Navier Stokes and the Millennium Problem

Based on the paper of Charles L. Fefferman

Nitesh Mathur
Dr. Lihe Wang

May 10, 2021

Table of Contents

Why Do We Care?

Why Do We Care?

Why Do We Care?

Why Do We Care?

Fluids

Fluids

Fluids

Fluids

Navier Stokes Equations - Overview

Navier Stokes Equations - Overview

- Basically, we are surrounded by fluids

Navier Stokes Equations - Overview

- Basically, we are surrounded by fluids
- Describe and predict fluid flows

Navier Stokes Equations - Overview

- Basically, we are surrounded by fluids
- Describe and predict fluid flows
- Conservation of momemtum and conservation of mass for Newtonian fluids

Navier Stokes Equations - Overview

- Basically, we are surrounded by fluids
- Describe and predict fluid flows
- Conservation of momemtum and conservation of mass for Newtonian fluids
- Newton's 2nd Law

Navier Stokes Equations - Overview

- Basically, we are surrounded by fluids
- Describe and predict fluid flows
- Conservation of momemtum and conservation of mass for Newtonian fluids
- Newton's 2nd Law
- Viscosity taken into account (as opposted to Euler's equations for inviscid flow)

Navier Stokes Equations - Overview

- Basically, we are surrounded by fluids
- Describe and predict fluid flows
- Conservation of momemtum and conservation of mass for Newtonian fluids
- Newton's 2nd Law
- Viscosity taken into account (as opposted to Euler's equations for inviscid flow)
- Air currents, ocean currents, water flow, video games, etc.

The Equation

The Equation

- $$\begin{aligned} \nabla \cdot u &= 0 \\ \rho \frac{du}{dt} &= -\nabla p + \mu \nabla^2 u + \mathbf{F} \end{aligned} \tag{1}$$

The Equation

- $$\begin{aligned} \nabla \cdot u &= 0 \\ \rho \frac{du}{dt} &= -\nabla p + \mu \nabla^2 u + \mathbf{F} \end{aligned} \tag{1}$$
- Newtonian (shear vs viscosity), incompressible, isothermal (no loss or gain of heat)

Let us break it down further - I

Let us break it down further - I

- u is the velocity vector field and first equation states that **mass is conserved** within the fluid.

Let us break it down further - I

- u is the velocity vector field and first equation states that **mass is conserved** within the fluid.
- Divergence of vector field explains how little or how much a point acts as a source for a fluid.

Part II - Derivation

Part II - Derivation

- $F = ma$

Part II - Derivation

- $F = ma$
- Consider ρ be the density and $\rho = m/v$.

Part II - Derivation

- $F = ma$
- Consider ρ be the density and $\rho = m/v$.
- Since u is the velocity vector field, $\frac{du}{dt}$ represents the acceleration.

Part II - Derivation

- $F = ma$
- Consider ρ be the density and $\rho = m/v$.
- Since u is the velocity vector field, $\dfrac{du}{dt}$ represents the acceleration.
- Now, we can substitute to get $\rho \dfrac{du}{dt} = \sum F$.

What forces are acting on it?

What forces are acting on it?

- Internal vs External forces

What forces are acting on it?

- Internal vs External forces
- Internal: (1) Pressure (high to low pressure) represented by $-\nabla p$.

What forces are acting on it?

- Internal vs External forces
- Internal: (1) Pressure (high to low pressure) represented by $-\nabla p$.
- Viscosity (friction) $\mu \nabla^2 u$.

What forces are acting on it?

- Internal vs External forces
- Internal: (1) Pressure (high to low pressure) represented by $-\nabla p$.
- Viscosity (friction) $\mu\nabla^2 u$.
- F - external force (If gravity is the only external force, then $F = \rho g$.

Table of Contents

The Official Millennium Problem - Charles Fefferman

The Official Millennium Problem - Charles Fefferman

- Given a position $x \in \mathbb{R}^n$, equations solves for unknown velocity vector $u(x, t) = (u_i(x, t))_{1 \leq i \leq n} \in \mathbb{R}^n$. and pressure $p(x, t) \in \mathbb{R}$.

The Official Millennium Problem - Charles Fefferman

- Given a position $x \in \mathbb{R}^n$, equations solves for unknown velocity vector $u(x,t) = (u_i(x,t))_{1 \leq i \leq n} \in \mathbb{R}^n$. and pressure $p(x,t) \in \mathbb{R}$.
- Restrict to incompressible fluids filling all of $\mathbb{R}^n$.

- Given a position $x \in \mathbb{R}^n$, equations solves for unknown velocity vector $u(x,t) = (u_i(x,t))_{1 \le i \le n} \in \mathbb{R}^n$. and pressure $p(x,t) \in \mathbb{R}$.
- Restrict to incompressible fluids filling all of $\mathbb{R}^n$.
- $$\frac{\partial}{\partial t} u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \quad (x \in \mathbb{R}^n, t \ge 0)$$
  $$\mathrm{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \quad (x \in \mathbb{R}^n, t \ge 0) \tag{2}$$

The Official Millennium Problem - Charles Fefferman

- Given a position $x \in \mathbb{R}^n$, equations solves for unknown velocity vector $u(x,t) = (u_i(x,t))_{1 \le i \le n} \in \mathbb{R}^n$. and pressure $p(x,t) \in \mathbb{R}$.
- Restrict to incompressible fluids filling all of $\mathbb{R}^n$.
- $$\frac{\partial}{\partial t} u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \quad (x \in \mathbb{R}^n, t \ge 0)$$
  $$\mathrm{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \quad (x \in \mathbb{R}^n, t \ge 0) \tag{2}$$
- With initial conditions $u(x,0) = u^o(x), \quad (x \in \mathbb{R}^n)$

Constraints

Constraints

- We want $u(x,t)$ does not grow large as $|x| \to \infty$ for physical relevance. Hence, we restrict:

Constraints

- We want $u(x,t)$ does not grow large as $|x| \to \infty$ for physical relevance. Hence, we restrict:
- $$|\partial_x^\alpha u^o(x)| \leq C_\alpha K(1+|x|)^{-K} \qquad \text{on } \mathbb{R}^n, \text{ for any } \alpha \text{ and } K \quad (3)$$

Constraints

- We want $u(x,t)$ does not grow large as $|x| \to \infty$ for physical relevance. Hence, we restrict:
- $$|\partial_x^\alpha u^o(x)| \leq C_\alpha K(1+|x|)^{-K} \quad \text{on } \mathbb{R}^n, \text{ for any } \alpha \text{ and } K \quad (3)$$
- $$|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K} \quad \text{on } \mathbb{R}^n \times [0,\infty) \text{ for any } \alpha, m, K. \quad (4)$$

Constraints

- We want $u(x,t)$ does not grow large as $|x| \to \infty$ for physical relevance. Hence, we restrict:
- $$|\partial_x^\alpha u^o(x)| \leq C_\alpha K(1+|x|)^{-K} \quad \text{on } \mathbb{R}^n, \text{ for any } \alpha \text{ and } K \tag{3}$$
- $$|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K} \quad \text{on } \mathbb{R}^n \times [0,\infty) \text{ for any } \alpha, m, K. \tag{4}$$
- Only accept solution if it satisfies

$$p, u \in C^\infty(\mathbb{R}^n \times [0,\infty)) \tag{5}$$

Periodic Solutions

Periodic Solutions

- $$\int_{\mathbb{R}^n} |u(x,t)|^2 \, dx < C \quad \text{for all } t \geq 0 \quad \text{(bounded energy)} \quad (6)$$

Periodic Solutions

- $$\int_{\mathbb{R}^n} |u(x,t)|^2 \, dx < C \quad \text{for all } t \geq 0 \quad \text{(bounded energy)} \quad (6)$$

- Alternatively, to rule out problems at infinity, look at spatially periodic solutions:

$$u^o(x+e_j) = u^o(x), \quad f(x+e_j, t) = f(x,t) \quad \text{for } 1 \leq j \leq n \quad (7)$$

Alternate Solution

Alternate Solution

- In place of (3) and (4), assume u^o is smooth and

$$|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|t|)^{-K} \quad \text{on } \mathbb{R}^3 \times [0,\infty) \text{ for any } \alpha, m, K. \tag{8}$$

Alternate Solution

- In place of (3) and (4), assume u^o is smooth and

$$|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|t|)^{-K} \quad \text{on } \mathbb{R}^3 \times [0,\infty) \text{ for any } \alpha, m, K. \tag{8}$$

- We then accept solution that are physically relevant if it satisfies:

$$u(x,t) = u(x+e_j,t) \qquad \text{on } \mathbb{R}^3 \times [0,\infty) \text{ for } 1 \leq j \leq n \tag{9}$$

Alternate Solution

- In place of (3) and (4), assume u^o is smooth and

$$|\partial_x^\alpha \partial_t^m f(x,t)| \le C_{\alpha m K}(1+|t|)^{-K} \quad \text{on } \mathbb{R}^3 \times [0,\infty) \text{ for any } \alpha, m, K. \tag{8}$$

- We then accept solution that are physically relevant if it satisfies:

$$u(x,t) = u(x+e_j,t) \qquad \text{on } \mathbb{R}^3 \times [0,\infty) \text{ for } 1 \le j \le n \tag{9}$$

- $$p, u \in C^\infty(\mathbb{R}^n \times [0,\infty)) \tag{10}$$

The Crux

A. **Existence and smoothness of Navier-Stokes solutions on** $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Let $u^o(x)$ be any smooth, divergence-free vector field satisfying (3). Take $f(x, t)$ to be identically zero. Then there exist smooth functions $p(x, t), u_i(x, t)$ on $\mathbb{R}^3 \times [0, \infty)$ that satisfy (2), (5), (6)

The Crux

A. **Existence and smoothness of Navier-Stokes solutions on $\mathbb{R}^3$.** Take $\nu > 0$ and $n = 3$. Let $u^o(x)$ be any smooth, divergence-free vector field satisfying (3). Take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t), u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (2), (5), (6)

B. **Existence and smoothness of Navier-Stokes equations in $\mathbb{R}^3/\mathbb{Z}^3$.** Take $\nu > 0$ and $n = 3$. Let $u^o(x)$ be any smooth, divergence-free vector field satisfying (7) and take $f(x,t)$ be identically zero. Then there exist smooth functions $p(x,t), u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (2), (9), (10).

Continued

C. **Breakdown of Navier-Stokes solutions on** $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^o(x)$ on $\mathbb{R}^3$ and a smooth $f(x, t)$ on $\mathbb{R}^3 \times [0, \infty)$ satisfying (3), (4), for where **there exist no solutions** (p, u) of (2), (5), (6) on $\mathbb{R}^3 \times [0, \infty)$.

Continued

C. **Breakdown of Navier-Stokes solutions on $\mathbb{R}^3$.** Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^o(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ satisfying (3), (4), for where **there exist no solutions** (p,u) of (2), (5), (6) on $\mathbb{R}^3 \times [0,\infty)$.

D. **Breakdown of Navier-Stokes Solutions on $\mathbb{R}^3/\mathbb{Z}^3$.** Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^o(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ satisfying (7), (8) for which there exist **no solutions** (p,u) of (2), (9), (10) on $\mathbb{R}^3 \times [0,\infty)$.

Table of Contents

Results so far

Results so far

- In 2D, solutions known for (A), (B) via **Ladyzhenskaya**

Results so far

- In 2D, solutions known for (A), (B) via **Ladyzhenskaya**
- In 3D, (A), (B) hold provided u^o satisfies smallness condition

Results so far

- In 2D, solutions known for (A), (B) via **Ladyzhenskaya**
- In 3D, (A), (B) hold provided u^o satisfies smallness condition
- For u^o not small, $(A), (B)$ hold if $[0, \infty)$ replaced by finite interval $[0, T)$. ("blowup time")

Issues

Issues

- If there is a solution with finite blowup time T, then velocity becomes **unbounded** near the blowup time.

Issues

- If there is a solution with finite blowup time T, then velocity becomes **unbounded** near the blowup time.
- For Euler equation ($\nu = 0$), Beale-Kato-Majda condition is satisfied.

Weak Solutions

Weak Solutions

- Leray (1934) - showed that N-S in 3D have weak solutions with suitable growth properties.

Weak Solutions

- Leray (1934) - showed that N-S in 3D have weak solutions with suitable growth properties.
- **Uniqueness not known.** (for Euler, uniqueness is false)

Weak Solutions

- Leray (1934) - showed that N-S in 3D have weak solutions with suitable growth properties.
- **Uniqueness not known.** (for Euler, uniqueness is false)
- Scheffer and Schnirelman exhibited weak solutions of Euler on $\mathbb{R}^2 \times \mathbb{R}$ with compact support.

Weak Solutions

- Leray (1934) - showed that N-S in 3D have weak solutions with suitable growth properties.
- **Uniqueness not known.** (for Euler, uniqueness is false)
- Scheffer and Schnirelman exhibited weak solutions of Euler on $\mathbb{R}^2 \times \mathbb{R}$ with compact support.
- Caffarelli-Kohn-Nirenberg improved Scheffer's results

Latest Work

Latest Work

- Escauriaza-Seregin-Sverak blowup criterion (2003)

Latest Work

- Escauriaza-Seregin-Sverak blowup criterion (2003)
- Seregin (2012), Phuc (2015), Gallagher-Koch-Planchon (2016), Albritton (2016) blowup criteria

Latest Work

- Escauriaza-Seregin-Sverak blowup criterion (2003)
- Seregin (2012), Phuc (2015), Gallagher-Koch-Planchon (2016), Albritton (2016) blowup criteria
- Terrance Tao (2016) - finite time blowup result of an **averaged** version of the 3D N-S

Latest Work

- Escauriaza-Seregin-Sverak blowup criterion (2003)
- Seregin (2012), Phuc (2015), Gallagher-Koch-Planchon (2016), Albritton (2016) blowup criteria
- Terrance Tao (2016) - finite time blowup result of an **averaged** version of the 3D N-S
- Buckmaster and Vicol (2019) - Weak solutions of 3D N-S equations not unique in the class of weak solutions with finite kinetic energy.

Latest Work

- Escauriaza-Seregin-Sverak blowup criterion (2003)
- Seregin (2012), Phuc (2015), Gallagher-Koch-Planchon (2016), Albritton (2016) blowup criteria
- Terrance Tao (2016) - finite time blowup result of an **averaged** version of the 3D N-S
- Buckmaster and Vicol (2019) - Weak solutions of 3D N-S equations not unique in the class of weak solutions with finite kinetic energy.
- Numerical work

The End

- Thank You!
- Questions?