Math 6000, Fall 2020 (Prof. Kinser), Homework 1

Nitesh Mathur

26 August 2020

Source Discussed with Zach Bryhtan after attempting myself and did proof analysis/proofreading with him after the initial write-up. (Other Sources: Wikipedia, Dummit and Foote, Google)

Problem 1 *Skills developed: working with properties of maps rather than choosing elements. This can simplify certain kinds of proofs by not introducing extra symbols too keep track of. It is particularly useful when you have several interacting maps.*

Let $f\colon A \to B$ be a morphism in an arbitrary category $\mathcal{C}$ (so the proofs should not make reference to "elements"). Prove each of the following:

(a) If f is a retraction, then f is an epimorphism.

(b) If f is a section, then f is a monomorphism.

(c) The morphism f is an isomorphism if and only if f is a monomorphism and a retraction. (This is if and only if f is an epimorphism and a section; the proof is similar so don't turn it in.)

Defs./Theorems

1. Let $\mathcal{C}$ be a category. A morphism $f : A \to B$ is a **monomorphism** if $fg_1 = fg_2 \Rightarrow g_1 = g_2$ for all $g_1, g_2 : Z \to A$. ("left-cancellation"; if object is a set, monomorphism $\iff$ injection/1-1).

2. An **epimorphism** if $g_1 f = g_2 f \Rightarrow g_1 = g_2$ for all $g_1, g_2 : B \to Z$. ("right-cancellation"; if object is a set, epipmorphism $\iff$ surjection/onto).

3. An **isomorphism** if $\exists\ g : B \to A$ such that

   $$\begin{cases} gf = 1_A \\ fg = 1_B \end{cases}$$

   ("has an inverse"; if object is a set, isomorphism $\iff$ bijection).

4. **Retraction** if $\exists\ g : B \to A$ such that $fg = 1_B$ ("has right inverse").

5. **Section** if $\exists\ g : B \to A$ such that $gf = 1_A$ ("has left inverse");

Proofs

(a) We assume that $f : A \to B$ is a retraction. Then, $\exists\, g : B \to A$ such that $fg = 1_B$.

Suppose that $\exists$ morphisms $g_1, g_2 : B \to Z$ such that $g_1 f = g_2 f$.

Show: $g_1 = g_2$.

$$\begin{aligned} g_1 f &= g_2 f \\ \Rightarrow g_1(f \circ g) &= g_2(f \circ g) \quad \text{(since } f \text{ is a retraction)} \\ g_1 \circ 1_B &= g_2 \circ 1_B \\ g_1 &= g_2 \end{aligned}$$

Since we have shown that $g_1 f = g_2 f \Rightarrow g_1 = g_2$, we have shown that f is an epimorphism.

(b) We assume that $f : A \to B$ is a section. Then, $\exists\, g : B \to A$ such that $gf = 1_A$.

Suppose that $\exists g_1, g_2 : Z \to A$ such that $fg_1 = fg_2$.

Show $g_1 = g_2$.

$$\begin{aligned} fg_1 &= fg_2 \\ \Rightarrow (g \circ f)g_1 &= (g \circ f)g_2 \quad \text{(since } f \text{ is a section)} \\ 1_A \circ g_1 &= 1_A \circ g_2 \\ g_1 &= g_2 \end{aligned}$$

Since we have shown that $fg_1 = fg_2 \Rightarrow g_1 = g_2$, we have shown that f is a monomorphism.

(c) $\Rightarrow$ We assume that f is an isomorphism. Then, $\exists\, g : B \to A$ such that

$$\begin{cases} gf = 1_A \\ fg = 1_B \end{cases}.$$

Suppose $\exists g_1, g_2 : Z \to A$ such that $fg_1 = fg_2$.

Show Show $g_1 = g_2$ and f is a retraction.

(i) Since f is an isomorphism, we know that $\exists\, g : B \to A$ such that $fg = 1_B$. Hence, f is also a retraction by definition.

(ii) Furthermore,

$$fg_1 = fg_2$$

$$\begin{aligned} \Rightarrow g \circ (fg_1) &= g \circ (fg_2) \\ (gf) \circ g_1 &= (gf) \circ g_2 \quad \text{(by Associativity)} \\ 1_A \circ g_1 &= 1_A \circ g_2 \\ g_1 &= g_2 \end{aligned}$$

Since we showed that $fg_1 = fg_2 \Rightarrow g_1 = g_2$ for all such g_1, g_2, we have shown that f is also a monomorphism.

$\therefore f$ isomorphism $\Rightarrow f$ is a retraction and a monomorphism.

(c) $\Leftarrow$ Suppose $fg_1 = fg_2 \Rightarrow g_1 = g_2$ for all morphisms, $g_1, g_2 : Z \to A$. We also assume that $\exists\, g : B \to A$ such that $fg = 1_B$.

Show $\exists\, g : B \to A$ such that $gf = 1_A$.

$$\begin{aligned} fg &= 1_B \quad \text{(since } f \text{ is a retraction)} \\ \Rightarrow fgf &= 1_B \circ f \\ fgf &= f \\ \Rightarrow fgf &= f \circ 1_A \end{aligned}$$

Note, fgf is a map from $A \to B$. Hence, since $fgf = f \circ 1_A, gf = 1_A$.

Since we have shown that $gf = 1_A$, we have shown that f is an isomorphism.

Problem 2 *Skills developed: practice applying definition of "functor" in a more familiar setting. Creating examples to understand abstract properties.*

Let **Groups** be the category of groups, and **Rings** be the category of rings with 1 (morphisms are ring homomorphisms sending 1 to 1).

(a) Given a group G, let $Ab(G)$ be the largest quotient of G which is abelian. Show that Ab is a functor from **Groups** to itself.

(b) Show that the map from **Rings** to **Groups**, defined on objects by sending a ring R to its group of units $R^\times$, extends to a functor between these categories. Show **by example** that it is neither faithful nor full.

Defs./Theorems **Note**, we will use $\leq$ to denote subgroup, and $\trianglelefteq$ for normal subgroup.

1. A **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ is given by:

(i) $\forall A \in Ob(\mathcal{C}), F(A) \in Ob(\mathcal{D})$

(ii) $\forall f \in Hom_{\mathcal{C}}(A, B), F(f) \in Hom_{\mathcal{D}}(F(A), F(B))$ such that $F(1_A) = 1_{F(A)}$.

(iii) $F(gf) = F(g) \cdot F(f)$ for all gf composable in $\mathcal{C}$

2. A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ is the same except **reverses** directions of morphisms: i.e. $F(gf) = F(f) \cdot F(g)$.

3. A functor $F : \mathcal{C} \to \mathcal{D}$ induces maps on Hom sets. $\forall A, B \in \mathcal{C}$, we have $F_{A,B} : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{D}}(F(A), F(B))$.

3a. F is **faithful** if $F_{A,B}$ is **injective** $\forall A, B$; **(3b.)** F is **full** if $F_{A,B}$ is **surjective** $\forall A, B$.

4. (D/F: Pg. 169) Let G be a group, and let A, B be nonempty subsets of G.

(a) Define $[x, y] = x^{-1}y^{-1}xy$, called the **commutator** of x and y.

(b) Define $[A, B] =< [a, b] > |a \in A, b \in B >$, the group generated by commutators of elements from A and from B.

(c) Define $G' =< [x, y]|x, y \in G >$, the subgroup of G generated by commutators of elements from G, called the **commutator subgroup** of G.

5. D/F (Prop. 7.4) G/G' is the **largest abelian quotient of** G**.**

(pg. 170 Proof) Suppose $H \trianglelefteq G$ and G/H is abelian. Then for all $x, y \in G$ we have $(xH)(yH) = (yH)(xH)$, so

$$\begin{aligned} 1H &= (xH)^{-1}(yH)^{-1}(xH)(yH) \\ &= x^{-1}y^{-1}xyH \\ &= [x, y]H \end{aligned}$$

Thus, $[x, y] \in H$ for all $x, y \in G$ so that $G' \leq H$.

Proofs

(a) Show Show that Ab is a functor, i.e.

(1) $\forall\ G \in$ **Groups**, $Ab(G) \in$ **Groups**.

(2) $\forall \phi \in Hom_{\textbf{Groups}}(G_1, G_2)$, $Ab(\phi) \in Hom_{\textbf{Groups}}(Ab(G_1), Ab(G_2))$ such that $Ab(1_G) = 1_{Ab(G)}$

(3) $Ab(\phi_2 \circ \phi_1) = Ab(\phi_2) \circ Ab(\phi_1)$ for all ϕ_2, ϕ_1 composable in **Groups**.

(a-i) Let $G \in$ **Groups**. Let G' be the commutator subgroup of G. Then, G/G' is the largest quotient of G which is Abelian as shown above.

$Ab(G) = G/G'$ forms a quotient, which is a group. Hence, $Ab(G) \in$ **Groups**. and (1) is satisfied.

(a-ii) Let $G_1, G_2 \in$ **Groups** and $\phi \in Hom_{\textbf{Groups}}(G_1, G_2)$.

Let $G'_1 \leq G_1, G'_2 \leq G_2$ be corresponding commutator subgroups. Then, $Ab(G_1) = G_1/G'_1$ and $Ab(G_2) = G_2/G'_2$ are the largest abelian quotients respectively.

Define $Ab(\phi) = \bar{\phi} \in Hom_{\textbf{Groups}}(Ab(G_1), Ab(G_2))$ as follows:

$$\bar{\phi} : Ab(G_1) \to Ab(G_2) \quad \text{by } \bar{\phi}(xG'_1) = \phi(x)G'_2$$

(a-iii) Show $Ab(1_G) = 1_{Ab(G)}$.

Let 1_G be the identity morphism on G, i.e. $1_G \in Hom_{\textbf{Groups}}(G, G)$ and $Ab(1_G) = \overline{1_G} \in Hom_{\textbf{Groups}}(G/G', G/G')$. Then,

$$\begin{aligned} Ab(1_G) &= \overline{1_G}(xG'_1) \\ &= 1_G(x)G'_1 \\ &= xG'_1 \\ &= 1_{Ab(G)} \end{aligned}$$

$\therefore Ab(1_G) = 1_{Ab(G)}$

(a-iv) For reference, here is the related diagram:

$$\begin{array}{ccc} G_1 & \xrightarrow{\ \phi\ } & G_2 \\ \big\downarrow{\scriptstyle Ab} & & \big\downarrow{\scriptstyle Ab} \\ Ab(G_1) = G/G'_1 & \xrightarrow{\ \bar{\phi}\ } & Ab(G_2) = G/G'_2 \end{array}$$

Note, $Ab(\phi) \dashrightarrow \bar{\phi}$ arrow should also be there for the induced morphism.

(a-v) Let $G_1, G_2, G_3 \in$ **Groups** with $\phi_1 \in Hom_{\textbf{Groups}}(G_1, G_2)$ and $\phi_2 \in Hom_{\textbf{Groups}}(G_2, G_3)$ being the correponding group homomorphisms.

Let $Ab(G_1) = G/G_1', Ab(G_2) = G/G_2'$ and $Ab(G_3) = G/G_3'$ be the corresponding objects after the functor Ab is applied. The induced morphisms are as follows: $Ab(\phi_1) = \bar{\phi}_1 \in Hom_{\textbf{Groups}}(Ab(G_1), Ab(G_2)), Ab(\phi_2) = \bar{\phi}_2 \in Hom_{\textbf{Groups}}(Ab(G_2), Ab(G_3))$.

Note that $Ab(\phi_2 \circ \phi_1) \in Hom_{\textbf{Groups}}(G/G_1', G/G_3')$. Then, we have the following:

$$\begin{aligned} Ab(\phi_2 \circ \phi_1)(xG_1') &= \overline{\phi_2 \circ \phi_1}(xG_1') \\ &= (\phi_2 \circ \phi_1)(x)G_3' \\ &= Ab(\phi_2) \circ \phi_1(x)G_2' \\ &= Ab(\phi_2) \circ Ab(\phi_1)(xG_1') \end{aligned}$$

Hence (3) is satisfied, and we have shown that Ab is a covariant functor.

Source (Pg. 913 Example 3 for inspiration)

(b) Let **Rings** and **Groups** be categories. Define a functor F between these two categories.

Show Show that F is a functor, i.e.

(1) $\forall R \in Ob(\textbf{Rings}), F(R) \in Ob(\textbf{Groups})$

(2) $\forall \phi \in Hom_{\textbf{Rings}}(R_1, R_2), F(\phi) \in Hom_{\textbf{Groups}}(F(R_1), F(R_2))$ such that $F(1_{R_1}) = 1_{F(R_1)}$

(3) $F(\phi_2 \circ \phi_1) = F(\phi_2) \circ F(\phi_1)$ for all ϕ_2, ϕ_1 composable in **Rings**.

Proof (b-i) Let $R \in Ob(\textbf{Rings})$. Define $F(R) = R^\times$ to be the group of units. Then, $F(R) \in Ob(\textbf{Groups})$.

(b-ii) Let $R_1, R_2 \in$ **Rings** and $\phi : R_1 \to R_2$ be a ring homomorphism with 1. (Note, $F(R_1) = R_1^\times$ and $F(R_2) = R_2^\times$ are the corresponding objects).

Define $F(\phi) = \bar{\phi} = \phi|_{R_1^\times} : R_1^\times \to R_2^\times$.

Since it is a restriction on a group of units, $\bar{\phi}$ is a group homomorphism.

(b-iii) Let 1_{R_1} be the identity morphism on R_1. Then,

$$\begin{aligned} F(1_{R_1}) &= \overline{1_{R_1}} \\ &= 1|_{R_1^\times} \\ &= 1|_{F(R_1)} \end{aligned}$$

$\therefore$, $F(1_{R_1}) = 1_{F(R_1)}$.

(b-iv) Here is the corresponding commutative diagram:

$$\begin{array}{ccc} R_1 & \xrightarrow{\phi} & R_2 \\ \downarrow F & & \downarrow F \\ F(R_1) = R_1^\times & \xrightarrow{\bar{\phi}} & F(R_2) = R_2^\times \end{array}$$

Note, we should have the induced morphism $\phi \dashrightarrow \bar{\phi}$ as well.

(b-v) Let $\phi_1 \in Hom_{\textbf{Rings}}(R_1, R_2), \phi_2 \in Hom_{\textbf{Rings}}(R_2, R_3)$. Then:

$$\begin{aligned} F(\phi_2\phi_1) &= \overline{\phi_2\phi_1} \\ &= (\phi_2\phi_1)|_{R_1^\times} \\ &= (\phi_2|_{R_2^\times}\phi_1)|_{R_1^\times} \\ &= (\phi_2|_{R_2^\times})(\phi_1|_{R_1^\times}) \quad \text{since } \bar{\phi}_1, \bar{\phi}_2 \text{ are group homomorphisms} \\ &= F(\phi_2)F(\phi_1) \end{aligned}$$

Hence, F is a covariant functor.

Note we used the following fact from line 2 to line 3: (From wikipedia) Restricting a function twice is the same as restricting it once, i.e. if $A \subset B \subset \text{dom} f$, $(f|_B)|_A = f|_A$.

(c) To show that $F_{A,B}$ is not injective, by counterexample, we need to exhibit that $Hom_{\mathcal{C}}(A, B) \neq Hom_{\mathcal{C}}(C, D)$, but $Hom_{\mathcal{D}}(F(A), F(B)) = Hom_{\mathcal{D}}(F(C), F(D))$.

(c - i) Consider rings $R_1 = \mathbb{Z}$ and $R_2 = \mathbb{Z}[x]$, the polynomial ring over $\mathbb{Z}$.

Let $\phi_1 \in Hom_{\textbf{Rings}}(\mathbb{Z}, \mathbb{Z}), \phi_2 \in Hom_{\textbf{Rings}}(\mathbb{Z}[x], \mathbb{Z}[x])$.

Yet, $F(\phi_1) = \bar{\phi}_1 \in Hom_{\textbf{Groups}}(\mathbb{Z}^\times, \mathbb{Z}^\times) = \bar{\phi}_2 = F(\phi_2)$.

$\therefore$, F_{R_1,R_2} is not injective and hence, F is not faithful.

(c-ii) To show that $F_{A,B}$ is not surjective, by counterexample, we need to exhibit that $\exists$ some map in $Hom_{\mathcal{D}}(F(A), F(B))$ such that there is no induced morphism coming from $Hom_{\mathcal{C}}(A, B)$.

Let $\bar{\phi} : \mathbb{Z}^\times \to \mathbb{Z}^\times$ by $\phi(\bar{1}) = -1$. Since we assumed that we have Rings with $1, \phi(1) = 1$.

Hence, the corresponding restriction map should also map $\bar{\phi}(1) = 1$, not -1.

$\therefore$, F_{R_1,R_2} is not surjective and hence, F is not full.

(Note: Motivation for part (c) - Zach)

Problem 3. *Skills developed: more practice with functors and getting used to passing back and forth between equivalent definitions (abstract vs. concrete).*

Recall that a group G determines a category with one object **G.** Let K be a field, and $\mathbf{Vec}_K$ the category of $K-$vector spaces. A *representation* of a group G on a $K-$vector space V is a group homomorphism $\rho : G \to GL(V)$.

(a) Given a representation ρ, define a functor $F_p : \mathbf{G} \to Vec_K$ in a natural way.

(b) Given a functor $F : \mathbf{G} \to \mathbf{Vec}_K$, define a representation of G on a vector space in a natural way.

(c) *Think about(write up is optional)* why these two processes are "inverses" to one another, so the concept of a representation is equivalent to this particular kind of functor. In general, a function from any category $\mathcal{C}$ to $\mathbf{Vec}_K$ can be interpreted (or defined) as a representation of the category $\mathcal{C}$.

Solution

Overview Let G be a group. Define a category **G** with 1 object: $\{\star\} = Ob(\mathbf{G})$ and $Hom_{\mathbf{G}}(\star, \star) = G$.

Define category $\mathbf{Vec}_K$ with objects being vector space V, and $Hom_{\mathbf{Vec}_K}(V, V) = GL(V)$.

Let $\rho : G \to GL(V)$, a representation of group G on a vector space V, be a group homomorphism defined as follows: For $g \in G, \rho(g) = T : V \to V \in GL(V)$.

$GL(V)$ is the general linear group, the set of all bijective linear transformations from V to V. Since $GL(V)$ is a group and needs to satisfy group axioms, $T \in GL(V)$ is invertible.

(a) Show To show F_ρ is a functor, we need to show:

(1)For $\star \in \mathbf{G}$, $F_\rho(\star) = V \in \mathbf{Vec}_K$.

(2) $\forall g \in Hom_{\mathbf{G}}(\star, \star), F_\rho(g) \in Hom_{\mathbf{Vec}_K}(F_\rho(\star), F_\rho(\star))$ such that $F_\rho(1_\star) = 1_{F_\rho(\star)}$

(3) $F_\rho(g_2 g_1) = F_\rho(g_2) \cdot F_\rho(G_1)$ for all $g_2 \circ g_1$ composable in **G**.

(a-i) Define $F_\rho(\star) = V \in \mathbf{Vec}_K$.

(a-ii) Let $g \in G = Hom_{\mathbf{G}}(\star, \star)$.

Define $F_\rho(g) = \rho(g) = T \in Hom_{\mathbf{Vec}_K}(F_\rho(\star), F_\rho(\star))$

(a-iii) Let $1_\star$ be the identity morphism in $Hom_{\mathbf{G}}(\star, \star)$.

Then, $F_\rho(1_\star) = \rho(1_\star) = 1_V = 1_{F_\rho(\star)}$, the identity linear transformation on V.

$\therefore$, $F_\rho(1_\star) = 1_{F_\rho(\star)}$

(a-iv) The commutative diagram is as follows:

$$\begin{array}{ccc} \star & \xrightarrow{\quad g \quad} & \star \\ \downarrow{\scriptstyle F_p} & & \downarrow{\scriptstyle F_p} \\ F_p(\star) = V & \xrightarrow{\ T\ } & F_p(\star) = V \end{array}$$

Note, we also have the induced morphism $F_p(g) \dashrightarrow T$.

(a-v) Let $g_1, g_2 \in Hom_{\mathbf{G}}(\star, \star)$.

$$\begin{aligned} F_\rho(g_2 g_1) &= \rho(g_2 g_1) \\ &= \rho(g_2) \circ \rho(g_1) \quad \text{(since } \rho \text{ is a group homomorphism)} \\ &= F_\rho(g_2) \circ F_p(g_1) \end{aligned}$$

Hence, $F_\rho : \mathbf{G} \to \mathbf{Vec}_K$ is a covariant functor.

(b) (From wikipedia) A **representation** of a group G on a vector space V over a field K is a group homomorphism from G to $GL(V)$.

Given $F : \mathbf{G} \to \mathbf{Vec}_K$ is a functor.

For object $\star \in \mathbf{G}$, $F(\star) = V \in \mathbf{Vec}_K$. For morphism $g \in Hom_{\mathbf{G}}(\star, \star)$, $F(g) \in Hom_{\mathbf{Vec}_K}(V, V)$.

Then we define the representation as follows: $\rho : g \to F(g)$

Show that ρ is a group homomorphism.

Let $g_1, g_2 \in G$. Then,

$$\begin{aligned} \rho(g_1 g_2) &= F(g_1 g_2) \\ &= F(g_1) \circ F(g_2) \quad \text{(since } F \text{ is a functor)} \\ &= \rho(g_1) \circ \rho(g_2) \end{aligned}$$

Hence, ρ is a group homomorphism, and we have defined the representation in a natural way.

(Since ρ is a group homomorphism and $F(g) \in Hom_{\mathbf{Vec}_K}(V, V)$, the identity will map to the identity linear transformation. $F(g)$ has to be invertible since V is an element of a group).

(c) These processes are "inverses" to one another because a representation is exactly how the functor maps morphisms to corresponding morphisms.

On the other hand, given a functor, the way morphisms are mapped happens to be the representation because it satisfies the properties of a group homomorphism.
