Global BV solution to a system of balance laws from traffic flow

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Traffic System

- Constructing global solutions and finding zero relaxation limits of traffic flow
- Roadways, Vehicles, Drivers
- Microscopic Vs Macroscopic
- We will be focusing on a specific macroscopic model

History of Traffic Flow

- Lighthill-Whitham-Richards (LWR) model [1955, 1956]
- Payne-Whitham (PW) model [1971, 1974]
- Viscous models studied by Kerner-Konhauser, Kuhne, Beckshulte, and Li [1984-1994, 2008]
- Aw-Rascle and Zhang's higher continuum (ARZ) models [2000, 2001]

Dafermos Theory

- Reading Research in Spring 2021 - Dafermos' *Hyperbolic conservation laws in continuum physics* [2]
- Aim - Existence and long time behavior of BV solutions to the Cauchy Problem
- Context - Strictly hyperbolic systems of balance laws
- Global existence of BV solutions based on damping
- For system of two balance laws, L^1-estimates derived by constructing convex entropies

Nonlinear Balance Laws

- Let $U \in \mathbb{R}^n$.
- $U = (u_1, u_2, ..., u_n), F(U) = (f_1(u), f_2(u), ..., f_n(u))$
- Consider the general conservation form

$$U_t + F(U)_x + P(U) = 0 \tag{1}$$

with initial data

$$U(x, 0) = U_0(x), \tag{2}$$

where $x \in \mathbb{R}, t > 0$.

The Model

- To analyze the 2×2 traffic flow model:

$$\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ v_t + (\frac{1}{2}v^2 + g(\rho))_x + \frac{v - v_\epsilon(\rho)}{\tau} &= 0, \end{aligned} \tag{3}$$

  with initial data

$$(\rho(x,0), v(x,0)) = (\rho_0(x), v_0(x)) \tag{4}$$

  where $x \in \mathbb{R}, t > 0, \tau > 0$.

- ρ - density, v - velocity, $v_\epsilon(\rho)$ - equilibrium velocity.
- $g(\rho)$ - anticipation factor and satisfies

$$g'(\rho) = \rho(v'_\epsilon(\rho)/\theta)^2, \tag{5}$$

  where $g'(\rho) \geq 0$, $0 < \theta < 1$.

LWR Model

- The equilibrium flow is described by Lighthill-Whitham-Richards (LWR) model [4, 5]

$$\rho_t + (\rho v_\epsilon(\rho))_x = 0, \qquad x \in \mathbb{R}, t > 0, \tag{6}$$

with initial data $\rho(x, 0) = \rho_0(x) > 0$.

- $q(\rho) = \rho v_\epsilon(\rho)$ is known as the fundamental diagram
- For our work, we let

$$v_\epsilon(\rho) = -a\rho + b, \tag{7}$$

where $a > 0, b > 0$.

- In our study, the equilibrium flux $q(\rho) = \rho(-a\rho + b)$ is a concave function of ρ.

Work Overview

- We showed in [1] the existence of a global BV solution for a system of balance laws arising in traffic flow in the framework of Dafermos [2]
- Showed the decay of L^1- and L^2-norms
- Computed entropy-entropy flux pair, Kawashima condition, sub-characteristic condition, and the partial dissipative inequality
- With these conditions we show the existence of BV solutions for the Cauchy problem

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First Transformation

- We want $U \equiv 0$ to be an equilibrium solution we need to do a change of variables $v = u + b$.
- Now we can rewrite (3) as follows

$$\begin{aligned} \rho_t + (\rho(u+b))_x &= 0 \\ u_t + (\frac{1}{2}(u+b)^2 + g(\rho))_x + \frac{u + b - v_\epsilon(\rho)}{\tau} &= 0 \end{aligned} \tag{8}$$

- In terms of the general form, we have

$$\begin{aligned} U &= (\rho, v - b) = (\rho, u) \\ F(U) &= (\rho(u+b), \frac{1}{2}u^2 + ub + g(\rho)))^T \\ P(U) &= (0, \frac{u + b - v_\epsilon(\rho)}{\tau}) \end{aligned} \tag{9}$$

Preliminaries

- The Jacobian is
$$\begin{bmatrix} u+b & \rho \\ g'(\rho) & u+b \end{bmatrix} \tag{10}$$
- Using (5) and (7), the eigenvalues are
$$\lambda_{1,2} = u + b \mp \frac{a}{\theta}\rho \tag{11}$$
- The corresponding right eigenvectors are
$$r_{1,2} = (\mp\frac{\theta}{a}, 1)^T. \tag{12}$$
- The system (8) is genuinely nonlinear since
$$\nabla\lambda_i \cdot r_i = \frac{q''(\rho)}{v_\epsilon'(\rho)} = 2 \neq 0, \qquad i = 1, 2. \tag{13}$$

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In order to apply Dafermos' theory [2], we had to

- Search for a convex entropy-entropy flux pair
- Verify conditions
- Transform system (3) once again into equivalent form
- Derive a global bound between the distance of the velocity and equilibrium velocity

Entropy-Entropy Flux Pair

- We need to find smooth entropy flux pair $(\eta, q)(U)$ where η is convex and has been normalized by $\eta(0), D\eta(0) = 0$.
- This is important since admissible solutions U must satisfy the entropy inequality

$$\partial_t \eta(U(x,t)) + \partial_x q(U(x,t)) + D\eta(U(x,t)) \cdot P(U(x,t)) \leq 0. \tag{14}$$

- We also want our system to be a **symmetrizable**, which means it needs to be endowed with nontrivial **companion balance laws**.
- So we also need to solve

$$\begin{aligned} DQ_1(U,X) &= B(U,X)^T DG_1(U,X) \\ DQ_2(U,x) &= B(U,X)^T DG_2(U,X), \end{aligned} \tag{15}$$

where $G_1 = U, G_2 = F(U), DQ_i = [\frac{\partial Q_i}{\partial \rho}, \frac{\partial Q_i}{\partial_u}], i = 1, 2.$

Continued

- After simplifying this computation, we derived a wave equation for η
- Solving the wave equation, we then constructed an explicit solution of a convex entropy-entropy flux pair

$$\begin{aligned} \eta(\rho, u) &= Q_1(\rho, u) = 2((s\rho))^2 + (u+b)^2), \\ q(\rho, u) &= Q_2(\rho, u) = 4(u+b)(s\rho)^2 + \frac{4}{3}(u+b)^3, \end{aligned} \tag{16}$$

  where $s = \dfrac{a}{\theta}$.
- With this entropy-entropy flux pair, the convexity conditions are satisfied

Partial Dissipative Inequality

- We assume that P is dissipative semidefinite relative to η, i.e.

$$D\eta(U) \cdot P(U) \geq \alpha |P(U)|^2, \tag{17}$$

  with $\alpha > 0$.
- For our system (8), we needed to find a condition such that

$$\begin{bmatrix} \frac{\partial \eta}{\partial \rho} & \frac{\partial \eta}{\partial u} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \frac{u + b - v_\epsilon(\rho)}{\tau} \end{bmatrix} \geq \alpha \left(\frac{u + b - v_\epsilon(\rho)}{\tau} \right)^2 \tag{18}$$

- After simplification, we get the following condition

$$\frac{|u + b - v_\epsilon(\rho)|}{u} \leq \frac{4\tau}{\alpha}, \tag{19}$$

- $v_\epsilon(\rho)$ - equilibrium speed, $u + b$ - speed
- Numerator of left-hand side of the inequality describes difference between the velocity and the equilibrium velocity.

Kawashima Condition

- The Kawashima condition is given by

$$DP(0)r_i(0) \neq 0, \qquad i = 1, ..., n. \tag{20}$$

- For our system, we have

$$DP(0)r_i(0) = \begin{bmatrix} 0 \\ \dfrac{\mp\theta + 1}{\tau} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{21}$$

since $0 < \theta < 1$.

Sub-characteristic condition

- The sub-characteristic is satisfied when

$$\lambda_1 < \lambda_* < \lambda_2. \tag{22}$$

- For $v = v_\epsilon(\rho)$,

$$\lambda * (\rho) = -2a\rho + b.$$

- The sub-characteristic condition is satisfied for (8) since we have

$$u + b - \frac{a}{\theta}\rho < -2a\rho + b < u + b + \frac{a}{\theta}\rho. \tag{23}$$

Equivalent Form

- In order to apply Dafermos' theory, we needed to convert (8) into an equivalent form

$$\begin{aligned} \partial_t V + \partial_x G(V,W) + X(V,W) &= 0 \\ \partial_t W + \partial_x H(V,W) + CW + Y(V,W) &= 0, \end{aligned} \tag{24}$$

where $x \in \mathbb{R}$, $t > 0$,and $\eta_{WW} C(0,0) > 0$.

- We had to do the following change of variables

$$\begin{aligned} V &= \rho \\ W &= a\rho + u \end{aligned} \tag{25}$$

- This transformed (8) to

$$\begin{aligned} V_t + [V(W - aV + b)]_x &= 0 \\ W_t + [\frac{1}{2}(W^2 - a^2V^2) + bW + g(\rho)]_x + \frac{1}{\tau}W &= 0 \end{aligned} \tag{26}$$

with initial conditions

$$Z_0 = (V_0, W_0) = (\rho_0, a\rho_0 + u_0). \tag{27}$$

Verification of Partial Dissipative Inequality

- We derive the following from our equivalent form

$$|W| = |W_0| e^{\frac{-1}{\tau} t} \tag{28}$$

- For the partial dissipative inequality to be satisfied under this transformation, we further required

$$||W_0||_{L_\infty} \leq \min\{\frac{1}{c}\delta_0, a\delta_0 \frac{4\tau}{\alpha + 4\tau}\} \tag{29}$$

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Theorem

Theorem (L^1-Estimates)

Consider the Cauchy problem (26) (27)*. Under condition* (29)*, the source is dissipative semidefinite* (17) *relative to the entropy* η *and the Kawashima condition* (17) *holds. Let* $Z = (V, W)$ *be an admissible* BV *solution as defined in* (14)*, with initial values* Z_0 *defined on the strip* $(-\infty, \infty) \times [0, T)$ *and taking values in a ball* $\mathcal{B})_\rho$ *of small radius* ρ*, centered at the origin. Suppose that* Z_0 *decays, as* $|x| \to \infty$*, sufficienty fast to render the integral*

$$\int_{-\infty}^{\infty} |Z_0(x)|^2 \, dx = \sigma^2, \tag{30}$$

where $\sigma > 0$ *finite. Furthermore, let*

$$\int_{-\infty}^{\infty} V_0(x) \, dx = 0. \tag{31}$$

Theorem (Continued)

Theorem

Then there is a $\sigma_0 > 0$, independent of T such that for $\sigma < \sigma_0$,

$$\int_{-\infty}^{\infty} |Z(x,t)| \, dx \leq b\sigma, 0 \leq t < T, \tag{32}$$

with b independent of T. Furthermore, if $T = \infty$,

$$\int_{-\infty}^{\infty} |Z(x,t)| \, dx \to 0, \quad \text{as } t \to \infty. \tag{33}$$

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Global BV Solution to a system of balance laws with *nonconcave* flux

- An important open problem is to extend the existing results to nonconcave fundamental diagrams $q(\rho)$ as suggested from the experiment data (See [5] and references there in)
- One of the characteristic fields of this system is neither linearly degenerate nor genuinely nonlinear
- Study of nonconcave flux is an open problem in the framework of Dafermos [2]
- We plan to apply the ideas from [1] but with a nonconcave flux

References I

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[2] C.M. Dafermos, *Hyperbolic conservation laws in continuum physics*. Fourth edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 325. Springer-Verlag, Berlin, 2016. xxxviii+826.

[3] H. Zhang, New Perspectives on Continuum Traffic Flow Models (special double issue on traffic flow theory), *Networks and Spatial Economics*, 1, (2001).

[4] T. Li, Global solutions and zero relaxation limit for a traffic flow model, *SIAM J. Appl. Math.*, 61, (2000), 1042–1061.

[5] T. Li, Global solutions of nonconcave hyperbolic conservation laws with relaxation arising from traffic flow, *J. Differential Equations*, 190, (2003), 131–149