Navier Stokes

Nonuniqueness of Weak Solutions to the Navier-Stokes Equation

Nitesh Mathur
Dr. Lihe Wang

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Introduction

1. “The Navier Stokes is a simple mathematical model to describe the way an incompressible visous fluid flows.”
2. “Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet.”
3. Explanation/Prediction of ‘Breeze and turbulence’ may be found by understanding the Navier-Stokes equation
4. Acceleration, Pressure, Viscosity (Convection Vs Diffusion Terms)

The Millennium Problem

1. $$\begin{cases} \frac{\partial}{\partial t} u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial u_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \quad (x \in \mathbb{R}^n, t \geq 0), \\ \text{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \quad (x \in \mathbb{R}^n, t \geq 0) \end{cases} \tag{1}$$

2. **Problem:** Does a **smooth**, physically reasonable solution exist for the Navier Stokes equation?
3. Proofs for Existence, Smoothness, and Breakdown on $\mathbb{R}^3$ or $\mathbb{R}^3/\mathbb{Z}^3 \Rightarrow \$1,000,000$.

Brief Historical Progress

1. **Analysis Idea:** Prove existence and regularity of solution of PDE by constructing a weak solution (then showing it is smooth)
2. (Leray 1934): Navier Stokes in 3-Space dimension always has a weak solution (although uniqueness is not known)
3. (Fujita and Kato 1964): Under certain conditions, there does exist solutions to the N-S
4. (Caffarelli, Kohn, and Nirenbeg 1982): Partial Regularity of Suitable Weak Solutions (based on Scheffer's work)
5. (Others) Olga Ladyzhenskaya, Serrin, and more

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Introduction

1. **Claim:** Weak solutions of the 3D Navier-Stokes equations are not unique in class of weak solutions with finite kinetic energy.
2. $$\begin{cases} \partial_t v + \text{div}(v \otimes v) + \nabla p - \nu \Delta v = 0 \\ \text{div} v = 0 \end{cases} \tag{2}$$
3. We say $v \in C^0(\mathbb{R}; L^2(\mathbb{T}^3))$ is a weak solution of N-S if for any $t \in \mathbb{R}$ the vector filed $v(\cdot, t)$ is weakly divergence free, has zero mean, and:

$$\int_{\mathbb{R}} \int_{\mathbb{T}^3} v \cdot (\partial_t \phi + (v \cdot \nabla)_\phi + \nu \Delta \phi) \; dx \; dt = 0 \tag{3}$$

Main Results

Theorem (Theorem 1.2)

(Nonuniqueness of weak solutions)
There exists $\beta > 0$*, such that for any nonnegative smooth function* $e(t) : [0, T] \to \mathbb{R}_{\geq 0}$*, there exists* $v \in C_t^0([0, T]; H_x^\beta(\mathbb{T}^3))$ *a weak solution of the Navier-Stokes equations, such that* $\int_{\mathbb{T}^3} |v(x, t)|^2 \, dx = e(t)$ *for all* $t \in [0, T]$*. Moreover, the associated vorticity* $\nabla \times v$ *lies in* $C_t^0([0, T]); L_x^1(\mathbb{T}^3))$*.*

Theorem 1.3

Theorem (Theorem 1.3)

(Dissipative Euler solutions arise in the vanishing viscosity limit)
For $\bar{\beta} > 0$ *let* $u \in C^{\tilde{\beta}}_{t,x}(\mathbb{T}^3 \times [-2T, 2T])$ *be a zero-mean weak solution of the Euler equations. Then there exists* $\beta > 0$, *a sequence* $\nu_n \to 0$, *and a uniformly bounded sequence* $v^{(\nu_n)} \in C^0_t([0,T]; H^\beta_x(\mathbb{T}^3))$ *of weak solutions to the Navier-Stokes equations, with* $v^{(\nu_n} \to u$ *strongly in* $C^0_t([0,T]; L^2_x(\mathbb{T}^3))$.

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Parameters

1. Consider the Navier-Stokes-Reynolds (NSR) system:

$$\begin{cases} \partial_t v_q + \mathrm{div}(v_1 \otimes v_q) + \nabla p_q - \nu \Delta v_q = \mathrm{div}\ \mathring{R}_q \\ \mathrm{div}\ v_q = 0 \end{cases} \tag{4}$$

2. Assume $\mathring{R}_q$ is a trace-free symmetric matrix.
3. For every $q \geq 0$, construct a solution $(v_q, p_q, \mathring{R}_q)$ to the NSR system.
4. $\mathring{R}_q$ - Reynold stress, v_q - Approximate solution, λ_q - frequency parameter, δ_q - amplitude parameter

Parameters/Questions

1. Fix sufficiently large constant $b \in 16\mathbb{N}$.
2. Fix ‘regularity parameter’ $\beta > 0$ such that $\beta b^2 \leq 4$ and $\beta b \leq \frac{1}{40}$.
3. Fix constant $M_e = ||e||_{C_t^1}$.
4. Remark: Sufficient to take $b = 2^9$ and $\beta = 2^{-16}$.
5. $\lambda_q = a^{(b^q)}, \delta_q = \lambda_1^{3\beta} \lambda_q^{-2\beta}$ for some integer $a >> 1$.

Inductive Estimates

1. Assume the following estimates on (4).
2. $||v_q||_{C^1_{x,t}} \leq \lambda_q^4, ||\mathring{R}_q||_{L^1} \leq \lambda_q^{-\epsilon R}\delta_{q+1}, ||\mathring{R}_q||_{C^1_{x,t}} \leq \lambda_q^{10}$.
3. Assume $0 \leq e(t) - \int_{\mathbb{T}^3} |v_q|^2 \, dx \leq \delta_{q+1}$ and
4. $e(t) - \int_{\mathbb{T}^3} |v_q(x,t)|^2 \, dx \leq \dfrac{\delta_{q+1}}{100} \Rightarrow v_q(\cdot, t) \equiv 0$ and $\mathring{R}_q(\cdot, t) \equiv 0$ for all $t \in [0, T]$.

Proposition

Theorem (Proposition 2.1)

There exists a universal constant $M > 0$, a sufficiently small parameter $\epsilon_R = \epsilon_R(b, \beta) > 0$ and a sufficiently large parameter $a_0 = a_0(b, \beta, \epsilon_R, M, M_e) > 0$ such that for any integer $a \geq a_0$, which is a multiple of the N_Γ. The following holds:
Let $(v_q, p_q, \mathring{R}_q)$ be a triple solving the NSR system in $\mathbb{T}^3 \times [0, T]$ satisfing the inductive estimates. Then there exists a second triple $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ solving NSR and satisfying the estimates with q replaced by $q + 1$. In addition,

$$||v_{q+1} - v_q||_{L^2} \leq M\delta_{q+1}^{1/2}.$$

Proof of Theorem 1.2 - Trivial Case

1. Denote H^β the L^2-based Sobolev space with regularly index β. Then, $C_t^0 H_x^\beta \subset C_t^0 L_x^2$.
2. For $q \equiv 0$, NSR is satisfies with $\mathring{R}_0 = 0$ and inductive assumptions hold.
3. For a sufficiently large, we may ensure that:

$$|e(t) \leq ||e||_{C_t^1} = M_e \leq \frac{\lambda_1^\beta}{100} = \frac{\delta_1}{100}.$$

Proof of Theorem 1.2 - Induction

1. For $q \geq 1$, we inductively apply Proposition 2.1

$$\sum_{q=0}^{\infty} ||v_{q+1} - v_q||_{H^{\beta'}} \lesssim \sum_{q=0}^{\infty} ||v_q + 1 - v_q||_{L^2}^{1-\beta'} (||v_{q+1}||_{C^1} + ||v_q||_{C^1})^{\beta'}$$

$$\lesssim \sum_{q=0}^{\infty} M^{1-\beta'} \lambda_1^{3\beta \frac{1-\beta'}{2}} \lambda_{q+1}^{-\beta \dfrac{1-\beta'}{2}} \lambda_{q+1}^{4\beta'}$$

$$\lesssim M^{1-\beta'} \lambda_1^{3\beta \frac{1-\beta'}{2}}$$

for $\beta' < \beta/(8+\beta)$.

2. Hence, $\{v_q\}_{q \geq 0}$ is uniformly bounded $C_t^0 H_x \beta'$ for such β'.

Proof of Theorem 1.2 - Part 1

1. From the inductive estimates, the uniform boundedness in $C_t^0 L_x^2$, The embedding $W_x^{2,1} \subset L_x^2$, we obtain that:
2. $$\begin{aligned} ||\partial_t v_q||_{H^{-3}} &\lesssim ||\mathbb{P}\text{div}(v_q \otimes v_q) - \nu \Delta v_q - \mathbb{P}\text{div}\mathring{R}_q||_{H^{-3}} \\ &\lesssim ||v_q \otimes v_q||_{L^1} + ||v_q||_{L^2} + ||\mathring{R}_q||_{L^1} \\ &\lesssim M^2 \lambda_1^{3\beta} \end{aligned}$$
3. Note: $\mathbb{P}$ is the Leray Projection (projection on the divergence-free vector fields) and is defined as:
4. $\mathbb{P}(u) = u - \nabla \Delta^{-1}(\nabla \cdot u)$, for vector fields u.

End of Proof of Theorem 1.2

1. Hence, $\{v_q\}_{q\geq 0}$ is uniformly bounded in $C_t^1 H_x^{-3}$.
2. Then, for any $0 < \beta'' < \beta', \sum_{q\geq 0}(v_q + 1 - v_q) =: v$ converges in $C_t^0 H_x^{\beta''}$.
3. Since $||\mathring{R}_q||_{L^1} \to 0$ as $q \to \infty, v$ is a $C_t^0 H_x^{\beta''}$ **weak solution of the Navier-Stokes Equation**.
4. Finally, kinetic energy of $v(\cdot, t)$ is given by $e(t)$ for all $t \in [0, T]$.

Proof of Theorem 1.3

1. Consider the incompressible Euler equations:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = 0 \\ \text{div } v = 0 \end{cases}$$

2. Fix $\bar{\beta} > 0$ and a weak solution $u \in C^{\bar{\beta}}_{t,x}$ to the Euler equation on $[-2T, 2T]$.
3. Existence guaranteed for $\bar{\beta} < 1/3$ and for $\bar{\beta} > 1$.
4. Let $M_u = ||u||_{C^{\bar{\beta}}}$ and pick $n \geq 1$.
5. Choose all parameters as Proposition (except $a \geq a_0$ and $0 < \beta' < \min(\bar{\beta}/2, \beta/(8 + \beta))$.

Theorem 1.3 Proof Setup

1. We may ensure that:

$$\sum_{q=n}^{\infty} M^{1-\beta'} \lambda_1^{3\beta\frac{1-\beta'}{2}} \lambda_{q+1}^{-\beta\frac{1-\beta'}{2}} \lambda_{q+1}^{4\beta'} \leq \frac{1}{2Cn}$$

2. Let $\{\phi_\epsilon\}_{\epsilon>0}$ be a family of standard compact support mollifiers on $\mathbb{R}^3$ and $\{\varphi_\epsilon\}_{\epsilon>0}$ be family of standard compact support mollifiers on $\mathbb{R}$.
3. Define $v_n = (u *_x \phi_{\lambda_n^{-1}} *_t \varphi_{\lambda_n^{-1}})$ to be mollification of u with time scale λ_n^{-1}, restricted to $[0, T]$.
4. Define the energy function $e(t) = \int_{\mathcal{T}^3} |v_n(x,t)|^2 \, dx + \frac{\delta_n}{2}$.

Proof

1. Since u is a solution to the Euler equations, there exists a **mean-free** p_n such that

$$\partial_t v_n + \operatorname{div}(v_n \otimes v_n) + \nabla p_n - \lambda_n^{-2} \Delta v_n = \operatorname{div}(\mathring{R}_n)$$

2. Here $\mathring{R}_n$ is the traceless symmetric part of the tensor:

$$(v_n \otimes v_n) - ((u \otimes u) *_x \phi_{\lambda_n^{-1}} *_t \varphi_{\lambda_n^{-1}} - \lambda_n^{-2} \nabla v_n.$$

3. By a commutator estimate, we have:

$$||\mathring{R}_n||_{L^1} \lesssim ||\mathring{R}_n||_{C^0} \lesssim \lambda_n^{-1} M_u + \lambda_n^{-2\bar{\beta}} M_u^2$$

4. Similarly, we have:
$||\mathring{R}_n||_{C^1_{t,x}} \lesssim M_u + \lambda_n^{1-2\bar{\beta}} M_u^2, \quad ||v_n||_{C^1_{t,x}} \lesssim \lambda_n^{1-\bar{\beta}} M_u.$

Conclude Proof of Theorem 1.3

1. Set $\nu := \nu_n := \lambda_n^{-1}$ with a sufficiently large, depending on M_u and $\bar{\beta}$, we may ensure $(v_n, \mathring{R}_n)$ obey inductive assumptions for $q = n$.
2. Choose a such that $\lambda_n^{\bar{\beta}-\beta'} Mu \leq \dfrac{1}{2n|\mathbb{T}^3|^{1/2}}$.
3. By Proposition 2.1, we obtain a weak solution, $u^{(\nu_n)}$ of the N-S equations with desired regularity such that:

$$||v^{(\nu_n)} - u||_{H^{\beta'}} \leq ||v^{(\nu_n)} - v_n||_{H^{\beta'}} + |\mathbb{T}^3|^{1/2} ||u - v_n||_{C^{\beta'}} \leq \frac{1}{n}$$

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Intermittent Beltrami Flows - Introduction

- Use building blocks for the convex integration scheme which are ‘intermittent.’
- Building blocks are spatially inhomogeneous and have different scaling in different L^p norms.
- Built by adding eigenfunctions of curl in a certain geometric manner.

Proposition 2.1

- Proposition 2.1 implies Theorems 1.2 and 1.3.
- New Idea: Construct perturbation $v_{q+1} - v_q$ as sum of the terms of the form
$$a_{(\xi)} \mathbb{W}_{(\xi)},$$
where $\mathbb{W}_{(\xi)}$ is an intermittent Beltrami wave.
- Fundamentally different from usual Beltrami flows since their L^1 norm is much smaller than their L^2 norm.
- 'Gain comes from the fact that Reynolds stress has to be estimated in L^1 rather than L^2, and that $v\Delta v$ is linear in v.'

Beltrami Waves - Proposition 3.1

- Given $\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3$, let $A_\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3$ such that

$$A_\xi \cdot \xi = 0, |A_\xi| = 1, A_{-\xi} = A_\xi$$

- Let $B_\xi = \frac{1}{\sqrt{2}}(A_\xi + i\xi \times A_\xi)$.
- Let Λ be a given finite subset of $\mathbb{S}^2 \cap \mathbb{Q}^3$ such that $-\Lambda = \Lambda$, and let $\lambda \in \mathbb{Z}$ be such that $\lambda\Lambda \subset \mathbb{Z}^3$.

Proposition 3.1 - Continued

- Then, for any choice of $a_\xi \in \mathbb{C}$ with $\bar{a}_\xi = a_{-\xi}$ the vector field:

$$W(x) = \sum_{\xi \in \Lambda} a_\xi B_\xi e^{i\lambda \xi \cdot x}$$

  is real-valued, divergence-free and satisfies $\operatorname{div}(W \otimes W) = \nabla \frac{|W|^2}{2}$.

- Also, since $B_\xi \otimes B_{-\xi} + B_{-\xi} \otimes B_\xi = \mathrm{Id} - \xi \otimes \xi$, we have

$$\fint_{\mathbb{T}} W \otimes W \, dx = \frac{1}{2} \sum_{\xi \in \Lambda} |a_\xi|^2 (\mathrm{Id} - \xi \otimes \xi).$$

Proposition 3.2

For every $N \in \mathbb{N}$, we can choose $\epsilon_\gamma > 0$ and $\lambda > 1$ with the following property. Let $B_{\epsilon,\gamma}$ (Id) denote the ball of symmetric 3×3 matrices, centered at Id, of radius ϵ_γ. Then, there exists pairwise disjoint subsets: $\Lambda_\alpha \subset \mathbb{S}^2 \cap \mathbb{Q}^3 \quad \alpha \in \{1, ..., N\}$, with $\lambda\Lambda_\alpha \in \mathbb{Z}^3$, and smooth positive functions :

$$\gamma_\epsilon^{(\alpha)} \in C^\infty(B_\epsilon(\mathrm{Id})) \quad \alpha \in \{1, ..., N\}, \xi \in \Lambda_\alpha,$$

with derivatives that are bounded independently of λ, such that

(a) $\xi \in \Lambda_\alpha$ implies $-\xi \in \Lambda_\alpha$ and $\gamma_\xi^{(\alpha)} = \gamma_{-\xi}^{(\alpha)}$;

(b) for each $R \in B_{\epsilon_\gamma}$ (Id), we have the identity:

$$R = \frac{1}{2} \sum_{\xi \in \Lambda_\alpha} (\gamma_\epsilon^{(\alpha)}(R))^2 (\mathrm{Id} - \xi \otimes \xi).$$

Main Idea

- Modify Beltrami waves by adding oscillations to mimic the structure of the kernels D_r.
- Then we can construct approximate Beltrami waves with L^p norm for p close to 1.

Intro to Intermittent Beltrami Waves

- Recall the Dirichlet Kernel is $D_n(x) = \sum_{\xi=-n}^{n} e^{ix\xi} = \dfrac{\sin(n+1/2)x}{\sin(x/2)}$.
- For $p > 1$, it obeys the estimate $||D_n||_{L^p} \sim n^{1-1/p}$.
- Consider a 3D integer cube $\Omega_r = \{\xi = (j,k,l) : j,k,l \in \{-r,...,r\}\}$
- Normalizing to unit size in L^2, we obtain a kernel: $D_r(x) := \dfrac{1}{(2r+1)^{3/2}} \sum_{\xi \in \Omega_r} e^{i\xi \cdot x} = \dfrac{1}{(2r+1)^{3/2}} \sum_{j,k,l \in \{-r,...,r\}} e^{i(jx_1+kx_2+lx_3)}$, such that for $1 < p \leq \infty$, we have:
- $||D_r||_{L^2}^2 = (2\pi)^3, \quad ||D_r||_{L^p} \lesssim r^{3/2-3/p}$

Constants

- Large Parameter r will parameterize the number of frequencies **along edges of the cube** Ω_r.
- We have small parameter σ such that $\lambda\sigma \in \mathbb{N}$ parametrizes the **spacing between frequencies.**
- $\mu \in (\lambda, \lambda^2)$ **measures the amount of temporal oscillation in our building blocks.**
- Universal constants $c_\Lambda \in (0,1)$, $N_\Lambda \geq 1$.
- Assume $\sigma r \leq c_\Lambda/(10 N_\Lambda)$

Brief Computations

- For $\xi \in \Lambda_\alpha$, vectors $\{\xi, A_\xi, \xi \times A_\xi\}$ form an orthonormal basis of $\mathbb{R}^3$.
- Define a **rescaled periodic Dirichlet kernel** by

$$\begin{aligned}\eta_{(\xi)}(x,t) &= \eta_{\xi,\lambda,\sigma,r,\mu}(x,t)\\ &= D_r(\lambda\sigma N_\Lambda(\xi\cdot x+\mu t), \lambda\sigma N_\Lambda A_\xi \cdot x, \lambda\sigma N_\Lambda(\xi\times A_\xi)\cdot x)\end{aligned}$$

- Identity: $\dfrac{1}{\mu}\partial_t \eta_{(\xi)}(x,t) = \pm(\xi\cdot\nabla)_{\eta(\xi)}(x,t)$

Intermittent Beltrami Wave

- The map $(x_1, x_2, x_3) \mapsto (\lambda\sigma N_\Lambda(\xi \cdot x + \mu t), \lambda\sigma N_\Lambda A_\xi \cdot x, \lambda\sigma N_\Lambda(\xi \times A_\xi) \cdot x)$ maps (e_1, e_2, e_3) to $(\xi, A_\xi, \xi \times A_\xi)$.
- These are all volume preserving transformations on $\mathbb{T}^3$ and by our choice of normalization, we have:

$$\fint \eta_{(\xi)}^2(x,t)\, dx = 1, \qquad ||\eta_{(\xi)}||_{L^p(\mathbb{T}^3)} \lesssim r^{3/2-3/p}$$

for all $1 < p \leq \infty$.

- Let $W_{(\xi)}(x) = W_{\xi,\lambda}(x) = B_\xi e^{i\lambda\xi\cdot x}$ be the Beltrami plane wave at frequency λ.

Continued

- We have curl $W_{(\xi)} = \lambda W_{(\xi)}, \qquad \text{div } W_{(\xi)} = 0.$
- Take λ to be a multiple of N_Λ, so that $W_{(\xi)}$ is $\mathbb{T}^3$-periodic.
- **Finally**, we define the **intermittent Beltrami Wave** $\mathbb{W}_{(\xi)}$ as

$$\begin{aligned}\mathbb{W}_{(\xi)}(x,t) &= \mathbb{W}_{\xi,\lambda,\sigma,r,\mu}(x,t)\\ &= \eta_{\xi,\lambda,\sigma,r,\mu}(x,t)W_{\xi,\lambda}(x)\\ &= \eta_{(\xi)}(x,t)W_{(\xi)}(x)\end{aligned}$$

- Note, $\mathbb{W}_\xi$ is not anymore divergence free, nor is it an eigenfunction of curl.

Proposition 3.4

- Let $\mathbb{W}_{(\xi)}$ be as defined above, and let $\Lambda_\alpha, \xi_\gamma, \gamma_{(\xi)} = \gamma_\xi^{(\alpha)}$ be as in Proposition 3.2.
- If $a_\xi \in \mathbb{C}$ are constants chosen such that $\bar{a}_\xi = a_{-\xi}$, the vector field
$$\sum_\alpha \sum_{\xi \in \Lambda_\alpha} a_\xi \mathbb{W}_{(\xi)}(x)$$
is real valued.
- Moreover, for each $R \in B_{\epsilon_\gamma}$ (Id), we have the identity

$$\sum_{\xi \in \Lambda_\alpha} (\gamma_{(\xi)}(R))^2 \fint_{\mathbb{T}^3} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(-\xi)} \, dxs = \sum_{\xi \in \Lambda_\alpha} (\gamma_{(\xi)}(R))^2 B_\xi \otimes B_{-\xi} = R.$$

Proposition 3.5

- Let $\mathbb{W}_{(\xi)}$ be defined as above. The bound:
- $$||\nabla^N \partial_t^K \mathbb{W}_{(\xi)}||_{L^p} \lesssim \lambda^N (\lambda\sigma\tau\mu)^K r^{3/2-3/p}$$
- $$||\nabla^N \partial_t^K \eta_{(\xi)}||_{L^p} \lesssim (\lambda\sigma\tau)^N (\lambda\sigma\tau\mu)^K r^{3/2-3/p}$$

L^p Decorrelation:

- For $f : \mathbb{T}^3 \to \mathbb{R}$, we have the trivial estimate: $||f\mathbb{W}_{(\xi)}||_{L^1} \lesssim ||f||_{L^2}||\mathbb{W}_{(\xi)}||_{L^2}$.
- This estimate does not take advange of $(2\pi\lambda\sigma)^{-1}$ periodic function $\mathbb{W}_{(\xi)}e^{-i\lambda\xi\cdot x}$.
- **Improved Estimates:** $||f\mathbb{W}_{(\xi)}||_{L^1} \leq ||f||_{L^1}||\mathbb{W}_{(\xi)}||_{L^1}$.

Lemma 3.7

- Fix integers $M, \kappa, \lambda \geq 1$ such that:
- $$\frac{2\pi\sqrt{3}\lambda}{\kappa} \leq \frac{1}{3} \text{ and } \lambda^4 \frac{(2\pi\sqrt{3}\lambda)^M}{\kappa^M} \leq 1.$$
- Let $p \in \{1,2\}$, and let f be a $\mathbb{T}^3$-periodic function such that there exists a constant C_f such that
  $$||D^j f||_{L^p} \leq C_f \lambda^j$$
  for all $1 \leq j \leq M+4$.
- In addition, let g be a $(\mathbb{T}/\lambda)^3$ periodic function. Then we have that
  $$||fg||_{L^p} \lesssim C_f ||g||_{L^p}$$
  holds, where the implicit constant is universal.

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Mollification of v_q

- (In order to avoid loss of derivative) We replace v_q by a mollified velocity field v_ℓ.
- Let $\{\phi_\epsilon\}_{\epsilon>0}$ be a family of mollifiers on $\mathbb{R}^3$ (space); $\{\varphi_\epsilon\}_{\epsilon>0}$ be a family of mollifiers on $\mathbb{R}$ (time)
- $$v_\ell = (v_q)_{*x}\phi_\ell)_{*t}\varphi_\ell$$
- $$\mathring{R}_\ell = (\mathring{R}_q)_{*x}\phi_\ell)_{*t}\varphi_\ell$$
- Then $(v_\ell, \mathring{R}_\ell)$ obey a modified NSR system.

Stress Cutoffs

- Because the Reynold stress $\mathring{R}_\ell$ is not spatially homogeneous, stress cutoff functions are introduced.
- Let $0 \leq \tilde{\chi}_0, \tilde{\chi} \leq 1$ be bump functions adapted to intervals $[0,4]$ and $[1/4,4]$ respectively, such that together they form a **partition of unity**.
- Define $\chi_{(i)}(x,t) = \chi_{i,q+1}(x,t) = \tilde{\chi}_i(< \frac{\mathring{R}_\ell(x,t)}{100\lambda_q^{-\epsilon R}\delta_{q+1}} >)$ for all $i \geq 0$.
- Here, $< A >= (1+|A|^2)^{1/2}$ where $|A|$ denotes the Euclidean norm of matrix A.

Definition of Velocity Increment

- $a_{(\xi)} = a_{(\xi,i,q+1)} := \rho_i^{1/2}\chi_{i,q+1}\gamma_{(\xi)}(Id - \frac{\text{Id} - \mathring{R}_\ell}{\rho_i(t)})$ for $i \geq 1$.
- Here $\rho_i := \lambda_q^{-\epsilon_R}\delta_{q+1}4^{i+c_0}$.
- Later, ρ_0 defined as $\rho_0 = ((\rho^{1/2} *_t \varphi_\ell)^2$.
- The **principal part of** w_{q+1} is defined as:

$$w_{q+1}^{(p)} := \sum_i \sum_{\xi \in \Lambda_{(i)}} a_{(\xi)} \mathbb{W}_{(\xi)}$$

  where sum is over $0 \leq i \leq i_{\max(q)}$.
- In this section, other constants, **incompressibility corrector** and **temporal corrector** also defined.

Lemmas: Estimates of the perturbation

- **Lemma 4.1** For $q \geq 0$, there exists $i_{\max}(q) \geq 0$ determined by

$$i_{max}(q) = \min\{i \geq 0 : 4^{i-2} \geq \lambda_q^{11}\delta_{q+1}^{-1}\}$$

such that

- $\chi_{(i)} \equiv 0 \quad$ for all $i > i_{\max}$.
- Moreover, for all $0 \leq i \leq i_{\max}$, $\rho_i \lesssim r^{i_{\max}} \lesssim \ell^{-1}$
- Moreover, $\sum_{i=0}^{i_{max}} \rho_i^{1/2} 2^{-i} \leq 3\delta_{q+1}^{1/2}$.

More Lemmas

- **Lemma 4.2** Let $0 \leq i \leq i_{\max}$. Then, we have
- $||\chi_{(i)}||_{L^2} \lesssim 2^{-i}$.
- $||\chi_i||_{C^N_{x,t}} \lesssim \lambda_q^{10} \ell^{1-N} \lesssim \ell^{-N}$ for all $N \geq 1$.
- **Lemma 4.3** The following lower and upper bounds hold: $\int_{\mathbb{T}^3} \chi_{(0)}^2 \, dx \geq \frac{|\mathbb{T}^3|}{2}$,
- $\sum_{i \geq 1} \rho_i \int_{\mathbb{T}^3} \chi_{(i)}^2(s,t) \, dx \lesssim \lambda_q^{-\epsilon_R} \delta_{q+1}$.

More Lemmas

- **Lemma 4.4** The bounds
- $$||a_\xi||_{L^2} \lesssim \rho_i^{1/2} 2^{-i} \lesssim \delta_{q+1}^{1/2}.$$
- $$||a_\xi||_{L^\infty} \lesssim \rho_i^{1/2} \lesssim \delta_{q+1}^{1/2} 2^i.$$
- $$||a_\xi||_{C^N_{x,t}} \lesssim \ell^{-N}$$

hold for all $0 \le i \le i_{\max}$ and $N \ge 1$.

Proposition 4.5

- The principal part of the velocity perturbation, incompressibility, and temporal correctors obey the bounds:
- $||w_{q+1}^{(p)}||_{L^2} \leq \frac{M}{2}\delta_{q+1}^{1/2}$
- $||w_{q+1}^{(c)}||_{L^2} + ||w_{q+1}^{(t)}||_{L^2} \lesssim r^{3/2}\ell^{-1}\mu^{-1}\delta_{q+1}^{1/2}$
- $||w_{q+1}^{(p)}||_{W^{1,p}} + ||w_{q+1}^{(c)}||_{W^{1,p}} + ||w_{q+1}^{(t)}||_{W^{1,p}} \lesssim \ell^{-2}\lambda_{q+1}r^{3/2-3/p}$
- $||\partial_t w_{q+1}^{(p)}||_{L^p} + ||\partial_t w_{q+1}^{(c)}||_{L^p} \lesssim \ell^{-2}\lambda_{q+1}\sigma\mu r^{5/2-3/p}$
- $||w_{q+1}^{(p)}||_{C_{x,t}^N} + ||w_{q+1}^{(c)}||_{C_{x,t}^N} + ||w_{q+1}^{(t)}||_{C_{x,t}^N} \leq \frac{1}{2}\lambda_{q+1}^{(3+5N)/2}$

Corollary 4.6

For $N \in \{0,1,2,3\}$ and $p > 1$, we have:

- $||w_{q+1}||_{L^2} \leq \frac{3M}{4}\delta_{q+1}^{1/2}$
- $||v_{q+1} - v_q||_{L^2} \leq M\delta_{q+1}^{1/2}$
- $||w_{q+1}||_{W^{1,p}} \lesssim \ell^{-2}\lambda_{q+1}r^{3/2-3/p}$
- $||w_{q+1}||_{C^N_{x,t}} \leq \frac{1}{2}\lambda_{q+1}^{(3+5N)/2}$
- $||v_{q+1}||_{C^N_{x,t}} \leq \lambda_{q+1}^{(3+5N)/2}$.
- Proves Proposition 2.1 claims.

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Proposition 5.1

Theorem

There exists a $p > 1$ *sufficiently close to* 1 *and an* $\epsilon_R > 0$ *sufficiently small, depending on b and* β *(in particular, independent of q) such that there exists a traceless symmetric 2 tensor* $\tilde{R}$ *and a scalar pressure field* $\tilde{p}$ *satisfying*

$$\partial_t v_{q+1} + div(v_{q+1} \otimes v_{q+1}) + \nabla \tilde{p} - \nu \Delta v_{q+1} = div\tilde{R} \tag{5}$$

and the bound $||\tilde{R}||_{L^p} \lesssim \lambda_{q+1}^{-2\epsilon R} \delta_{q+2}$*, where the constant depends on the choice of p and* ϵ_R.

Consequences

- The inductive estimates defined in Theorem 2 Setup hold for a suitably defined Reynold stress $\mathring{R}_{q+1}$.
- Compared to $\tilde{R}$, the stress $\mathring{R}_{q+1}$ constructed below also obeys a satisfactory C^1 estimate.

Corollary 5.2

- There exists a traceless symmetric 2 tensor $\mathring{R}_{q+1}$ and a scalar pressure field p_{q+1} such that:
- $$\partial_t v_{q+1} + \text{div}(v_{q+1} \otimes v_{q+1}) + \nabla \tilde{p} - \nu \Delta v_{q+1} = \text{div} \mathring{R}_{q+1}. \quad (6)$$

  Moreover,
- $$||\mathring{R}_{q+1}||_{L^1} \leq \lambda_{q+1}^{-\epsilon_R} \delta_{q+2}$$
- $$||\mathring{R}_{q+1}||_{C^1_{x,t}} \leq \lambda_{q+1}^{10}.$$

Prior Knowledge

- The proof of the corollary will use the following information:
- The 2-tensor valued elliptic operator $\mathcal{R}$ that has the property that $\mathcal{R}v(x)$ is a symmetric trace-free matrix for each $x \in \mathbb{T}^3$ and $\mathcal{R}$ is a right inverse of the div operator, i.e.,
- $$\text{div}\mathcal{R}v = v - \fint_{\mathbb{T}^3} v(x)\ dx$$

  for any smooth v.
- Classical Calderon-Zygmund bound $|||\nabla|\mathcal{R}||_{L^p \to L^p} \lesssim 1$.
- Schauder estimates $||\mathcal{R}||_{L^p \to L^p} + ||\mathcal{R}||_{C^0 \to C^0} \lesssim 1$ for $p \in (1, \infty)$.

Proof of Corollary 5.2

- Recall $\tilde{R}$ is a traceless symmetric 2 tensor and $\tilde{p}$ is a scalar pressure field.
- Let $\mathring{R}_{q+1} = \mathcal{R}(\mathbb{P}_H \text{div} \tilde{R}), p_{q+1} = \tilde{p} - \Delta^{-1} \text{ div div} \tilde{R}$
- Since $||\mathcal{R} \text{ div}||_{L^p \to L^p} \lesssim 1$, we can directly bound:
- $$||\mathring{R}_{q+1}||_{L^1} \lesssim ||\mathring{R}_{q+1}||_{L^p} \lesssim ||\tilde{R}||_{L^p} \lesssim \lambda_{q+1}^{-2\epsilon_R} \delta_{q+2}$$
- The estimate $||\mathring{R}_{q+1}||_{L^1} \leq \lambda_{q+1}^{-\epsilon_R} \delta_{q+2}$ follows.

Continued

- Now we need to prove $||\mathring{R}_{q+1}||_{C^1_{x,t}} \leq \lambda_{q+1}^{10}$.
- Using (6) and Corollary 4.6, we get:
-

$$\begin{aligned} ||\mathring{R}_{q+1}||_{C^1} &= ||\mathcal{R}\mathbb{P}_H(\operatorname{div}\tilde{R})||_{C^1} \\ &\lesssim ||\partial_t v_{q+1} + \operatorname{div}(v_{q+1} \otimes v_{q+1}) - \nu\Delta v_{q+1}||_{C^1} \\ &\lesssim ||\partial_t v_{q+1}||_{C^1} + ||v_{q+1} \otimes v_{q+1}||_{C^2} + ||v_{q+1}||_{C^3} \\ \lesssim \lambda_{q+1}^9 \end{aligned} \tag{7}$$

by using Schauder estimates $||\mathcal{R}\mathbb{P}_H||_{C^0 \to C^0} \lesssim 1$.

Finally

- $$\begin{aligned}||\partial_t \mathring{R}_{q+1}||_{L^\infty} &\lesssim ||\partial_t(\partial_t v_{q+1} + \text{ div } (v_{q+1} \otimes v_{q+1}) - \nu \Delta v_{q+1}||_{C^0} \\ &\lesssim ||\partial_t^2 v_{q+1}||_{C^0} + ||\partial_t v_{q+1} \otimes v_{q+1}||_{C^1} + ||\partial_t v_{q+1}||_{C^2} \\ &\lesssim \lambda_{q+1}^9\end{aligned} \tag{8}$$

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Lemma 6.1

- For all t and t' satisfying $|t - t'| \leq 2\ell$ and all $i \geq 0$, we have:
- $|e(t') - e(t'') \lesssim \ell^{1/2}$.
- $| \int_{\mathbb{T}^3} |v_q(x,t)|^2 \, dx - \int_{\mathbb{T}^3} |v_q(x,t')|^2 \, dx| \lesssim \ell^{1/2}$.
- $| \int_{\mathbb{T}^3} (\chi_i^2(x,t) - \chi_i^2(x,t')) \, dx| \lesssim \ell^{1/2}$
- $|\rho(t) - \rho(t')| \lesssim \ell^{1/2}$

Lemmas

- **Lemma 6.2** If $\rho_0(t) \neq 0$, then the energy of v_{q+1} satisfies:

$$|e(t) - \int_{\mathbb{T}^3} |v_{q+1}(x,t)|^2 \, dx - \frac{\delta_{q+2}}{2}| \leq \frac{\delta_{q+2}}{4}.$$

- Lemma implies that if $\rho_0(t) \neq 0$, then,

$$e(t)0 \int_{\mathbb{R}^3} |v_q(x,t)|^2 \, dx > \frac{\delta_{q+1}}{100}$$

- **Lemma 6.3** Since $\rho_0(t) = 0$, then $v_{q+1}(\cdot,t) \equiv 0$, $\mathring{R}_{q+1}(\cdot,t) \equiv 0$ and

$$e(t) - \int_{\mathbb{T}} |v_{q+1}(x,t)|^2 \leq \frac{3\delta_{q+2}}{4}.$$

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Main Idea

- Proven that weak solutions of 3D Navier-Stokes equations are **not unique** in the class of weak solutions with finite kinetic energy
- Holder continuous dissipative weak solutions of the 3D Euler equations may be obtained as a strong **vanishing viscosity limit** of a sequence of finite energy weak solutions of the 3D Navier-Stokes equations.
- Use *intermittent Beltrami flows* as a tool

- Paper - cited in at least 48 articles
- Convex Integration Construction in Hydrodynamics, ▸ Link
- Zhang, X. Zhao, G. Stochastic Lagrangian Path for Leray's Solutions of 3D Navier-Stokes Equation (2021) ▸ Link
- Novack, Matthew. Nonuniquess of Weak Solutions to the 3 Dimensional Quasi-Geostrophic Equations (2020).
- Colombo, M., de Rosa, L. , Forcella L. Regularity results for rough solutions of the incompressible Euler equations via interpolation methods (2020).

What Did I learn/Question

- Inequalities and regularity are important.
- Learned about the main issues behind the Navier-Stokes Problem and why this is a complex problem
- Schauder's Estimates good tool to have
- What is Onsager's Conjecture?
- "Threshold regularity for the validity of energy conservation of weak solutions of the incompressible Euler equations is the exponent $1/3$."

The End

- Thank You!
- Questions?