Math 6000, Fall 2020 (Prof. Kinser), Homework 3

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Source Discussed solutions with Zach after thinking about the hw myself and then did proof analysis of each other's paper after we had written our own initial drafts.

Problem 1. *Skills developed: practice with categories defined by compound constructions.*

Fix a homomorphism of groups $f : A \to B$. Let $\mathcal{C}$ be the category whose objects are pairs (X, ϕ) such that $\phi : X \to A$ is a group homomorphism satisfying $f\phi = 0$. A morphism $(X, \phi) \to (Y, \psi)$ in $\mathcal{C}$ is given by a group homomorphism $g : X \to Y$ satisfying $\phi = \psi g$. Prove that $\mathcal{C}$ has a terminal object by explicitly describing it.

Edit Typo: the symbol "0" has no meaning the general context of groups. The defining condition for objects in this category should be that the composition is the trivial homomorphism, instead of 0.

Defs/Theorems

1. An object I in a category $\mathcal{C}$ is **initial** if for every $X \in \mathcal{C}$, $\exists! I \to X$ in $\mathcal{C}$ (This does not necessary mean there exists one).
2. A **terminal** object is defined dually (initial object in $\mathcal{C}^{op}$).
3. Let $f : G \to H$ be a group homomorphism. The **kernel** of f is the set K such that

$$K = \{x \in G : f(x) = e\}$$

Proof Let e be the identity in B.

Let (X, ϕ), $(Y, \psi) \in \mathcal{C} \Rightarrow \phi : X \to A$ is a group homomorphism satisfying $f\phi = e$ and $\psi : Y \to A$ is a group homomorphism satisfying $f\psi = e$.

We also have $\phi = \psi g$, where $g : X \to Y$ is a group homomorphisms. Then,

$$\begin{aligned} f\phi &= e \\ \Rightarrow f(\psi g) &= e \end{aligned}$$

Since we are dealing with group homomorphisms and compositions, the terminal object will consist of a group and a group homomorphism mapping to the identity. In particular,

(i) Consider $T = \ker(A) \trianglelefteq A$ and define $\overline{\phi} : T \to A$ be the inclusion morphism. We assert that$(T, \overline{\phi})$ is the terminal object in $\mathcal{C}$.

(ii) Check that $(T, \overline{\phi})$ is an object in $\mathcal{C}$. Consider the following:

$$\begin{aligned} f\overline{\phi}(T) &= f(T) \quad \text{(since } \overline{\phi} \text{ is just the inclusion)} \\ &= e \end{aligned}$$

So, $\overline{\phi} : T \to A$ is a group homomorphism satisfying $f\overline{\phi} = e$.

Hence $(T, \overline{\phi}) \in \mathcal{C}$.

(ii) A morphism $(X, \phi) \to (T, \overline{\phi})$ is given by a group homomorphism $\overline{g} : X \to T$ defined by $\overline{g} = \phi|_{\ker(A)}$.

We have $\overline{g} : X \to T, \overline{\phi} : T \to A$. Then,

$$\begin{aligned} \overline{\phi} \circ \overline{g} &= \overline{\phi} \circ \phi|_{\ker(A)} \\ &= \phi \end{aligned}$$

(iii) Let (X, ϕ) be an arbitrary object in $\mathcal{C}$. Now show that $(X, \phi) \to (T, \overline{\phi})$ is unique.

Let $\overline{g} : X \to T$. Suppose there is another morphism $g_2 : X \to T$.

By definition, g_2 satisfies $\phi = \overline{\phi} \circ g_2$.

Then, we have the following:

$$\begin{aligned} \overline{\phi} \circ g_2 &= \phi = \overline{\phi} \circ \overline{g} \\ \Rightarrow f\overline{\phi} \circ g_2 &= f\phi = f\overline{\phi} \circ \overline{g} \\ \underbrace{f\overline{\phi}}_{e} \circ g_2 &= e = \underbrace{f\overline{\phi}}_{e} \circ \overline{g} \\ \Rightarrow g_2 &= \overline{g} \end{aligned}$$

Therefore, $\overline{g} : X \to T$ is unique, and $(T, \overline{\phi})$ is the terminal object we are looking for.

Problem 2. *Skills developed: practice with functors and fundamentals of modules*

Let R be a ring, and $M \in R-\mathrm{Mod}$. Define,

$$\text{Tor}(M) = \{m \in M | rm = 0 \text{ for some nonzero } r \in R\}.$$

(a) Prove that if R is an integral domain (a commutative ring with no zero divisors), then $Tor(M)$ is a submodule of M, called the *torsion submodule.*

(b) Prove that $Tor : R-\mathrm{Mod} \to R-\mathrm{Mod}$ is a functor when R is an integral domain.

(c) Let R be the ring of 2×2 matrices over a field. Show that $Tor(R)$ is **not** a submodule of R

Hint: you don't even have to specify the field because you will only need the elements 0 and 1.

Defs/Theorems

1. **Submodule Criterion** Let R be a ring and let M be an $R-$module. A subset N of M is a submodule of m iff

   (1) $N \neq \emptyset$

   2) $x + ry \in N$ for all $r \in R$ and for all $x, y \in N$.

Proof(a) We will employ the submodule criterion for this problem.

(a-i) Show that $Tor(M) \neq \emptyset$.

By definition, a module M is an Abelian group under addition. Hence, it has an additive identity 0. Therefore, for any $r \in R, r \cdot 0 = 0 \Rightarrow Tor(M) \neq \emptyset$.

(a-ii) Let $x, y \in Tor(M)$.

$x, y \in Tor(M) \Rightarrow r \cdot x = 0, s \cdot y = 0$, for some nonzero $r, s \in R$ respectively.

Show Let R be an integral domain (and hence a commutative ring).

For $r' \in R$, we will show $x - r'y \in Tor(M)$. Multiply rs to the left of $x - r'y$ to get:

$$\begin{aligned} rs(x - r'y) &= rsx - rsr'y \\ &= srx - rr'sy \quad \text{(since we are in a commutative ring)} \\ &= s(rx) = rr'(sy) \\ &= s \cdot 0 - rr' \cdot 0 \quad \text{(by assumption,} rx = 0, sy = 0) \\ &= 0 \end{aligned}$$

Since $r, s \in R$, by closure, $rs \in R$.

Let $\tilde{r} = rs \in R$ and $\tilde{m} = x - r'y$.

We have shown that $\tilde{r}\tilde{m} = 0$ for some nonzero $\tilde{r} \in R \Rightarrow x - r'y \in$ Tor(M).

Hence, by the submodule criterion, Tor(M) is a submodule of M.

(**Source:** Did a similar problem in last year's hw)

(b) Show Show that Tor(M) is a functor, i.e.

(1) $\forall\, M \in Ob(R - \text{Mod}), \text{Tor}(M) \in Ob(R - \text{Mod})$

(2) $\forall\, \phi \in Hom_{R-\text{Mod}}(M, N), \text{Tor}(\phi) \in Hom_{R-\text{Mod}}(\text{Tor}(M), \text{Tor}(N))$ such that $\text{Tor}(1_M) = 1_{\text{Tor}(M)}$.

(3) $\text{Tor}(\phi_2 \circ \phi_1) = \text{Tor}(\phi_2) \circ \text{Tor}(\phi_1)$ for all ϕ_2, ϕ_1 composable in $R - \text{Mod}$.

(b-i) Let $M \in R - \text{Mod}$. In part (a), we showed that Tor(M) is a submodule M since R is an integral domain. In particular, Tor$(M) \in R - \text{Mod}$.

(b-ii) Let $\phi : M \to N$ to be a $R-$module homomorphism. In particular $\phi(rx+y) = r\phi(x)+\phi(y)$ for all $x, y \in M$ and all $r \in R$.

Then, define the induced morphism, Tor(ϕ) as follows:

$$\text{Tor}(\phi) = \phi|_{\text{Tor}(M)} : \text{Tor}(M) \to \text{Tor}(N)$$

(b-iii) Let 1_M be the identity module homomorphism on M. Show $\text{Tor}(1_M) = 1_{\text{Tor}(M)}$.

$$\begin{aligned} Tor(1_M) &= 1_M|_{Tor(M)} \\ &= 1_{Tor(M)} \end{aligned}$$

$\therefore$, $\text{Tor}(1_M) = 1_{\text{Tor}(M)}$.

(b-iv) Here is the related commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ | & & | \\ Tor(M) & \xrightarrow{Tor(\phi)} & Tor(N) \end{array}$$

Note, there should be an arrow indicating the induced morphism, $\phi \dashrightarrow$ Tor(ϕ).

(b-v) Let $\phi_1 : M \to N, \phi_2 : N \to O$ be $R-$module homomorphisms. Then,

$$\begin{aligned} \text{Tor}(\phi_2\phi_1) &= (\phi_2\phi_1)|_{\text{Tor}(M)} \\ &= (\phi_2|_{\text{Tor}(N)}\phi_1)|_{\text{Tor}(M)} \quad \text{(retricting twice is same as restricting once)} \\ &= (\phi_2|_{\text{Tor}(N)})(\phi_1)|_{\text{Tor}(M)} \quad \text{(since } \phi_1, \phi_2 \text{ are module homomorphisms)} \\ &= \text{Tor}(\phi_2)\text{Tor}(\phi_1) \end{aligned}$$

Hence, Tor is a covariant functor.

(c) We need to find a counterexample. First note that matrices are abelian groups, under addition. So any submodule would contain 0, and therefore,will be nonempty.

Hence, we need to then see if we can pick $x, y \in 2\times 2$ matrices such that for all $r \in R, x+ry$ is not in the submodule.

Let $r, s \in R$.

Consider $r = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Then, $r \cdot x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow r \in$ Tor(R).

Now, take $s = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then, $s \cdot y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow s \in$ Tor(R).

Then, $r + s = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin$ Tor(R).

The final assertion follows since multiplying by the identity would get the same r back, unless r was 0, but that would be a contradiction (since we assume $r \in R$ is nonzero).

Hence, Tor(R) is **not** a submodule of R if R is the ring of 2×2 matrix.

Problem 3. *Skills developed: practice with functors and fundamentals of modules.*

Fix a ring homomorphism $\phi : R \to S$, and let M be a left $S-$module Recall from class that M can also be considered a left $R-$module by *restriction of scalars:* $r \cdot m = \phi(r)m$ for $r \in R, m \in M$. The notations $_S(M)$ and $_RM$ can be used to clarify whether M is being considered as an $S-$ or $R-$module at any given point, but it is always the same set (this is OK since ϕ is fixed, otherwise ϕ needs to be in the notation if several ring homomorphisms $R \to S$ are relevant). Prove that ϕ induces a functor $\phi^* : S - \text{Mod} \to R - \text{Mod}$.

Hint: ϕ^* doesn't do anything to elements of a module, it just sends $_SM$ to $)_RM$, and it doesn't do anything to morphisms, they are the same maps of sets. The only thing to check is that if $f : M \to N$ is an $S-$module homomorphism, then $\phi^*(f)$ is actually an $R-$module homomorphism.

Proof We are given that $\phi : R \to S$ is a ring homomorphism and a left $S-$ module can be considered as a left $R-$module via $r \cdot m = \phi(r) \cdot m$.

Let $f : M \to N$ be a $S-$module homomorphism. Since f is a $S-$module homomorphism, we know that

$$f(sx + y) = sf(x) + f(y)$$

for all $x, y \in M$ and all $s \in S$.

Show Show ϕ^* is a functor, i.e.

(a) $\forall\ _SM \in Ob(S - \text{Mod}), \phi^*(_SM) = {}_RM \in Ob(R - \text{Mod})$

(b) $\forall\ f \in Hom(_SM, {}_SN), \phi^*(f) \in Hom(\phi^*(_SM), \phi^*(_SN))$ such that $\phi * (1_{_SM}) = 1_{\phi*(_SM)}$

(c) $\phi^*(f_2 \circ f_1) = \phi^*(f_2) \circ \phi^*(f_1)$

(a) Let $_SM$ be a left $S-$ module. Define the mapping of objects as $\phi^*(_SM) = {}_RM$, which is a left $R-$module.

(b-i) (By the hint, since ϕ^* does not do anything to morphisms) $\phi^*(f) = f$.

Show We want to show that $\phi^*(f) : \phi^*(_SM) \to \phi^*(_SN)$ is an $R-$ module homomorphism (where the corresponding objects are $_RM$ and $_RN$ respectively).

(b-ii) Let $r \in R, x, y \in {}_RM$. Then, we have the following:

$$\begin{aligned}
\phi * (f)(rx + y) &= f(rx + y) \\
&= f(\phi(r)x + y) && \text{(restriction of scalars) } r \cdot x = \phi(r) \cdot x \\
&= f(\phi(r)x) + f(y) && \text{since } f \text{ is a S-module homomorphism} \\
&= \phi(r)f(x) + f(y) \\
&= r\phi^*(f)(x) + \phi^*(f)y
\end{aligned}$$

Hence, $\phi^*(f)$ is a $R-$module homomorphism.

(Note, in the second equality we applied the *restriction of scalars* and think in terms of left $S-$module. Since we assumed that f is a $S-$module homomorphism, we could break it up as we did above).

(b-iii) Let $1_S M$ be the identity module homomorphism on ${}_S M$. Show that $\phi^*(1_{_S M}) = 1_{\phi*(_S M)}$

$$\begin{aligned}
\phi^*(1_{_S M})(r \cdot m) &= 1_{_S M}(\phi(r) \cdot m) \\
&= \phi(r) \cdot m \\
&= r \cdot m
\end{aligned}$$

$\therefore, \phi^*(1_{_S M}) = 1_{\phi*(_S M)}$

(b-iv) Here is the related commutative diagram:

$$\begin{tikzcd}
{}_S M \arrow[r, "f"] \arrow[d, dash] & {}_S N \arrow[d, dash] \\
\phi^*({}_S M) = {}_R M \arrow[r, "\phi^*(f)=f"] & \phi^*({}_S N) = {}_R N
\end{tikzcd}$$

(c) Let $f_1 : {}_S M \to {}_S N, f_2 : {}_S N \to {}_S O$. Let $r \in R, m \in {}_R M$. Then,

$$\begin{aligned}
\phi^*(f_2 \circ f_1)(r \cdot m) &= (f_2 \circ f_1)(\phi(r) \cdot m)) \\
&= \phi(r)(f_2 \circ f_1)(m)) \\
&= \phi^*(r(f_2(f_1(m))) \\
&= \phi^*(f_2(rf_1(m)) \\
&= \phi^*((f_2) \circ f_1(\phi(r) \cdot m)) \\
&= \phi^*((f_2) \circ \phi^*(f_1))(r \cdot m)
\end{aligned}$$

Hence, ϕ^* is a covariant functor.