module.

Construct morphisms of abelian groups in a natural way:

$\alpha : \text{Hom}_R(M, {}_RN) \to \text{Hom}_S(S \otimes_R M, N)$ and $\beta : \text{Hom}_S(S \otimes_R M, N) \to \text{Hom}_R(M, {}_RN)$

satisfying that $\alpha$ and $\beta$ are inverse to one another (and **prove** that this is the case).

**Optional** (somewhat tedious **Challenge**) Show that $\alpha$ and $\beta$ are functorial in $M$ and $N$, separately.

**Defs/Thms**

**1.** Let $L$ be an abelian group say $\phi : M \times N \to L$ is R-balanced if

(i) $\phi(m_1 + m_2, n) = \phi(m_1, n) + \phi(m_2, n)$

(ii) $\phi(m, n_1 + n_2) = \phi(m, n_1) + \phi(m, n_2)$

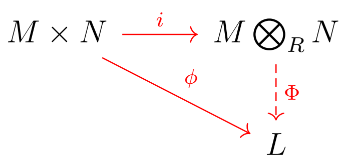
(iii) $\phi(mr, n) = \phi(m, rn)$

for all $m, m_1, m_2 \in M, n_1, n_2 \in N$, and $r \in R$.

**2.** D and F (pg. 345) Suppose $R$ is a ring with 1, $M$ is a right $R-$module, and $N$ is a left $R$-module. Let $M \bigotimes_R N$ be the tensor product of $M$ and $N$ over $R$ and let $i : M \times N \to M \bigotimes_R N$ be the $R-$ balanced map defined above.

(1) If $\Phi : M \bigotimes_R N \to L$ is any group homomorphism from $M \bigotimes_R N$ to an abelian group $L$, then the composite map $\phi = \Psi \circ i$ is an $R-$balanced map from $M \times N$ to $L$.

(2) Conversely, suppose $L$ is an abelian group and $\phi : M \times N \to L$ is any $R-$balanced map. Then there is a unique group homomorphism $\Phi : M \bigotimes_R N \to L$ such that $\phi$ factors through $i$, i.e. $\phi = \Phi \circ i$. Equivalently $\phi \iff \Phi$ in the commutative diagram:




establishes a bijection:

$R$- balanced maps ($\phi : M \times N \to L$) $\iff$ Group homomorphisms ($\Phi : M \bigotimes_R N \to L$)

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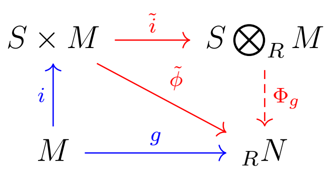
**1. Proof** It is given that $\phi : R \to S$ is a ring homomorphism.

In order for $S \otimes_R M$ to make sense, we need $S$ to be a right $R$-module and $M$ to be a left $R$-module.

From the the last homework, by restriction of scalars, we can consider $S$ as a right-$R$ module as follows: $s \cdot r = s \cdot \phi(r)$ for $r \in R, s \in S$. Similarly, $N$ can be considered as a left $R-$module by $r \cdot n = \phi(r)n$ for $r \in R, n \in N$.

Let $g \in \mathrm{Hom}(M, {}_RN)$ be a right module homomorphism. Consider the following commutative diagram:




(Note, in the diagram above, ${}_RN$ is considered as a $R$ module when coming from $g$, the $R-$module homomorphism, and can be considered a $S$-module $N$ when coming from group homomorphism $\Phi$, which we will describe below).

**Claim** We claim that $\alpha(g) = \Phi_g$ is the morphism of abelian groups.

**(a-Show)** We want to show that the $\tilde{\phi}$ is an $R-$balanced map so we can get the induced morphism from $S \otimes_R M$ to $N$ by the **Universal Mapping Property**.

Let $(s, m) \in S \times M$. Define $\tilde{\phi}(s, m) = sg(m)$.

**(a-i)** Then, consider $\tilde{\phi}(s_1 + s_2, m)$.

$$\begin{aligned}\tilde{\phi}(s_1 + s_2, m) &= (s_1 + s_2)g(m) \\ &= s_1 g(m) + s_2 g(m) \\ &= \tilde{\phi}(s_1, m) + \tilde{\phi}(s_2, m)\end{aligned}$$

**(a-ii)** Consider $\tilde{\phi}(s, m_1 + m_2)$.

$$\tilde{\phi}(s, m_1 + m_2) = sg(m_1 + m_2)$$

$$
\begin{aligned}
&= s[g(m_1) + g(m_2)] \quad \text{since } g \text{ is a R-module homomorphism} \\
&= sg(m_1) + sg(m_2) \\
&= \tilde{\phi}(s, m_1) + \tilde{\phi}(s, m_2)
\end{aligned}
$$

**(a-iii)** Consider $\tilde{\phi}(sr, m)$. for $r \in R$.

$$
\begin{aligned}
\tilde{\phi}(sr, m) &= (s.r)g(m) \\
&= s\phi(r)g(m) \\
&= sg(rm) \quad \text{since } g \text{ is a R-module homomorphism} \\
&= \tilde{\phi}(s, rm)
\end{aligned}
$$

Note, in the second and third equality we used the fact that $s$ can be considered as a right $R-$module and $g(rm) \in N$ can be considered a left $R-$module due to the restriction of scalars.

Hence, we have shown that $\phi : S \times M \to N$ is $R-$balanced. Then, by the universal mapping property, there is a unique group homomorphism $\Phi : S \otimes_R M \to N$ such that $\tilde{\phi}$ factors through $i$, i.e. $\tilde{\phi} = \Phi \circ i$.

In particular, for simple tensor $(s \otimes m) \in (S, M)$, $\Phi_g(s \otimes m) = s \cdot g(m)$.

**(a-iv)** Show that $\alpha$ is a **homomorphism** of abelian groups. Suppose $g_1, g_2 \in \operatorname{Hom}(M, {}_R N)$. Then, we have

$$
\begin{aligned}
\alpha(g_1 + g_2)(s \otimes m) &= \Phi_{g_1+g_2}(s \otimes m) \\
&= s(g_1 + g_2)(m) \\
&= s(g_1(m) + g_2(m)) \\
&= sg_1(m) + sg_2(m) \quad \text{(since } g_1, g_2 \text{ are R-module homomorphisms)} \\
&= \alpha(g_1)(s \otimes m) + \alpha(g_2)(s \otimes m)
\end{aligned}
$$

Hence, $\alpha$ is a homomorphism.

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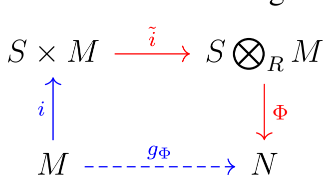
**(b) Show** Define a map $\beta : \operatorname{Hom}_S(S \otimes_R M, N) \to \operatorname{Hom}_R(M, {}_R N)$. Suppose $\Phi \in \operatorname{Hom}(S \otimes_R M, N)$.

**(b) Claim** $\beta(\Phi) = g_\Phi$.

**(b) Proof** Consider the following commutative diagram, similar to (a):




Define $\beta(\Phi) = g_\Phi$, where $g_\Phi = \Phi \circ \tilde{i} \circ i$.

**(b-i)** Show that $\beta$ is a homomorphism.

Let $\Phi_1, \Phi_2 \in \mathrm{Hom}_{S \otimes_R M, N}$. Then,

$$\begin{aligned}\beta(\Phi_1 + \Phi_2) &= g_{\Phi_1 + \Phi_2} \\ &= (\Phi_1 + \Phi_2) \circ \tilde{i} \circ i \\ &= (\Phi_1 \circ \tilde{i} \circ i) + (\Phi_2 \circ \tilde{i} \circ i) \\ &= g_{\Phi_1} + g_{\Phi_2} \\ &= \beta(\Phi_1) + \beta(\Phi_2)\end{aligned}$$

Hence, $\beta$ is also a homomorphism.

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**(c)Show** Show that $\alpha, \beta$ are actually inverses.

**(c)-i** Consider $\alpha \circ \beta$. Then:

$$\begin{aligned}\alpha \circ \beta(\Phi_1) &= \alpha \circ (\Phi_1 \circ i \circ \tilde{i}) \\ &= \alpha(g_{\Phi_1}) \\ &= \Phi_1\end{aligned}$$

**(c)-ii** Consider $\beta \circ \alpha$. Then,

$$\begin{aligned}\beta \circ \alpha(g) &= \beta(\Phi_g) \\ &= \Phi \circ \tilde{i} \circ i \\ &= g_{\Phi} \\ &= g\end{aligned}$$

Therefore, we have shown that $\alpha$ and $\beta$ are actually inverses.

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**Problem 2.** *Skills developed: practice manipulating tensors.* Let $K \subset L$ be a field extension of finite degree $d$.

**(a)** Determine (with proof) a necessary and sufficient condition on $d$ for $L \bigotimes_K L \cong L \bigotimes_L L$ as $K-$ vector spaces.

**(b)** Compute (with proof) the dimension of $L \bigotimes_K L$ as an $L-$vector space, where we use the induced left action of $L$ on the tensor product.

**(c)** Prove that $\mathbb{Q} \bigotimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as a $\mathbb{Q}$ vector space (this is independent of the first two parts, since $\mathbb{Z}$ is not a field).

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**Defs/Thms**

1. The degree of a field extension $K/F$ denoted $[K : F]$ is the dimension of $K$ as a vector space over $F$ i.e. $[K : F] = \dim_F K$. The extension is said to be finite if $[K : F]$ is finite.

2. (D and F Corollary 13, pg. 502) If the extension $K/F$ is finite, then it is algebraic.

3. (From notes/ D and F Corollary 11 pg. 396) The dimension of $\operatorname{Hom}_F(V, W)$ is $(\dim V)(\dim W)$.

4. (D and F pg. 343 Example 1) For any ring $R$ and any left $R-$module $R \otimes_R N \cong N$ (so "extending scalars from $R$ to $R$" does not change the module).

5. (D and F pg. 503 Theorem 14) Let $F \subset K \subset L$. Then,

$$[L : F] = [L : K] \cdot [K : F]$$

(Tower of Fields)

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**(a)** Suppose $L \otimes_K L \cong L \otimes_L L$.

In particular, $L \otimes_K L \cong L \otimes_L L \iff \dim_K(L \otimes_K L) = \dim_K(L \otimes_L L)$.

**(a-i)** It is given that $\dim_K L = [L : K] = d$. Then, we have:

$$\begin{aligned}
[L \otimes_K L : K] &= \dim_K(L \otimes_K L) \\
&= (\dim_K L) \cdot (\dim_K L) \\
&= [L : K] \cdot [L : K] \\
&= d \cdot d \\
&= d^2
\end{aligned}$$

**(a-ii)** (By the fact above that $R \otimes_R N \cong N$), we can see that $L \otimes_L L \cong L$.

(In particular, let $R = L$ and $N = L$ can be considered as a module over itself. Hence, we have $L \otimes_L L \cong L$). Then,

$$\begin{aligned}\dim_K(L\otimes_L L) &= \dim_K L\\ &= [L:K]\\ &= d\end{aligned}$$

**(a-iii) Conclude** Therefore,

$$\begin{aligned} L\otimes_K L &\cong L\otimes_L L\\ \iff \dim_K(L\otimes_K L) &= \dim_K(L\otimes_L L)\\ \Rightarrow d^2 &= d\\ d^2-d &= 0\\ d(d-1) &= 0\\ \Rightarrow d &= 0,1\end{aligned}$$

The necessary and sufficient condition on $d$ is that $d$ is either $0$ or $1$.

If $d=0, L=K$ would both be the $0$ vector space, while $d=1\Rightarrow [L:K]=1\Rightarrow L=K$.

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**(b)** Suppose $K\subset L\subset L\otimes_K L$. Then, by the tower of fields, we have:

$$\begin{aligned}[L\otimes_K L:K] &= [L\otimes_K L:L]\cdot[L:K]\\ \Rightarrow d^2 &= [L\otimes_K L:L]\cdot d \qquad \text{by part (a)}\\ \Rightarrow d &= [L\otimes_K :L]\\ \Rightarrow d &= \dim_L(L\otimes_K L)\end{aligned}$$

The conclusion follows if the assumption is true. $K\subset L$ is given. Show that $L\subset L\otimes_K L$.

Consider $L\xrightarrow{i} L\times L\xrightarrow{\tilde{i}} L\otimes_K L$ by $\ell\mapsto(\ell,1)\mapsto \ell\otimes 1$.

Since inclusions are injective and the composition of injective functions is injective, we can conclude that $L\subset L\otimes_K L$ and invoke the tower of fields argument.

(since injective field homomorphisms is an embedding).

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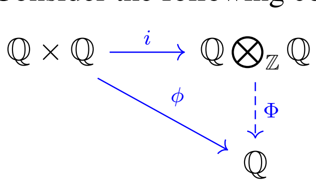
**(c)** Consider the following commutative diagram:




Let $(q_1,q_2)\in\mathbb{Q}\times\mathbb{Q}$. Define $\phi(q_1,q_2)=q_1\cdot q_2$. Show $\phi$ is $R-$ balanced:

Let $q,q_1,q_2\in\mathbb{Q}$. Then,

**(c-i)**

$$\begin{aligned}\phi(q_1+q_2,q) &= (q_1+q_2)\cdot q\\ &= q_1q+q_2q\\ &= \phi(q_1,q)+\phi(q_2,q)\end{aligned}$$

**(c-ii)**

$$\begin{aligned}\phi(q,q_1+q_2) &= q\cdot(q_1+q_2)\\ &= qq_1+qq_2\\ &= \phi(q,q_1)+\phi(q,q_2)\end{aligned}$$

**(c-iii)** Let $z \in \mathbb{Z}$. Then, we have

$$\begin{aligned}\phi(q_1z,q_2) &= (q_1z)q_2\\ &= q_1(zq_2) \quad \text{(by Associativity)}\\ &= \phi(q_1,zq_2)\end{aligned}$$

Hence, $\phi$ is $R-$balanced. Therefore, by the universal mapping property, we can define $\Phi : \mathbb{Q} \otimes_Z \mathbb{Q} \to \mathbb{Q}$ such that $\phi = \Phi \circ i$.

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As shown above, by the universal mapping property, $\Phi : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ is also a group homomorphism. In order to show isomorphism, show that $\Phi$ is surjective and injective.

**(c-iv)** Show that $\Phi : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ is surjective.

Let $q$ be an arbitrary element in $\mathbb{Q}$.. Then, consider $(1,q) \in \mathbb{Q}\times\mathbb{Q} \Rightarrow \phi(1,q) = \Phi(1\otimes q) = q$.

Since we have found $(1 \otimes q) \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $q \in \mathbb{Q}$ was arbitrary, it is shown that $\Phi$ is surjective.

**(c-v)** Show that $\Phi$ is injective.

Let $a = q_1 \otimes q_2 \in \mathbb{Q} \otimes_Z \mathbb{Q}, b = q_3 \otimes q_4 \in \mathbb{Q} \otimes_Z \mathbb{Q}$. Then,

$$\begin{aligned}\Phi(a) &= \Phi(b)\\ \Phi(q_1 \otimes q_2) &= \Phi(q_3 \otimes q_4)\\ q_1 \cdot q_2 &= q_3 \cdot q_4\\ &\Rightarrow a = b\end{aligned}$$

Therefore, $\Phi$ is an isomorphism and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

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