The Factorial Function and Generalization
Based on the Paper of Manjul Bhargava

Nitesh Mathur

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About the Author

Paper Published in 2000
Since worked on Higher Composition Laws, 15 and 290 Theorems, and Average Rank of Elliptic Curves
Fields Medal Recipient in 2014
Doctoral Advisor: Andrew Wiles
Professor at Princeton, Leiden University, and adjunct professor in several others.
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Definition:

\[ n! = \prod_{k=1}^{n} k = n(n-1)(n-2)\ldots(3)(2)(1) \]

Examples: \(5! = (5)(4)(3)(2)(1) = 120\)

The Gamma Function

\[ \Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} \, dt \]

\[ \Gamma(n) = (n-1)! \]

\(\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\)
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The Gamma Function

\( \Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} \, dt \)

\( \Gamma(n) = (n-1)! \)

\( \Gamma(5) = 4! = 24, \Gamma(1/2) = \sqrt{\pi} \)
Factorial Function in Number Theory

Theorem 1
For any nonnegative integers, \( k \) and \( l \), \((k + l)! \) is a multiple of \( k! \cdot l! \).

Theorem 2
Let \( f \) be a primitive polynomial of degree \( k \) and let \( d(Z, f) = \gcd \{ f(a) : a \in \mathbb{Z} \} \). Then, \( d(Z, f) \) divides \( k! \).

Theorem 3
Let \( a_0, a_1, ..., a_n \in \mathbb{Z} \) be any \( n+1 \) integers. Then their product of their pairwise differences \( \prod_{i < j} (a_i - a_j) \) is a multiple of \( 0! \cdot 1! \cdot ... \cdot n! \).

Theorem 4
The number of polynomial functions from \( \mathbb{Z} \) to \( \mathbb{Z}/n\mathbb{Z} \) is given by \( n - 1 \prod_{k=0}^{n} \gcd(n, k!) \).
Theorem 1
For any nonnegative integers, \( k \) and \( l \), \((k + l)!\) is a multiple of \( k!l!\).
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For any nonnegative integers, $k$ and $l$, $(k + l)!$ is a multiple of $k!!$.

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Let \( f \) be a primitive polynomial of degree \( k \) and let \( d(\mathbb{Z}, f) = \gcd\{f(a) : a \in \mathbb{Z}\} \) Then, \( d(\mathbb{Z}, f) \) divides \( k! \).

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Let \( a_0, a_1, \ldots, a_n \in \mathbb{Z} \) be any \( n + 1 \) integers. Then their product of their pairwise differences

\[
\prod_{i<j}(a_i - a_j)
\]

is a multiple of \( 0!1! \ldots n! \).
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For any nonnegative integers, \( k \) and \( l \), \((k + l)!\) is a multiple of \( k!l! \).

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\[ d(\mathbb{Z}, f) = \gcd\{f(a) : a \in \mathbb{Z}\} \] Then, \( d(\mathbb{Z}, f) \) divides \( k! \).

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Let \( a_0, a_1, \ldots, a_n \in \mathbb{Z} \) be any \( n + 1 \) integers. Then their product of their pairwise differences
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\]
is a multiple of \( 0!1! \ldots n! \).

Theorem 4 The number of polynomial functions from \( \mathbb{Z} \) to \( \mathbb{Z}/n\mathbb{Z} \) is given by
\[
\prod_{k=0}^{n-1} \frac{n}{\gcd(n, k!)}
\]
These theorems are true on $\mathbb{Z}$.

Is there a "Generalized Factorial Function" so that for any subset $S$ of $\mathbb{Z}$, the theorems mentioned above still remain true?
p-Ordering

Let $S \subset \mathbb{Z}$ and fix a prime $p$. 

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- Choose $a_0 \in S$
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- Choose $a_0 \in S$
- Choose $a_1 \in S$ that minimizes the highest power of $p$ dividing $a_1 - a_0$
p-Ordering

Let \( S \subset \mathbb{Z} \) and fix a prime \( p \).

- Choose \( a_0 \in S \)
- Choose \( a_1 \in S \) that minimizes the highest power of \( p \) dividing \( a_1 - a_0 \)
- Choose an element \( a_2 \in S \) that minimizes the highest power of \( p \) dividing \((a_2 - a_0)(a_2 - a_1)\)
Let $S \subset \mathbb{Z}$ and fix a prime $p$.

- Choose $a_0 \in S$
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- Choose an element $a_2 \in S$ that minimizes the highest power of $p$ dividing $(a_2 - a_0)(a_2 - a_1)$
- For the $k^{th}$ step, choose an element $a_k \in S$ that minimizes the highest power of $p$ dividing $(a_k - a_0)(a_k - a_1) \cdot \ldots \cdot (a_k - a_{k-1})$
p-Ordering

Let $S \subset \mathbb{Z}$ and fix a prime $p$.

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- For the $k^{th}$ step, choose an element $a_k \in S$ that minimizes the highest power of $p$ dividing $(a_k - a_0)(a_k - a_1) \cdot \ldots \cdot (a_k - a_{k-1})$
- Notation: For each $k$, $\nu_k(S, p)$ represents the highest power of $p$ that fulfills the above expression \{\nu_0(S, p), \nu_1(S, p), ..\}
Example

Let $S$ be the set of all primes. $S = \{2, 3, 5, 7 \ldots\}$ and fix prime $p = 2$
Example

Let $S$ be the set of all primes. $S = \{2, 3, 5, 7...\}$ and fix prime $p = 2$

- Let $a_0 = 19$
Example

Let $S$ be the set of all primes. $S = \{2, 3, 5, 7...\}$ and fix prime $p = 2$

- Let $a_0 = 19$
- We need to pick $a_1$.
  - The highest power of $p$ that divides $2 - a_0 = -17$ is $2^0 = 1$
Example

Let $S$ be the set of all primes. $S = \{2, 3, 5, 7...\}$ and fix prime $p = 2$

- Let $a_0 = 19$

- We need to pick $a_1$.
  The highest power of $p$ that divides $2 - a_0 = -17$ is $2^0 = 1$

- Let's pick $a_2$ $(a_2 - 19)(a_2 - 2)$. Pick $a_2 = 5 \Rightarrow (5 - 19)(5 - 2) = (-14)(3) = (2 \cdot -7)(3)$
  The highest power of $p$ that divides $(a_2 - 19)(a_2 - 2)$ is $2^1 = 2$. 
Example

Let $S$ be the set of all primes. $S = \{2, 3, 5, 7\ldots\}$ and fix prime $p = 2$

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  $a_2 = 5 \Rightarrow (5 - 19)(5 - 2) = (-14)(3) = (2 \cdot -7)(3)$
  The highest power of $p$ that divides $(a_2 - 19)(a_2 - 2)$ is $2^1 = 2$.
- Similarly, for $a_3$, we need $(a_3 - 19)(a_3 - 2)(a_3 - 5)$. In this case, the highest power of $p$ that divides the product above is $a_3 = 17$
  $(17 - 19)(17 - 2)(17 - 5) = (-2)(15)(2^2 \cdot 3)$ The corresponding power here is $2^3 = 8$. 

Nitesh Mathur (TU)
Short title
March 9, 2018 7 / 19
Let $S$ be the set of all primes. $S = \{2, 3, 5, 7, \ldots\}$ and fix prime $p = 2$

- Let $a_0 = 19$

- We need to pick $a_1$.
  The highest power of $p$ that divides $2 - a_0 = -17$ is $2^0 = 1$

- Let's pick $a_2 (a_2 - 19)(a_2 - 2)$. Pick $a_2 = 5 \Rightarrow (5 - 19)(5 - 2) = (-14)(3) = (2 \cdot -7)(3)$
  The highest power of $p$ that divides $(a_2 - 19)(a_2 - 2)$ is $2^1 = 2$.

- Similarly, for $a_3$, we need $(a_3 - 19)(a_3 - 2)(a_3 - 5)$. In this case, the highest power of $p$ that divides the product above is $a_3 = 17$
  $(17 - 19)(17 - 2)(17 - 5) = (-2)(15)(2^2 \cdot 3)$ The corresponding power here is $2^3 = 8$.

- Similarly for the rest $a_k$
The $p$-ordering for $p = 2$ is as follows: $\{19, 2, 5, 17, 23, 31, \ldots, \}$ and its corresponding $p$-sequence is as follows, $\{1, 1, 2, 8, 16, 128, \ldots\}$
Construct such a $p$-ordering for every $p$ (Note: Not unique)

Punchline 1: The associated $p$-sequence of $S$ is independent of the choice of $p$-ordering.

Punchline 2: Let $S$ be any subset of $\mathbb{Z}$. Then the factorial function of $S$, denoted by $k_!^S$ is defined by

$$k_!^S = \prod p_{v_k(S, p)}$$
Construct such a \( p \) ordering for every \( p \) (Note: Not unique)
Construct such a $p$ ordering for every $p$ (Note: Not unique)

**Punchline 1**: The associated $p$-sequence of $S$ is independent of the choice of $p$-ordering.
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**Punchline 1**: The associated $p$-sequence of $S$ is independent of the choice of $p$-ordering.

**Punchline 2** Let $S$ be any subset of $\mathbb{Z}$. Then the *factorial function* of $S$, denoted by $k!_S$ is defined by

$$k!_S = \prod_{p} v_k(S, p)$$
The p-ordering for the prime subset of $\mathbb{Z}$ is as follows:

\[ p = 2 \]
\[
\{ 19, 2, 5, 17, 23, 31, \ldots \}
\]

\[ p = 3 \]
\[
\{ 2, 3, 7, 13, 17, 19, \ldots \}
\]

\[ p-sequence \]
\[
\{ 1, 1, 2, 8, 16, 128, \ldots \}
\]

\[ p-sequence \]
\[
\{ 1, 1, 1, 3, 3, 9, \ldots \}
\]
The $p$-ordering for the prime subset of $\mathbb{Z}$ is as follows:

$p = 2$
$p$-ordering:
\{19, 2, 5, 17, 23, 31, ...\}

$p$-sequence:
\{1, 1, 2, 8, 16, 128, ...\}

$p = 3$
$p$-ordering:
\{2, 3, 7, 5, 13, 17, 19, ...\}

$p$-sequence:
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  - p-ordering: $\{19, 2, 5, 17, 23, 31, \ldots, \}$
  - p-sequence is as follows, $\{1, 1, 2, 8, 16, 128, \ldots\}$

- $p = 3$
  - p-ordering: $\{2, 3, 7, 5, 13, 17, 19, \ldots\}$
  - p-sequence: $\{1, 1, 1, 3, 3, 9, \ldots\}$
Examples

- \(4!_p = 48, \ 6!_p = 11520, \ldots\)
- Notice, one has to multiply across. Each \(k\) represents an index in each \(p\)-sequence.

<table>
<thead>
<tr>
<th>(p = 2)</th>
<th>(p = 3)</th>
<th>(p = 5)</th>
<th>(p = 7)</th>
<th>(p = 11)</th>
<th>(k!_p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k = 0)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(k = 1)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(k = 2)</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(k = 3)</td>
<td>8</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(k = 4)</td>
<td>16</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(k = 5)</td>
<td>128</td>
<td>9</td>
<td>5</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(k = 6)</td>
<td>256</td>
<td>9</td>
<td>5</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
The Natural Numbers

Consider \( N \subset \mathbb{Z} \). The natural ordering of \( N = \{1, 2, 3, \ldots\} \) is a \( p \)-ordering of \( N \).

The \( p \)-sequences of \( N \) are as follows:

- \( p = 2 \):
  \[
  \{1, 1, 2, 2, 8, 8, 16, 16, \ldots\}
  \]

- \( p = 3 \):
  \[
  \{1, 1, 1, 3, 3, 3, 9, 9, 9, \ldots\}
  \]

- \( p = 5 \):
  \[
  \{1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 25, \ldots\}
  \]

- \( p = 7 \):
  \[
  \{1, 1, 1, 1, 1, 1, 1, 7, 7, 7, 7, 7, 7, 25, \ldots\}
  \]

Check Your Results:

- \( 0! \) \( N \) = 1
- \( 2! \) \( N \) = 2
- \( 3! \) \( N \) = 6
- \( 7! \) \( N \) = 720
Consider $\mathbb{N} \subset \mathbb{Z}$ The natural ordering of $\mathbb{N} = \{1, 2, 3, \ldots\}$ is a $p$-ordering of $\mathbb{N}$.

The $p$-sequences of $\mathbb{N}$ are as follows:

- $p = 2$: $\{1, 1, 2, 2, 8, 8, 16, 16, \ldots\}$
- $p = 3$: $\{1, 1, 1, 3, 3, 3, 9, 9, 9, \ldots\}$
- $p = 5$: $\{1, 1, 1, 1, 5, 5, 5, 5, 5, 25, \ldots\}$
- $p = 7$: $\{1, 1, 1, 1, 1, 1, 1, 7, 7, 7, 7, 7, 7, \ldots\}$

Check Your Results:

- $2!$: $\mathbb{N} = 2 \times 1 \times 1 \times 1 \times 1 \times \cdots = 2$
- $3!$: $\mathbb{N} = 2 \times 3 \times 1 \times 1 \times 1 \times \cdots = 6$
- $7!$: $\mathbb{N} = 16 \times 9 \times 5 \times 1 \times \cdots = 720$
Consider \( \mathbb{N} \subset \mathbb{Z} \) The natural ordering of \( \mathbb{N} = \{1, 2, 3, \ldots\} \) is a p-ordering of \( \mathbb{N} \).
The p-sequences of \( \mathbb{N} \) are as follows:

- \( p = 2 \): \( \{1, 1, 2, 2, 8, 8, 16, 16, \ldots\} \)
- \( p = 3 \): \( \{1, 1, 1, 3, 3, 3, 9, 9, 9, \ldots\} \)
- \( p = 5 \): \( \{1, 1, 1, 1, 5, 5, 5, 5, 5, 25, \ldots\} \)
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Check Your Results:
- \( 0! \) \( \mathbb{N} = 1 \times 1 \times 1 \times 1 \times 1 \times \ldots = 1 \)
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- \( 3! \) \( \mathbb{N} = 2 \times 3 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times \ldots = 6 \)
- \( 7! \) \( \mathbb{N} = 16 \times 9 \times 5 \times 1 \times 1 \times \ldots = 720 \)
Consider $\mathbb{N} \subseteq \mathbb{Z}$ The natural ordering of $\mathbb{N} = \{1, 2, 3, \ldots\}$ is a $p$-ordering of $\mathbb{N}$.

The $p$-sequences of $\mathbb{N}$ are as follows:

- $p = 2$: $\{1, 1, 2, 2, 8, 8, 16, 16, \ldots\}$
- $p = 3$: $\{1, 1, 1, 3, 3, 3, 9, 9, 9, \ldots\}$
Consider $\mathbb{N} \subset \mathbb{Z}$. The natural ordering of $\mathbb{N} = \{1, 2, 3, \ldots\}$ is a $p$-ordering of $\mathbb{N}$.

The $p$-sequences of $\mathbb{N}$ are as follows:

- $p = 2$: $\{1, 1, 2, 2, 8, 8, 16, 16, \ldots\}$
- $p = 3$: $\{1, 1, 1, 3, 3, 3, 9, 9, 9, \ldots\}$
- $p = 7$: $\{1, 1, 1, 1, 5, 5, 5, 5, 5, 25, \ldots\}$
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- Consider $\mathbb{N} \subset \mathbb{Z}$ The natural ordering of $\mathbb{N} = \{1, 2, 3, \ldots\}$ is a $p$-ordering of $\mathbb{N}$. The $p$-sequences of $\mathbb{N}$ are as follows:
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  - $p = 3$: $\{1, 1, 1, 3, 3, 3, 9, 9, 9, \ldots\}$
  - $p = 5$: $\{1, 1, 1, 1, 1, 5, 5, 5, 5, 25, \ldots\}$
  - $p = 7$: $\{1, 1, 1, 1, 1, 1, 7, 7, 7, 7, \ldots\}$

Check Your Results:
The Natural Numbers

- Consider \( \mathbb{N} \subset \mathbb{Z} \). The natural ordering of \( \mathbb{N} = \{1, 2, 3, \ldots\} \) is a p-ordering of \( \mathbb{N} \).

  The p-sequences of \( \mathbb{N} \) are as follows:

  - \( p = 2 \): \{1, 1, 2, 2, 8, 8, 16, 16, \ldots\}
  - \( p = 3 \): \{1, 1, 1, 3, 3, 3, 9, 9, 9, \ldots\}
  - \( p = 5 \): \{1, 1, 1, 1, 5, 5, 5, 5, 5, 25, \ldots\}
  - \( p = 7 \): \{1, 1, 1, 1, 1, 1, 7, 7, 7, 7, 7, \ldots\}

  Check Your Results:

  - \( 0!_{\mathbb{N}} = 1 \times 1 \times 1 \times 1 \times 1 \ldots = 1 \)
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- $p = 5$: $\{1, 1, 1, 1, 1, 5, 5, 5, 5, 25, \ldots\}$
- $p = 7$: $\{1, 1, 1, 1, 1, 1, 7, 7, 7, 7, 7, \ldots\}$

Check Your Results:

- $0!_{\mathbb{N}} = 1 \times 1 \times 1 \times 1 \times 1 \ldots = 1$
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- \( p = 3 \): \( \{1, 1, 1, 3, 3, 3, 9, 9, 9, \ldots\} \)
- \( p = 5 \): \( \{1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 25, \ldots\} \)
- \( p = 7 \): \( \{1, 1, 1, 1, 1, 1, 7, 7, 7, 7, 7, 7, \ldots\} \)

Check Your Results:

- \( 0!_{\mathbb{N}} = 1 \times 1 \times 1 \times 1 \times 1 \ldots = 1 \)
- \( 2!_{\mathbb{N}} = 2 \times 1 \times 1 \times 1 \times 1 \ldots = 2 \)
- \( 3!_{\mathbb{N}} = 2 \times 3 \times 1 \times 1 \times 1 \ldots = 6 \)
The Natural Numbers

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- \( p = 2: \{1, 1, 2, 2, 8, 8, 16, 16, \ldots\} \)
- \( p = 3: \{1, 1, 1, 3, 3, 3, 9, 9, 9, \ldots\} \)
- \( p = : \{1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 25, \ldots\} \)
- \( p = 7: \{1, 1, 1, 1, 1, 1, 1, 7, 7, 7, 7, 7, \ldots\} \)

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- \( 0!_{\mathbb{N}} = 1 \times 1 \times 1 \times 1 \times 1 \ldots = 1 \)
- \( 2!_{\mathbb{N}} = 2 \times 1 \times 1 \times 1 \times 1 \ldots = 2 \)
- \( 3!_{\mathbb{N}} = 2 \times 3 \times 1 \times 1 \times 1 \ldots = 6 \)
- \( 7!_{\mathbb{N}} = 16 \times 9 \times 5 \times 1 \ldots = 720 \)
### More Examples

<table>
<thead>
<tr>
<th>Sl. No.</th>
<th>Set $S$</th>
<th>$k!S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Set of natural numbers</td>
<td>$k!$</td>
</tr>
<tr>
<td>2</td>
<td>Set of even integers</td>
<td>$2^k \times k!$</td>
</tr>
<tr>
<td>3</td>
<td>Set of integers of the form $an + b$</td>
<td>$a^k \times k!$</td>
</tr>
<tr>
<td>4</td>
<td>Set of integers of the form $2^n$</td>
<td>$(2^k - 1)(2^k - 2) \ldots (2^k - 2^{k-1})$</td>
</tr>
<tr>
<td>5</td>
<td>Set of integers of the form $q^n$ for some prime $q$</td>
<td>$(q^k - 1)(q^k - 2) \ldots (q^k - q^{k-1})$</td>
</tr>
<tr>
<td>6</td>
<td>Set of squares of integers</td>
<td>$(2k)!/2$</td>
</tr>
</tbody>
</table>
Theorem 1
For any nonnegative integers, \(k\) and \(l\), \((k + l)! / S\) is a multiple of \(k! / S\) and \(l! / S\).

Theorem 2
Let \(f\) be a primitive polynomial of degree \(k\) and let \(d_{\{S, f\}} = \gcd\{f(a) : a \in S\}\)
Then, \(d_{\{S, f\}}\) divides \(k! / S\).

Theorem 3
Let \(a_0, a_1, ..., a_n \in S\) be any \(n + 1\) integers. Then their product of their pairwise differences
\[ \prod_{i < j} (a_i - a_j) \]
is a multiple of \(0! / S\) and \(1! / S\) and ... and \(n! / S\).

Theorem 4
The number of polynomial functions from \(S\) to \(\mathbb{Z}/n\mathbb{Z}\) is given by
\[ n - 1 \prod_{k = 0}^{n} \gcd(n, k! / S) \]
Theorem 1
For any nonnegative integers, $k$ and $l$, $(k + l)!_S$ is a multiple of $k!_Sl!_S$. 
Theorem 1
For any nonnegative integers, $k$ and $l$, $(k + l)!_S$ is a multiple of $k!_S l!_S$.

Theorem 2 Let $f$ be a primitive polynomial of degree $k$ and let $d(S, f) = \gcd\{f(a) : a \in S\}$ Then, $d(S, f)$ divides $k!_S$.
Revisit Theorems

- **Theorem 1**
  For any nonnegative integers, \( k \) and \( l \), \((k + l)!_S\) is a multiple of \( k!_S/l!_S\).

- **Theorem 2** Let \( f \) be a primitive polynomial of degree \( k \) and let \( d(S, f) = \gcd \{ f(a) : a \in S \} \) Then, \( d(S, f) \) divides \( k!_S \).

- **Theorem 3**
  Let \( a_0, a_1, \ldots, a_n \in S \) be any \( n + 1 \) integers. Then their product of their pairwise differences
  \[
  \prod_{i<j}(a_i - a_j)
  \]
  is a multiple of \( 0!_S 1!_S \ldots n!_S \).
• **Theorem 1**  
  For any nonnegative integers, $k$ and $l$, $(k + l)!_S$ is a multiple of $k!_S / l!_S$.

• **Theorem 2** Let $f$ be a primitive polynomial of degree $k$ and let $d(S, f) = \gcd\{f(a) : a \in S\}$ Then, $d(S, f)$ divides $k!_S$.

• **Theorem 3**  
  Let $a_0, a_1, \ldots a_n \in S$ be any $n + 1$ integers. Then their product of their pairwise differences 
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• **Theorem 4** The number of polynomial functions from $S$ to $\mathbb{Z}/n\mathbb{Z}$ is given by
  \[ \prod_{k=0}^{n-1} \frac{n}{\gcd(n, k!_S)} \]
The Rest of the Paper

A bunch of proofs.

Generalization to Dedekind Rings.

Generalization to Higher Dimensions.

Applications
A bunch of proofs.
The Rest of the Paper

- A bunch of proofs.
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- A bunch of proofs.
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- Applications
Posed Questions

For a subset $S \subset \mathbb{Z}$, is there a natural combinatorial interpretation of $k!^S$.

What is the natural combinatorial interpretation for

$$\binom{n}{k}^S = \frac{n!^S}{k!^S (n-k)!^S}$$

coefficients?

What is the "binomial theorem" for generalized binomial?
For a subset $S \subset \mathbb{Z}$, is there a natural combinatorial interpretation of $k!_S$. 

What is the natural combinatorial interpretation for $(n \choose k)_S = \frac{n!_S}{k!_S (n-k)!_S}$ coefficients?

What is the "binomial theorem" for generalized binomial?
Posed Questions

- For a subset $S \subset \mathbb{Z}$, is there a natural combinatorial interpretation of $k!_S$.
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  $$(n)_S = \frac{n!_S}{k!_S(n-k)!_S}$$
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Posed Questions

- For a subset $S \subset \mathbb{Z}$, is there a natural combinatorial interpretation of $k!_S$.
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  \[(n)_S = \frac{n!_S}{k!_S (n-k)!_S} \text{ coefficients?} \]
- What is the "binomial theorem" for generalized binomial?
Thank You!

- Dr. O’Neil
- Dr. Donahue
- Jon Bolin
- Journal Club
Questions?