Number Theory Seminar

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1 Riemann Zeta Function

1.1 Euler Products

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$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

*Runs over all numbers

Exercises

1. Convergent for $\sigma > 1$
2. Uniformly convergent on any region $\sigma \geq 1 + \delta$

Hint Think About Weirstrauss M-Test. (It is analytic function in the region, term by term, think about ζ', etc.

$$\prod_{p} (1 - \frac{1}{p^s})^{-1},$$

where $\delta \in \mathbf{C}, S = \sigma + it$.

*Runs only over primes.

3. This is absolutely convergent for $\sigma > 1$.

Hint Convergence about infinite product? Take log of product, and then the product becomes a lot. Then, we can think about when does the new sum converge?

$-\sigma \log(1 - \frac{1}{p^s})$, then expand it using power series expansion for log. Finally, we will get a zeta function looking expression (which converges) so just the prime terms in the zeta function has to converge.

Claim $\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - \frac{1}{p^s})^{-1} - \zeta(s)$

Leading Questions

1. Why should these be related? (Unique Prime factorization).
2. Why don't we have overcounting?
3. Equivalent statement to UPF of $\mathbf{Z}$.

Proof Hint Start with finite number of terms and think about $\frac{1}{1-a}$, the geometric series.

Proof.

$$\begin{aligned}\prod_{p \leq P} (1 - \frac{1}{p^s})^{-1} &= \prod_{p \leq P} (1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + ...) \\ &= 1 + \frac{1}{n_1^s} + \frac{1}{n_2^s}\end{aligned}$$

(integers whose primefactorization are $\leq P$).

Now consider

$$
\begin{aligned}
|\zeta(s) = \prod_{p \leq P} (1 - \frac{1}{p^2})^{-1}| \\
&= |\zeta(s) - 1 - \frac{1}{n_1^s} - \frac{1}{n_2^s} - ...| \\
&\leq \frac{1}{(p+1)^\sigma} + \frac{1}{(p+2)^\sigma} + ... \\
&\to 0 \text{ as } p \to \infty
\end{aligned}
$$

(since $\sigma > 1$). □

Conversation "Dirichlet series," but first done by Euler (of course).

Called the **Euler Products**.

Then came is Riemann and said, "Why not make it more complex?" (literally). Hence, it is called the Riemann Zeta function (the summation).

No non-trivial root of ζ function for great 0.

Claim $\zeta(s)$ has no root for $\sigma > 1$.

(Do the same thing. Finite sum, take it to the left hand side...)

Proof.

$$(1 - \frac{1}{2^s})(1 - \frac{1}{3^s})....(1 - \frac{1}{1-p^s})\zeta(s) = \sum_{i \in I} \frac{1}{n_i^S} \quad \text{(all prime factors strictly greater than } P\text{)}$$

$$
\begin{aligned}
|(1 - \frac{1}{2^s})(1 - \frac{1}{3^s})....(1 - \frac{1}{1-p^s})| \cdot |\zeta(s)| &> 0 \\
|\zeta(s)| &> 0
\end{aligned}
$$

□

1.2 Prime Number Theorem

Consider $\Pi(x) =$ (number of primes) $\leq x$, where $x \in \mathbf{R}_+$.

1. Is there a relation between $\zeta(s)$ and $\Pi(x)$.

$$
\begin{aligned}
\log(\zeta(s)) &= -\sum_p \log(1 - \frac{1}{p^s}) \\
&= -\sum_{n=2}^{\infty} \{\Pi(n) - \Pi(n-1)\} \log(1 - \frac{1}{n^s}) \\
&= -\sum_{n=2}^{\infty} \Pi(n) \{\log(1 - \frac{1}{1-n^s}) - \log(1 - \frac{1}{(n+1)^2})\} \quad \text{(Then COV)} \\
&= -\sum_{n=2}^{\infty} \Pi(n) \log(1 - \frac{1}{1-n^s}) + \sum_{n=2}^{\infty} \Pi(n-1) \log(1 - \frac{1}{1-(n+1)^s})
\end{aligned}
$$

For the second to last line, this is basically like FTC of integration i.e.

$$
\begin{aligned}
-\sum_{n=2}^{\infty}\Pi(n)\{\log(1-\frac{1}{1-n^s})-\log(1-\frac{1}{(n+1)^2})\} &= \sum_{n=2}^{\infty}\Pi(n)(\log(1-\frac{1}{x^s})|_n^{n+1} \\
&= \sum_{n=2}^{\infty}\Pi(n)\int_n^{n+1}\frac{s}{x(x^s-1)}\ dx \\
&= s\int_2^{\infty}\frac{\Pi(x)\ dx}{x(x^s-1)}
\end{aligned}
$$

1.3 Mobius Function

Consider the reciprocal of the zeta function:

$$
\begin{aligned}
\frac{1}{\zeta(s)} &= \prod_p(1-\frac{1}{p^s}) \\
&= 1+\sum_{p\text{prime}}(\frac{-1}{p^s})+\sum_{p_1,p_2}(\frac{-1}{p_1^s}-\frac{1}{p_2^2})+\sum_{p_1p_2p_3}\frac{(-1)^3}{p_1^sp_2^sp_3^s}+... \\
&= \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s},
\end{aligned}
$$

where $\mu(n)$ is the Mobius function, $\mu(1)=1$ and

$$
\begin{cases}
\mu(n) & =(-1)^k\text{if n is product of distinct k prime factors} \\
0 & = \text{ if } n \text{ not square free}
\end{cases} \tag{1}
$$

Proposition.

$$
\sum_{d|n}\mu(d)=\begin{cases}1 & \text{n =1} \\ 0 & \text{otherwise}\end{cases} \tag{2}
$$

$$
\begin{aligned}
1 &= \zeta(s)\cdot\frac{1}{\zeta(s)} \\
&= \sum_{m=1}^{\infty}\frac{1}{m^s}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \\
&= \sum_{p=1}^{\infty}\frac{1}{p^s}\sum_{d|\phi}\mu(\phi)
\end{aligned}
$$

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Recall Definition of a Riemann Zeta function, can be expressed in two different ways.

$$
\begin{aligned}
\zeta(s) &= \sum_{n=1}^{\infty}\frac{1}{n^s} && \text{(Dirichlet series)} \\
&= \prod_{p\text{ primes}}(1-\frac{1}{p^s})^{-1} && \text{(Euler product)}
\end{aligned}
$$

where $s=\sigma+it\in\mathbf{C}$.

Check out Praneel's blog here!

Remind **Mobius Function** (Arithmetic function) - Tells you when number is square free.

$$\mu(n) = \begin{cases} (-1)^k & , n \text{ square free}, n = p_1 \cdot ...p_k \text{ all distinct prime factors} \\ 0 & , \text{otherwise} \end{cases}$$

Now consider $\sum_{d|n} \mu(d)$ (will only study finite sums)

Theorem 1.

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

Proof.

$$1 = \zeta(s) \cdot \frac{1}{\zeta(s)}$$

Recall, $\frac{1}{\zeta(s)}$ had a series from involving $\mu(d)$. □

Now, let us try an alternative proof.

Proof. Let $n = p_1^{k_1} p_2^{k_2} ... p_n^{k_n}$.

Let $d = p_1^{\ell_1} \cdot ... p_m^{\ell_m}, \quad 0 \leq \ell_1 \leq k_1$

Observe $\sum_{d|n} \mu(d) = \sum_{d|n} \mu(d), \ell_i = 0$.

$\mu(d) = 1$ when even number of prime factors. Number of ways to choose them, i.e. binomial coefficients.

$$\begin{aligned} \mu(d) &= -1 \quad d = p_i p_j \\ \text{number of ways} \Rightarrow & \binom{n}{2} \\ &= -1 \text{ if } d = p_i p_j p_k \\ \text{number of ways} \Rightarrow & \binom{m}{3} \\ \sum_{d|m} \mu(d) &= 1 + \binom{m}{1}(-1) + \binom{m}{2}(-1)^2 + ... \binom{m}{3}(-1)^3 + ... \\ &= (1-1)^n \\ &= 0 \end{aligned}$$

□

Quote "Analytic number theory is all about tools...because you have to, you have, you have to know the tools to do number theory."

"Otherwise, number theory is basically Calc II."

Proposition (Mobius Inversion Formula). f known, g unknown, can use this inversion:

$$\begin{aligned} f_n &= \sum_{d|n} g(d) \\ \iff g(m) &= \sum_{e|m} \mu(\frac{m}{e}) f(e) \\ g(m) &= \sum_{e|m} \mu(\frac{m}{e}) \sum_{d|e} g(d) \\ &= \sum_{d|m} g(d) \underbrace{\sum_{c|\frac{m}{d}} \mu(c)}_{=0 \text{ unless } m=d} \\ &= g(m) \end{aligned}$$

Quotes of the Day ”**Lock them in a room and let them think hard**.” - Curtis

Praneel ”How well can you rearrange sums?”

i.e. ”Sum is basically a Lebesgue integral with respect to the counting measure feat Fubini.”

Example 2 (Von Mangoldt's function). Define

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$\begin{aligned} n &= p_1^{k_1} \cdot ... \cdot p_m^{k_m} \\ \Rightarrow \log(n) &= k_1 \log(p_1) + .. + k_n \log(p_n) \\ &= k_1 \Lambda(p_1) + ... + k_n \Lambda(p_n) \\ &= \sum_{d|n} \Lambda(d) \end{aligned}$$

Observe Logarithmic differentiation comes into play.

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= \sum_p \frac{\log(p)}{p^s}(1 - \frac{1}{p^s})^{-1} \\ &= -\sum_p \log(p) \sum_{m=1}^{\infty} \frac{1}{p^{ms}} \\ &= -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} \end{aligned}$$

1.4 Divisor Function and Ramanujan Sums

$$\sigma_a(n) = \sum_{d|n} d^a \tag{4}$$

The 'well-known' divisor function has $a = 0$.

Consider

$$\begin{aligned} \zeta(s)\zeta(s-a) &= \sum_{\mu=1}^{\infty} \frac{1}{\mu^s} \sum_{\nu=1}^{\infty} \frac{\nu^a}{\nu^s} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\nu|n} \nu^a \\ &= \sigma_a(\nu)\zeta(s) \end{aligned}$$

(since $n = \mu \cdot \nu$. Hence, it runs over all divisors of n.)

Let's denote $d(n) := \sigma_0(n)$, the number of divisors of n, Then

$$\begin{aligned}\zeta^2(s) &= \sum_{k=1}^{\infty} \frac{1}{k^s} \sum_{l=1}^{\infty} \frac{1}{l^s} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{kl=n} 1 \\ &= \sum_{1}^{\infty} \frac{d(n)}{n^s}\end{aligned}$$

1.5 Ramanujan Sums

Let

$$\begin{aligned}c_k(n) &= \sum_{h \le n, (h,k)=1} e^{-2nhi\pi}/k \\ &= \sum_{h \le n, (h,k)=1} \cos \frac{2nh\pi}{k}\end{aligned}$$

We will see one example of application of Mobius inversion to find an alternate expression for $c_k(n)$ involving the mobius function. We consider the sum

$$\eta_k(n) = \sum_{m=0}^{k-1} e^{-2nmi\pi/k}$$

which is equal to k when $k|n$ and 0 otherwise, being sum of roots of unity.

Consider

$$\sum_{d|k} c_d(n) = \sum_{d|k} \sum_{(r,d)=1, r<d} e^{-2nri\pi/d} = \eta_k(n)$$

Thus by Mobius inversion,

$$c_k(n) = \sum_{d|k} \mu(\frac{k}{d}) \eta_d(n) = \sum_{d|k, d|n} \mu(\frac{k}{d}) d.$$

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Speaker Curtis

1.6 Analytic Continuation

Ultimate Goal Recall, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ only converges when $\mathrm{Re}(s) > 1$.

1.6.1 Gamma Function

Step 1. Let us consider the Γ function.

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} \, dt \tag{5}$$

$$\begin{aligned}\Gamma(s) &= \int_0^{\infty} e^{-t} t^{s-1} \, dt \\ &= \int_0^{1} e^{-t} t^{s-1} \, dt + \int_1^{\infty} e^{-t} t^{s=1} \, dt\end{aligned}$$

We want to avoid $\int \frac{1}{t}\ dt$, so we need $s > 0$, where $s \in \mathbf{R}$.

Claim $\Gamma(s)$ extends to an analytic function on all of $\mathrm{Re}(s) > 0$, where $s \in \mathbb{C}$.

If $\mathrm{Re}(s) > 0$, then

$$\begin{aligned} |e^{-t}t^{s-1}| &= e^{-1}t^{\sigma-1} \\ &\Rightarrow \text{ Integral still converges on } \mathrm{Re}(s) > 0 \end{aligned}$$

Let $F_\epsilon(s) = \int_\epsilon^{1/\epsilon} e^{-t}t^{s-1}\ dt$. These are analytic as functions of s and $F_\epsilon(s) \xrightarrow{\epsilon\to 0} \Gamma(s)$.

Lemma 3.

$$\Gamma(s+1) = s\Gamma(s), \tag{6}$$

so $\Gamma(n+1) = n!$, where $n \in \mathbb{N}$.

Proof. Follows from Integration by Parts, where $u = t^{s-1}, dv = e^{-t}\ dt, v = -e^{-t}, du = (s-1)st^{s-2}\ dt$. □

We can rearrange this as follows:

$$\begin{aligned} \Gamma(s+1) &= s\Gamma(s) \\ \Rightarrow \Gamma(s) &= \frac{\Gamma(s+1)}{\Gamma(s)}, \end{aligned}$$

where $\mathrm{Re}(s+1) > 0$, i.e. $\mathrm{Re}(s) > -1$ and $s \neq 0$.

Repeat as necessary. Then, we have extended $\Gamma(s)$ to all of $\mathbb{C}$ meromorphically to $\mathbb{C}$ (meromorphic since poles at $s = 0, -1, -2, ...$, etc.)

Theorem 4.

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \tag{7}$$

$\Gamma(s)$ has poles at $0, -1, -2, ...$ and $\Gamma(1-s)$ has poles at $1, 2, 3....$ Hence LHS has a pole at all integers.

We need $0 < \mathrm{Re}(s) < 1$. (critical strip - More on this, *later*).

Proof. Contour Integration. □

1.6.2 Gamma And Zeta Together

Step 2. Now we relate $\zeta(s)$ to $\Gamma(s)$.

Observe $\frac{1}{e^t - 1} = \frac{e^{-t}}{1-e^{-t}} = \sum_{n=1}^\infty e^{-nt}$ by geometric series.

Now consider $\int_0^\infty \frac{t^{s-1}}{e^t - 1}\ dt$.

$$\begin{aligned} \int_0^\infty \frac{t^{s-1}}{e^t-1}\ dt &= \int_0^\infty t^{s-1} \sum_{n=1}^\infty e^{-nt}\ dt \\ &= \sum_{n=1}^\infty \int_0^\infty t^{s-1}e^{-nt}\ dt \qquad \text{(via Fubini)} \end{aligned}$$

U-sub Let $u = nt \Rightarrow t = \dfrac{u}{n}$ and $du = n\ dt \Rightarrow dt = \dfrac{du}{n}$. Then, we have

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} \frac{1}{n} \int_0^{\infty} u^{s-1} e^{-u}\ du \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{s}} \int_0^{\infty} u^{s-1} e^{-u}\ du \\
&= \zeta(s)\Gamma(s)
\end{aligned}$$

Step 3. Now we move on to the next portion of our analysis.

$$\begin{aligned}
\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\
&= \frac{1}{1^s} + \frac{1}{2^s} + ... \\
&= (\frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} + ...) + \frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} + ... \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} + \frac{2}{2^s}(1 + \frac{1}{2^s} + \frac{1}{3^s} + ... \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} + 2^{1-s}\zeta(s) \\
\Rightarrow \zeta(s) &= \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}
\end{aligned}$$

By the alternating series test, we only need $\dfrac{1}{n^s} \to 0$, i.e. this converges for $\mathrm{Re}(s) > 0$.

Now we can talk about $\zeta(s)$ in the critical strip.

Recall $\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1}\ dt$.

Notation Let $\omega(x) = \sum_{n=1}^{\infty} e^{-n\pi^2 x}$.

COV Let $t = \pi n^2 x \Rightarrow dt = \pi n^2\ dx$. Then, we have

$$\begin{aligned}
\Gamma(s) &= \int_0^{\infty} e^{-\pi n^2 x} (\pi n^2 x)^{s-1}\ \pi n^2\ dx \\
&= \pi^s n^{2s} \int_0^{\infty} e^{-\pi n^2 x} x^{s-1}\ dx \\
\Rightarrow \pi^{-s/2}\Gamma(s/2) n^{-s} &= \int_0^{\infty} x^{s/2-1} e^{-\pi n^2 x}\ dx \\
\Rightarrow \sum_{n=1}^{\infty} \Rightarrow \pi^{-s/2}\Gamma(s/2) n^{-s} &= \sum_{n=1}^{\infty} \int_0^{\infty} x^{s/2-1} e^{-\pi n^2 x}\ dx \\
\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \int_0^{\infty} x^{s/2-1} \omega(x)\ dx
\end{aligned}$$

Identities

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n\pi^2 x} \tag{8}$$

$$\theta(x^{-1} = x^{1/2}\theta(x) \tag{9}$$

$$\omega(x^{-1}) = \frac{-1}{2} + \frac{1}{2}x^{1/2} + x^{1/2}\omega(x) \tag{10}$$

Continuing our computation and using the above identities, we get

$$\begin{aligned}
\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \int_0^1 x^{s/2-1}\omega(x)\ dx + \int_1^\infty x^{s/2-1}\omega(x)\ dx \qquad (\text{Substitute } x \mapsto \frac{1}{x}) \\
&= \int_1^\infty x^{-s/2-1}\omega(1/x)\ dx + \int_1^\infty x^{s/2-1}\omega(x)\ dx \\
&= \int_1^\infty x^{-s/2-1}(\frac{-1}{2} + \frac{1}{2}x^{1/2} + x^{1/2}\omega(x)) + x^{s/2-1}\omega(x)\ dx \\
&= \frac{-1}{s(1-s} + \int_1^\infty \omega(x)(x^{s/2-1} + x^{\frac{1-s}{2}-1})\ dx
\end{aligned}$$

Let $\zeta(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then, $\zeta(s) = \zeta(1-s)$.

This is the **completed zeta function**.

1.7 Functional Equation

1.8 Riemann Hypothesis