Functional Analysis Notes

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Contents

1 Introduction

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1.1 Banach & C^* algebras

Suppose A is a **Banach Algebra** over $\mathbb{C}$ with algebraic structure $(A, +, \cdot, \lambda)$ (assume **unital** i.e. $1 \in A$). Furthermore, it is an algebra with a topology on it (induced by norms).

*Note: All finite dimensional norms induces the same topology. Estimates might be different, but open sets are the same.

Definition 1.1 (Norm)**.** Norm is a mapping $|| \;\; || : A \mapsto \mathbb{R}_+$ such that
(i) $||c \cdot x|| = |c| \cdot ||x|| \; \forall x \in A, c \in \mathbb{C}$
(ii) $||x + y|| \leq ||x|| + ||y||$ (sub-additive; Triangle Inequality)
(iii) $||x|| = 0 \iff x = 0_A$.
(iv) $||xy|| \leq ||x|| \cdot ||y||$ (sub-multiplicative)
(v) $(A, || \cdot ||$ is **complete** i.e. $\forall || \cdot ||$-Cauchy sequences converge.
(vi) $||x^* x|| = ||x||^2$, where $*$ is the **involution**.

- Properties (i)-(iii) form a **normed space**.
- Property (iii) is known as the 'faithfulness' property. If we do not have faithfulness, then we have a **semi-norm**.
- With the addition of property (iv), we have a **Norm Algebra**
- With (i)-(v), we have a **Banach Algebra**.
- A Banach Algebra with an involution map is known as an **Involutive Banach Algebra**.
- Finally, all the six properties combine to define the **C^*-Algebra**.

Definition 1.2 (Involution)**.** $* : A \to A$ is an involution if
(i) $(ax + by)^* = \bar{a}x^* + \bar{b}y^*$ (Conjugate linear in $\mathbb{C}$ or linear in $\mathbb{R}$).
(ii) $(x^*)^* = x$
(iii) $(xy)^* = y^* x^*$

Example 1 (Matrices). Consider $M_n(\mathbb{C}), k \in \mathbb{N}$.
It is equipped with the norm $||A||_\infty = \sup_{||v||_2 \leq 1} ||Av||_2$, where $|| \cdot ||_2$ denotes the Euclidean norm.
Also notice that we have $A^* = \bar{A}^T$ (conjugate transpose).
$M_n(\mathbb{C})$ satisfies all the conditions of a C^*-algebra.

Example 2 (Set of Continuous Functions). Similarly consider $C([0, 1] = \{f : [0, 1] \to \mathbb{C} \text{ continuous}\}$.

One can add continuous functions, do pointwise multiplications, and add by scalar, i.e. we have $(C[0, 1], +, \cdot, \lambda)$. Furthermore continuous functions of compact sets are bounded, so supremums are well-defined. Hence, it is equipped with the following norm:
$||f||_\infty = \sup_{x \in [0,1]} |f(x)| < \infty$.
Finally, $f^*(x) = \bar{f}(x)$ (complex conjugate).
All the conditions above are satisfied. For example, the sub-multiplicative condition can be shown as follows:

$$\begin{aligned} |(fg)(x)| &= |f(x)g(x)| \\ &= |f(x)| \cdot |g(x)| \\ &\leq ||f||_\infty \cdot ||g||_\infty \end{aligned}$$

Taking sups of both sides, we get the desired conclusion.

Example 3 (ℓ^∞ functions). Similarly, ℓ^∞ is also a C^*-algebra.
Define $\ell^\infty([0, 1]) = \{f : [0, 1] \to \mathbb{R} \text{ essentially bounded}\}$.

*Note: Commutativity is NOT assumed.

1.2 Spectral Theory

Definition 1.3 ($G(A)$)**.** $G(A) =$ Set of all invertible elments of $A = \{x \in A : \ \exists \text{ two side inverses} x^{-1}\}$

Definition 1.4 (Spectrum)**.** Fix $x \in A, \sigma_A(x) = \{\lambda \in \mathbb{C}, \quad x - \lambda(\text{or}\lambda 1 - x \notin G(A)\}$.

Example 4 (Matrices). Let $A = M_n(\mathbb{C}), x \in A$. Then,

$$\begin{aligned} &\lambda I_n - X \notin G(A) \\ &\iff \det(\lambda I_n - X) = 0 \\ &\iff \lambda \text{ is an eigenvalue} \end{aligned}$$

Hence, λ is a spectrum if it is the root of the the characteristic polynomial i.e.

$$\text{Spectrum } \sigma_A(X) = \{\text{eigenvalues}\}.$$

(Later, we will find in our first big theorem that Spectrum is countable).

Lemma 5. Let A be a Banach algebra. $x, y \in A$. Then,

$$\sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\}.$$

Proof. $1 - xy \in G(A) \iff 1 - yx \in G(A)$. We will prove the $(\Leftarrow)$ implication.

First note that

$$(1 - xy)^{-1} = 1 + x(1 - yx)^{-1}y. \tag{1}$$

Then, we can check that

$$\begin{aligned} 1 &= (1 - xy) \cdot [1 + x(1 - yx)^{-1}y] \qquad \text{from (1)} \\ &= (1 - xy) + (1 - xy)(x(1 - yx)^{-1}y) \\ &= (1 - xy) + x(1 - yx)^{-1}y - xyx(1 - yx)^{-1}y \\ &= 1 - xy + (1 - yx)^{-1}y - x(1 - (1 - yx))(1 - yx)^{-1}y \\ &= 1 - xy + x(1 - yx)^{-1}y - x(1 - yx)^{-1}y - x\underbrace{(1 - yx)^1(1 - yx)^{-1}}_{1}y \\ &= 1 - xy + xy \\ &= 1 \end{aligned}$$

□

1.2.1 Derivation of the Inverse

Recall the power series

$$(1 - z)^{-1} = \frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + ... = (1 - z)^{-1} \tag{2}$$

We can apply this as follows:

$$\begin{aligned} (1 - xy)^{-1} = \frac{1}{1 - xy} &= \sum_{n=0}^{\infty} (xy)^n \\ &= 1 + xy + xyxy + xyxyxy + ... \\ &= 1 + x(1 + yx + (yx)^2 + ..)y \\ &= 1 + x\frac{1}{1 - yx}y \\ &= 1 + x(1 - yx)^{-1}y \end{aligned}$$

(Very Cool).

Lemma 6. Let A be a Banach Algebra. $1 \in A, x \in A, ||x|| < 1$. Then, $1 - x \in G(A)$.

Proof. Let $y_n = 1 + x + x^2 + ...x^n$.

$$\begin{aligned}(1-x)^{-1} &= \sum_{n=0}^{\infty} x^n \\ &= 1 + x + x^2 + ...\end{aligned}$$

We need to take the limit of both sides. Before we can do that, consider WLOG for $n > m$,

$$\begin{aligned}||y_n - y_m|| &= ||\sum_{k=0}^{\infty} x^k - \sum_{l=0}^{m} x^l \\ &= ||\sum_{k=n}^{m} x^k|| \\ &\leq \sum_{k=m+1}^{n} ||x^k|| \\ &\leq \sum_{k=m+1}^{n} ||x|| \to 0.\end{aligned}$$

The last line follows since this is a tail of a power series. Hence, we have

$$\begin{aligned}(1-x)y_n &= (1-x)(1 + x + x^2 + ...) \\ &= 1 - x^{n-1} \qquad \text{Taking limits of both sides, we get}\end{aligned}$$

$$\begin{aligned}(1-x)(\sum_{k=0}^{\infty} x^n &= 1 \\ \therefore (1-x)^{-1} &= \sum_{k=0}^{\infty} x^n\end{aligned}$$

□

Corollary 6.1. $||1 - x|| < 1 \Rightarrow x \in G(A)$.

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$1 \in A, x \in A$ such that $||x|| < 1 \Rightarrow 1 - x \in G(A)$ and also $(1-x)^{-1} = \sum_{i=0}^{\infty} x^n$.

Corollary 6.2. If $||1 - y|| < 1 \Rightarrow y \in G(A)$.

This is true since $B(1, 1) \subset G(A)$ and $y^{-1} = \sum_{i=0}^{\infty}(1-y)^n$.

Theorem 7. $1 \in A$. Then the following properties hold:

a) $G(A) = \{x \in A, \ \exists x^{-1}\}$ is open.

b) The map $x \mapsto x^-1$ is continuous, where $x, x^{-1} \in G(A)$.

Here is the proof of a).

Proof. Fix $b \in G(A)$. Then, we claim that

$\forall a \in A$ such that $||a - b|| < \dfrac{1}{||b^{-1}||} \geq a \in G(A)$.

Note that $B(b, \dfrac{1}{||b^{-1}||} \in G(A)$, where $||b^{-1}|| > 0$. Then,

$$\begin{aligned}||1 - ab^{-1}|| &= ||bb^{-1} - ab^{-1}|| \\ &= ||(b-a)b^{-1}|| \\ &\leq ||b - a|| \cdot ||b^{-1|}|| \text{ since Normed Algebra} \\ &< \frac{1}{||b^{-1}||} \cdot ||b^{-1}|| \\ &= 1\end{aligned}$$

$\Rightarrow ||1 - ab^{-1}|| = 1 \Rightarrow ab^{-1} \in G(A) \Rightarrow b \in G(A)$ (by Corollary 6.2) $\Rightarrow a = (ab)^{-1}b \in G(A)$. □

Now we prove (b). To do this, we first need an estimate:

Proof.

$$\begin{aligned}
||(ab^{-1})^{-1}|| &= ||\sum_{n=0}^{\infty}(1-ab^{-1})^n|| \\
&\leq \sum_{n=0}^{\infty}||(1-ab^{-1})^n|| \quad \text{(by triangle inequality)} \\
&\leq \sum_{n=0}^{\infty}||((1-ab)^{-1})^n|| \quad \text{(Normed Algebra)} \\
&\leq \sum_{n=0}^{\infty}||1-ab^{-1}||^n \quad \text{(sub-multiplicative)} \\
&= \sum_{n=0}^{\infty}||bb^{-1}-ab^{-1}||^n \\
&= \sum_{n=0}^{\infty}||(b-a)b^{-1}||^n \\
&\leq \sum_{n=0}^{\infty}||b-a||^n \cdot ||b^{-1}||^n \\
&= \sum_{n=0}^{\infty}||b-a||||b^{-1}||^n \\
&= \frac{1}{1-||b-a||\cdot||b^{-1}||}
\end{aligned}$$

The last line follows since $||a-b|| < \dfrac{1}{||b^{-1}||}$.

Then, we have the following:

$$\begin{aligned}
||a^{-1}-b^{-1}|| &= ||a^{-1}\cdot 1 - 1\cdot b^{-1}|| \\
&= ||a^{-1}bb^{-1} - a^{-1}ab^{-1}|| \\
&= ||a^{-1}(b-a)b^{-1}|| \\
&= ||b^{-1}(ab^{-1})^{-1}(b-a)b^{-1}|| \\
&\leq ||b^{-1}||||ab^{-1}||||b-a||||b^{-1||} \quad \text{(Normed Algebra)} \\
&= ||(ab^{-1})^{-1}||||b-a||||b^{-1}||^2 \\
&\leq \frac{1}{1-||b-a||b^{-1}}\cdot||b-a||\cdot||b^{-1}||^2 \quad \text{(by Estimate)}
\end{aligned}$$

Note that $t \mapsto \dfrac{||b^{-1}||^2 t}{1-t||b^{-1}||} \to 0$ when $t=0$ i.e. when $||b-a|| \to 0$
$\iff ||a^{-1}-b^{-1}|| \to 0 \Rightarrow$ Continuous ∎

Definition 1.5. $x \in A$, (spectrum) $\sigma_A(x) \subset \mathbb{C}, \lambda \in \mathbb{C}, x - \lambda 1 \notin G(A)$.

Theorem 8. $1 \in A, x \in A \Rightarrow \sigma_A(x)$ is nonempty compact subset of $\mathbb{C}$.

Theorem 9. σ_A is closed.

(a) $\sigma_A(x) \subset \bar{D}(0, ||x||)$.

Proof. Pick $\lambda > ||x||$. Then,

$$\begin{aligned}
||\frac{1}{\lambda}x|| &= \frac{1}{\lambda}||x|| < 1 \\
\Rightarrow 1 - \frac{1}{\lambda}||x|| &\in G(A) \text{ (by Lemma)} \\
\Rightarrow \lambda 1 - x &\in G(A) \\
\Rightarrow \lambda &\notin \sigma_A(x) \\
\sigma_A &\text{ is bounded}
\end{aligned}$$

Since we have closed and bounded, it is compact. □

(Fact; Every Banach Algebra has a dual functional).
$\phi \in A^* = \{\phi A \to \mathbb{C}\}$.
$\lambda \in \mathbb{C} \mapsto \phi((\lambda 1 - x)^{-1}) \in \mathbb{C}$. Define $\Psi = (\lambda 1 - x)^{-1}$.

Proof. Assume by way of contradiction that $\sigma_A(x) = \emptyset$. Then, the resolvent set $\rho_A(x) = \mathbb{C}$.
$\forall \lambda \in \mathbb{C}, \lambda 1 - x \in G(A)$. Then, we have the following:

$$\begin{aligned}
&\Psi(\lambda) - \Psi(\lambda_0) \\
&= \phi((\lambda - x)^{-1} - (\lambda_0 1 - x)^{-1}) \\
&= \phi((\lambda 1 - x)^{-1}(\lambda 1 - x - = -(\lambda_0 1 - x)(\lambda_0 1 - x)^{-1})) \\
&= (\lambda - \lambda_0)\phi(\lambda 1 - x)^{-1}(\lambda_0 1 - x)^{-1})) \\
\lim_{\lambda \to \lambda_0} \frac{\Psi(\lambda 1) - \Psi(\lambda_0)}{\lambda - \lambda_0} &= \phi(\lambda 1 - x)^{-1}(\lambda_0 1 - x)^{-1} \\
&= \phi(\lambda_0 1 - x)^{-2})
\end{aligned}$$

□

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30 August 2021 We continue with the proof.

Recall $1 \in A, \forall x \in A, \sigma_A(x) = \{\lambda \in \mathbb{C} : (\lambda 1 - x) \notin G(A)\}$. Then, $\sigma_A(x) \neq 0$ and **compact**.

(Continued) $\rho_A(x) = \mathbb{C} - \sigma_A(x)$. Fix $\phi \in A^*$, where the dual $A^* = \{\phi A \to \mathbb{C}$ is linear, continuous $\}$. Note that

Let $\lambda \in \rho_A(x) \subset \mathbb{C} \to \mathbb{C}$.
Note that $|||\phi||| = \sup_{||x|| \leq 1} |\phi(x)|$. $(A^*, ||| \cdot |||)$ is a Banach space.

Proof. (Continued)
Take $(\lambda 1 - x) \in \rho_A(x)$. Then,
$\phi(\lambda 1 - x)^{-1}$ i.e. $\lambda \mapsto \phi(\lambda 1 - x)^{-1}$.

Observe

$$\begin{aligned}
(\lambda 1 - x)^{-1} - (\lambda 1 - x)^{-1} &= (\lambda_0 1 - x)^{-1}(\lambda_0 1 - x)^{-1}(\lambda 1 - x)^{-1} \\
\Rightarrow \phi(\lambda_0 1 - x)^{-1} - (\lambda 1 - x)^{-1}) &= \phi(\lambda - \lambda_0)(\lambda_0 1 - x)^{-1}(\lambda 1 - x)^{-}) \\
&= (\lambda - \lambda_0)\phi((\lambda_0 1 - x)^{-1}(\lambda 1 - x)^{-1}) \\
\Rightarrow \phi((\lambda_0 1 - x)^{-1}) \cdot \phi(\lambda 1 - x)^{-1} &= \phi(\lambda - \lambda - \lambda_0)(\lambda_0 1 - x)^{-1}(\lambda 1 - x) \\
&= (\lambda - \lambda_0)\phi(x_0 1 - \lambda)^{-1})(\lambda_1 - x)^{-1}
\end{aligned}$$

Then, we have

$$
\begin{aligned}
\frac{\Psi(\lambda 1)-\Psi(\lambda)}{\lambda-\lambda_0} &= \frac{\phi(\lambda_0 1-x)^{-1}-\phi(\lambda 1-x)^{-1})}{\lambda-\lambda_0} \\
&= \phi((\lambda_0 1-x)^{-1}(\lambda-x)^{-1}) \\
\Rightarrow \lim_{\lambda\to\lambda_0}\frac{\Psi(\lambda)-\Psi(\lambda_0}{\lambda-\lambda_0} &= \lim_{\lambda\to\lambda_0}\phi(\lambda_0 1-x)^{-1}(\lambda 1-x)^{-1})) \\
&= \phi(\lambda_0 1-x)^{-1})^2 \\
&= \Psi \in H(\rho_A(x)),
\end{aligned}
$$

where H is the family of **holomorphic** functions.

Now we show that $\rho_A(x) \neq \emptyset$. □

Proof. Assume by contradiction, $\sigma_A(x)=\emptyset \Rightarrow \rho_A(x)=\mathbb{C}$.

Ψ is entire. Next, we note that

$$
\begin{aligned}
(\lambda 1-x)^{-1} &= \lambda^{-1}(1-\frac{x}{\lambda})^{-1} \\
&= \lambda^{-1}\sum_{n=0}^{\infty}(\frac{x}{\lambda})^n \\
&= \sum_{n=1}^{\infty}\frac{x^{n-1}}{\lambda^n} \\
\Rightarrow \phi(\lambda 1-x)^{-1})) = \Psi(\sum_{n=1}^{\infty}\frac{x^{n-1}}{\lambda^n}) &= \sum_{i=1}^{n}(\phi(\frac{x^{n-1}}{\lambda^n})) \\
&= \sum_{i=1}^{n}\frac{1}{\lambda^n}\phi(x^{n-1})
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
||\phi(\lambda)| &= |\sum\frac{1}{\lambda^n}\phi(x^{n-1})| \\
&= \sum_{n=1}^{\infty}|\frac{1}{\lambda^n}\phi(x^{n-1})| \\
&\leq \sum_{n=1}^{\infty}\frac{1}{\lambda}|||\phi|||\cdot||x^{n-1}||
\end{aligned}
$$

Side Note

$$
\begin{aligned}
|\phi(x)| &\leq |||\phi||| \quad \forall x \leq 1 \\
|\phi(\frac{y}{||y||})|| &\leq |||\phi||| \\
\frac{1}{|||y|||}\cdot|\phi(y)| &\leq |||\phi||| \\
|\phi(y)| &\leq |||\phi|||\cdot||y|| \\
&\leq \sum_{n=1}^{\infty}\frac{1}{|\lambda|^n}|||\phi|||\cdot||x||^{n-1}
\end{aligned}
$$

Then, by geometric series

$$|\phi(\lambda_1)| \leq \frac{|||\phi|||}{|\lambda|} \cdot \sum_{n=0}^{\infty} (\frac{||\lambda||}{\lambda})^n$$
$$= \frac{|||\phi|||}{|\lambda|} \cdot \frac{1}{1 - \frac{||x||}{\lambda}}$$
$$\Rightarrow \lim_{x \to \infty} |\phi(\lambda 1)| \leq \lim_{x \to 0} \underbrace{\frac{|||\phi|||}{|\lambda}}_{\to 0} \cdot \underbrace{\frac{1}{1 - \frac{||x||}{\lambda}}}_{\to 1}$$

By Liouville Theorem, Ψ is constant $\Rightarrow \Psi = 0$.

Then, $\phi(\lambda 1 - x)^{-1} = 0$ for every ϕ. This contradicts Hahn-Banach theorem (since we will have $\phi = \emptyset$ for a nonempty input).

$\therefore \sigma_A(x) \neq \emptyset$. $\square$

Corollary 9.1. $1 \in A$ such that $G(A) = A - \{0\}$.

Proof. Pick $x \in A \Rightarrow \sigma_A(x) \neq \emptyset$.

Then, $\exists \lambda \in \sigma_A(x) \Rightarrow \lambda 1 - x \notin G(A) \Rightarrow \lambda 1 - x = 0 \iff x = \lambda 1$. $\square$

Theorem 10. $f(\lambda) = \sum_{n=0}^{k} a_n x^n$.

$x \in A, f(x) = \sum_{n=0}^{k} a_n x^n$.

Theorem 11. $x \in A, f(\sigma_A(x)) = \sigma_A(f(x))$.

Proof. $\lambda \notin \sigma_A(x) \Rightarrow$

$$f(x) - f(\lambda) = \sum_{n=0}^{k} a_n x^n - \sum a_n \lambda^n$$
$$= \sum_{n=0}^{k} a_n (x^n - \lambda^n)$$
$$= \sum_{n=0}^{k} a)n(x - \lambda 1)(\sum_{j=0}^{n-1} x^j x^{n-1-j})$$
$$\Rightarrow f(x) - f(\lambda)1 = (x - \lambda 1) \sum_{n=0}^{\infty} a_n (\sum_{j=0}^{n-1} x^j x^{n-1-j})$$

$\lambda \in \sigma_A(x) = (x - \lambda 1) \notin G(A) \Rightarrow f(x) - f(\lambda)1 \notin G(A) \Rightarrow f(\lambda) \notin \sigma_A(f(x)) \Rightarrow f(\sigma_A(x)) \subset \sigma_A(f(x))$.

Pick $\mu \notin f(\sigma_A(x))$. The,

$f(x) - \mu 1 = a_n (x - \lambda_1)(x - \lambda_2 ...$ (can factor a polynomial)

$\lambda \in \sigma_A(x)$. Then,

$f(\lambda) - \mu \neq 0 \iff \lambda_1, \lambda_2, ... \neq 0 \iff$ All functions invertible $\iff f(x) - \mu(1)$. $\square$

1.3 Three Pillars of Functional Analysis

1.3.1 Hahn-Banach

$x \neq 0$ and $\phi \in A^*$ and Re $(\phi(x)) \not\to 0$. (Can separate points).

1.3.2 Banach-Steinhaus or Uniform boundedness principle

1.3.3 Open Mapping Theorem

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Theorem 12. $1 \in A$ Banach Algebra, $x \in A$, then

$$r(x) = \lim_{n\to\infty} ||x^n||^{1/n}; \quad r(x) = \sup_{\lambda\in\sigma_A} (x)$$

Proof. Observe that $r(x^n) = (r(x))^n$.

Definition 1.6.

$$\begin{aligned} r(x^n) &= \sup_{\lambda\imath\sigma_A(x^n)} |\lambda| \\ &= \sup_{\lambda\in(\sigma_A(x))^n} \\ &= \sup_{\lambda_0\in\sigma_A(x)} |\lambda_0^n| \\ &= \sup_{\lambda_0\in\sigma_A(x)} |\lambda_0|^n \\ &= (\sup_{\lambda_0\in\sigma_A(x)} \lambda_0|)^n \\ &= r(x)^n \end{aligned}$$

$\Rightarrow (r(x))^n = r(x^n) \leq ||x^n||$
$\Rightarrow r(x) \leq ||x^n||^{1/n} \Rightarrow r(x) \leq \liminf_{n\to\infty} ||x^n||^{1/n}$.
(We also need to show that $r(x) \geq \limsup_{n\to\infty} ||x^n||^{1/n}$. □

Let $\Sigma = \{z \in \mathbb{C} | |z| > r(x)\} = \mathbb{C} - \overline{B(0, r(x))}$
Fix $\phi \in A^*$.

$$\begin{array}{ccc} \lambda & \longrightarrow & (\lambda - x)^{-1} \\ & & \downarrow \\ & & \phi((\lambda - x)^{-1}) \end{array}$$

$\mathbb{C} \to \mathbb{C}$ $\lambda \mapsto \phi(\lambda - x)^{-1}) \in H(\rho_A(x))$
$\lambda| > ||x||$

$$\begin{aligned} (\lambda - x)^{-1} &= \frac{1}{\lambda}\sum_{n=0}^{\infty}(\frac{x}{\lambda})^n = \sum_{n=0}^{\infty}\frac{x^{n-1}}{\lambda^n} \\ \phi((\lambda - x)^{-1}) &= \phi(\sum_{n=0}^{\infty}\frac{x^{n-1}}{\lambda^n}) \\ &= \sum_{n=0}^{\infty}\frac{\phi(x^{n-1}}{\lambda^n} \\ &= \sum_{n=1}^{\infty}\frac{a_n}{\lambda^n} \end{aligned}$$

$\Rightarrow \lim_{n\to\infty} |\frac{\phi(x^{n-1})}{\lambda^n}| = 0 \iff \lim_{n\to\infty} |\phi(\frac{x^{n-1}}{\lambda^n})| = 0.$
Let $y_n = \dfrac{x^{n-1}}{\lambda^n} \in A$. Then,
$\lim_{n\to\infty} \phi(y_n) = 0$ for all $\phi \in A^*$.

$$A^* : \{\phi : A \to \mathbb{C} | \text{ continuous, linear}\}$$

$A \subset A^{**}$.

$x \in A \rightsquigarrow x \in A^{**}$.

$x(\phi) := \phi(x)$

$$\begin{aligned}
||||x|||| &= \sup_{||||x||||\leq 1} |x(\phi)| \\
&= \sup_{||||x||||\leq 1} ||phi(x)| \\
&\leq \sup_{||||x||||\leq 1} ||||\phi(x)||| \cdot ||x|| \\
&= ||x||
\end{aligned}$$

2 Banach-Steinhaus Theorem

Theorem 13. $T_n : X \to Y$ linear, bounded.

$\forall x \in X, \exists C_x \geq 0$ such that $||T_n(x)||_Y \leq C_x \ \forall n \Rightarrow \exists c_1$ such that $||T_n|| < c \forall n$.

Proof. Fix $n, k \in \mathbb{N}$.

$\bar{A_k} = A_k = \{x \in X | T_n(x) \subset K \ \forall n\}$

$\forall x \in X, \ \exists k > 0, x \in A_k$.

$X = \cup A_k$ second Baire Category Theorem (A complete metric space cannot be expressed as a countable union of nowhere dense subsets).

$\exists k$ such that $A_k^o \neq \emptyset$.

$\exists r > 0, x_0 \in X$ such that $B(x_0, r) \subset A_k$.

$||T_n(x)|| < k \ \forall x \in B(x_0, r)$.

$x \in B(x_0, r) \Rightarrow x = x_0 + ry, \ ||y|| \leq 1$. Then,

$$\begin{aligned}
||T_n(x_0 + ry)|| &\leq k \quad \forall ||y|| \leq 1 \\
||T_n(ry)|| - ||T_n(x_0)|| &\leq k \\
|r|||T_n(y)|| &\leq k + ||T_n(x_0)|| \\
||T_n(y)|| &\leq \frac{k + T_n(x_0)||}{|r|} \\
\Rightarrow \sup_{||y||\leq 1} ||T_n(y)|| &\leq \frac{k + T_n(x_0)}{|r|}
\end{aligned}$$

Note that $||T_n(x_0)|| < k$. Then,

$\lim_{n\to} y_n(\phi) = 0$ where $y_n : A^* \to \mathbb{C}$.

Banach-Steinhaus: $\exists c > 0, ||y_n|| = ||||y_n|||| \leq c \forall n \in \mathbb{N}$.

$$\begin{aligned}
c &\geq \|\frac{x^{n-1}}{\lambda^n}\| \ \forall n \\
c &\geq \|\frac{x^{n-1}}{|\lambda|^n}\| \\
c\|x| \ &\geq \frac{\|x^n\|}{|\lambda|^n} \\
c|\lambda|^n\|x\| &\geq \|x^n\| \quad \forall n \\
c^{1/n}|\lambda|\|x\|^{1/n} &\geq \|x^n\|^{1/n} \quad \forall n \\
\limsup_{n\to\infty} c^{1/n}|\lambda|\|x\|^{1/n} &\geq \limsup_{n\to\infty} \|x^n\|^{1/n} \\
|\lambda| &\geq \limsup_{n\to\infty} \|x^n\|^{1/n} \quad \forall|\lambda| > r(x) \\
r(x) &\geq \limsup_{n\to\infty} \|x^n\|^{1/n}
\end{aligned}$$

$\square$

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3 Normal, Self-Adjoint, Isometry

Recall $1 \in A$ Banach Algebra. and $r(x) = \lim_{x \to 0} ||x^n|^{1/n}$.

Definition 3.1. $1 \in A$. (with an involution).

$x \in A$ is called **normal** iff $xx^* = x^*x$.

$x \in A$ is called **self-adjoint** iff $x = x^*$.

Exercise $x \in A, x^*x, xx^* \in A$, then we have

$(x^*x)^* = x^* \cdot (x^*)^* = x^*x$.

Proposition 1. $1 \in A$ (involutive Banach algebra, $x \in A$ is normal.

If in addition, A is a C^*-algebra, then $r(x^*x) = r(x)$.

Proof.

$$\begin{aligned}
(x^*x)^n &= (x^*)^n x^n \\
\lim_{n\to\infty} ||(x^*x)^n||^{1/n} &= \lim_{n\to\infty} ||x^{*n}x^n||^{1/n} \\
&\leq \lim_{n\to\infty} ||(x^*)^n|| \cdot ||x^n||^{1/n} \\
&= \lim_{n\to\infty} ||(x^n)^*|| \cdot ||x^n||^{1/n} \\
- &= \lim_{n\to\infty} ||x^n||^{2/n} \\
&= \lim_{n\to\infty} (||x^n||^{1/n})^2 \\
&= r(x)^2
\end{aligned}$$

$\iff r(x^*x = \lim_{x\to\infty} ||x^n||^{1/n}$. ☐

Proposition 2. A a C^*-algebra. If $x \in A$ is normal, $r(x) = ||x||$.

Proof. Suppose $x \in A^n \Rightarrow \iff x = x^*$ (self-adjoint).

Then, we have $||x^*x|| = ||x||^2$ (since it is C^*).

$\Rightarrow ||x^2|| = ||x||^2$.

By induction, we have $||x^{2^n} = ||x||^{2^n} \Rightarrow ||x^{2^n}||^{1/2^n} = ||x|| \; \forall n$.

Taking the limit, we get, $\lim_{n\to\infty} ||x^{2^n}||^{1/2^n} = r(x) = ||x||$ when x is self-adjoint $x = x^*$.

Finally, we have the following proof:

$$\begin{aligned}
r(x)^2 &= r(x^*x) \\
&= ||x^*x|| \\
&= ||x||^2
\end{aligned}$$

$\iff r(x) = ||x||$ when x is normal. ☐

Proposition 3. Suppose A, B any two unital C^*-algebras. Let $\Psi : A \to B$ be any $*$-homomorphism (preserves composition, one-to-one, and $(\Psi(x^*)) = (\Psi(x))^*$.

Contraction Then, $\forall x \in A, ||\Psi(x)||_B \leq ||x||_A$.

Moreover, if Ψ is a $*$-isomorphism, then $||\Psi(x)|| = ||x||$ for all x is an **isometry**.

Proof. Note that $\Psi(G(A)) \subset G(B)$.
Fix $x \in A$. Let $\lambda \notin \sigma_A(x)$. Then, $\lambda \in \rho_A(x)$.

$$\begin{aligned}
\lambda 1 - x &\in G(A) \\
\Rightarrow \Psi(\lambda 1 - x) &\in G(B) \\
\Rightarrow \lambda\Psi(1) - \Psi(x) = \lambda 1_B - \Psi(x) &\in G(B) \\
\Rightarrow \lambda &\notin \sigma_B(\Psi(x))
\end{aligned}$$

Then, we take the complement. So,

$$\sigma_B(\Psi(x)) \subset \sigma_A(x)$$

.

$$\sup_{\lambda \in \sigma_B(\Psi(x)} |\lambda| \leq \sup_{\lambda \in \sigma_A(x))} |\lambda| = r(x).$$

Trick

$$\begin{aligned}
||\Psi(x)||^2 &= ||\Psi(x))^*\Psi(x)|| \\
&= ||\Psi(x^*)\Psi(x)|| \\
&= ||\Psi(x^*x)|| \text{ (since self-adjoint)} \\
&= ||r(\Psi(x^*x))|| \\
&\leq r(x^*x) \\
&= ||x^*|| \\
&= ||x||^2 \\
\Rightarrow ||\Psi(x)|| \leq ||x|| \ \forall x \\
\Rightarrow ||\Psi^{-1}(x)|| \leq ||x||
\end{aligned}$$

□

Corollary 13.1. $\exists$ at most $1C^*$-norm on a given involutive Banach algebra.

Proof. Suppose $(A, ||\cdot||_1 \xrightarrow{\text{Id}} (A, ||\cdot||_2$
Id-map is a $*$-homomorphism. By Proposition 3, $||x||_2 = ||\text{Id}||_2 = ||x||_1$. □

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Proposition 4. Suppose $1 \in A$ is a C^*-algebra, $x \in A^n$. Then spectrum $\sigma_A(x) \subset \mathbb{R}$.

Proof. Fix $\lambda \in \sigma_A(x) = \alpha + i\beta$.
Fix $t \in \mathbb{R}$ and consider $y_t = x - \alpha + it \in A; y(t)$ is normal. Then,

$$\begin{aligned}
y_t * y_t &= (x - \alpha + it)^*(x - \alpha + it) \\
&= ((x - \alpha - it)(x - \alpha + it) \\
y_t y_t^* &= (x - \alpha + it)(x - \alpha - it)
\end{aligned}$$

Then,

$$\begin{aligned}
|i(t+\beta)|^2 &\leq r^2(x) \in \sigma_A(y_t) \\
|t+\beta|^2 &\leq r^2(x) = ||y_t||^2 = ||y_t^* y_t|| \\
&= ||(x - \alpha - it)(x - \alpha + it)|| \\
&= (x-\alpha)^2 + t^2|| \\
&\leq ||(x-\alpha)^2|| + t^2 \\
\Rightarrow t^2 + 2\beta t + \beta^2 &\leq ||x - \alpha||^2 + t^2 \qquad \forall t \in \mathbb{R} \\
2\beta t + \beta^2 &\leq ||x - \alpha||^2 \qquad \forall
\end{aligned}$$

This is true only when $\beta = 0$. $\square$

Lemma 14. $1 \in A, C^*$-algebra, $x \notin G(A), \exists \{x_n\}_n \subset G(A), x_n \to x$. Then, $\lim_{n\to\infty} ||x_n^{-1}|| = \infty$.

Proof. Assume by contradiction, $\exists c > 0$ such that $||x_n^{-1}|| \leq c,\ \forall n$.

$$\begin{aligned} ||1 - x_n^{-1}x|| &= ||x_n^{-1}x_n - x_n^{-1}x|| \\ &\leq ||x_n^{-1}||||x_n - x|| \\ &\leq c||x_n - x|| \end{aligned}$$

$\exists n_0 > 0$ such that $||1 - x_{n_0}^{-1}x|| \leq c||x_{n_0} - x|| < 1$
$\Rightarrow x_{n_0}^{-1}x \in G(A) \Rightarrow x \in G(A)$. $\square$

4 Examples of C^*-algebra

Example 15. $C([0,1]) = \{f : [0,1] \to \mathbb{C}|$ continuous $\}$ with norm $||f||_\infty = \sup|f(x)|$.

Example 16. $B(H) = \{T : H \to H|$ linear, bounded$\}$, where H is a Hilbert space. Then, $(B(H), +, \lambda, \circ)$ is an algebra.

$||T||_\infty = \sup_{||\xi||\leq 1} \sup ||T(\xi)||$.

$||T \circ U(\xi)|| \leq ||T|| \cdot ||U(\xi)|| \leq ||T|| \cdot ||U|| \cdot ||\xi||$

$\Rightarrow ||T \circ U|| \leq ||T|| \cdot ||U||$

$\forall T : H \to H \rightsquigarrow T^* : H \to T$

$< T\xi, \eta > = < \eta . T^*\eta > \ \ \forall \xi, \eta \in H$.

Theorem 17 (Riesz Representation Theorem). $f : H \to \mathbb{C}$ linear, continuous $\exists! z \in H$ such that $f(x) = < x, z >$.

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$(B(H), +, \lambda, \cdot, *, ||\cdot||_\infty)$ Banach algebra.

Theorem 18. Operator $T \in B(H) \mapsto T^* \in B(H)$.

Theorem 19 (Riesz Representation Theorem). $\forall$ functional $\phi \in H^*$ (dual) $\exists v \in H$ such that $\phi(z) =< z, v > \ \forall z \in H$.

Proof. Assume $\phi \neq 0$.

Dual $H^* = \{\phi : H \to \mathbb{C}|$ continuous , linear $\}$

$\ker(phi) = \phi^{-1}(\{0\})$.

Closed Subspace Let $K = \ker(\phi) = \overline{\ker(\phi)}$. Then,

$K^\perp = \{w \in H| < w, k >= 0\ \forall kinK >$.

$H = K \bigoplus K^\perp$. □

Theorem 20. H is a **Hilber t space** and $K \subset H$, closed, and convex

Definition 4.1 (Convex). $\forall k_1, k_2 \in K, tK + (1-t)k_2 \in K\ \forall 0 \leq t \leq 1$.

Corollary 20.1. $\forall K \leq \overline{K} \leq H, H = K \bigoplus K^\perp$

Proof. Let $x \in H$. By Theorejm, choose $P_k(x)$ (not necessarily linear) such that

$x = P_k(x) + \underbrace{(x - P_k(x))}_{\in K^\perp}$

(Note that uniqueness follows: $y_1 + y_2 = x = x_1 + x_2$.

Let us look at $||x - P_k(x) \leq \inf ||x - n|\forall n \in K$.

Write $||x - P_k(x)| \leq ||x - P_k(x) + m||\ \forall m \in K$. Then,

$$\begin{aligned}
||x - P_k||^2 &\leq ||x - P_k + m||^2 \\
\iff ||x - P_k(x)||^2 &\leq ||x - P_k||^2 + 2 \text{ Re} < x - P_k(x), m > + ||m||^2\ \forall m \in K \\
\iff -2 \text{ Re} < x - P_k(x), tm > &\leq ||tm||^2\ \forall m \in K, t \in \mathbb{C} \\
\Rightarrow t \text{ Re } < x - P_k(x), m > &\leq |t|^2|m|^2 \\
\Rightarrow < x - P_k(t)(x), m > &= 0\ \forall m
\end{aligned}$$

□

Also ntoe that $T^* : H \to H$ is linear.

$$\begin{aligned}
| < z, T^* y > | &\leq || < Tz, y > || \text{ bounded operator} \\
&\leq ||Tz|| \cdot ||y|| \\
&\leq ||T||_\infty ||z|| ||y||\ \forall z, y \\
\Rightarrow | < z, T^* y > | &\leq ||T||_\infty \cdot ||y|| \\
\therefore ||T^*||_\infty &\leq ||T||_\infty
\end{aligned}$$

Show $T = (T^*)^*$.

$$\begin{aligned}
< T^* z, y > &=< z, (T^*)^* y > \\
< y, T^* z > &= \overline{< Ty, z >} \\
&=< z, Ty >
\end{aligned}$$

$\Rightarrow (T^*)^* y = Ty \Rightarrow (T^*)^* = T$. It is an involution.

C^*-axiom $||T||^2 = ||T^*T|| \leq ||T^*|| \cdot ||T|| \geq ||T||^2$.

$$\begin{aligned}
||T||^2 &= \sup_{||z|| \leq 1} ||Tz||^2 \\
&= \sup_{||z|| \leq 1} < Tz, Tz > \\
&\leq \sup_{||z||, ||\eta|| \leq 1} < Tz, T\eta > \\
&= \sup_{||z||, ||\eta|| \leq 1} < T^*Tz, \overline{\eta} > || \\
&= \sup_{||z|| \leq 1} ||T^*Tz|| \\
&= ||T^*T||
\end{aligned}$$

$\therefore, B(H)$ is a C^*-algebra.

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Exercises

1. Show that $(B(H), ||\cdot||_\infty)$ is a complete space.
2. Suppose $(A, ||\cdot||)$ is a Banach Algebra.
   $\bar{I} = I \subset A$ closed ideal. Then show that $(A/I, ||\cdot||_I$ is a Banach Algebra, where

$$||\alpha + I||_I = \inf_{x\in I} ||\alpha + x||_A \leq ||\alpha||.$$

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5 Gelfand Transform

Today we will discuss the spectrum of an algebra instead of just a n element.

Suppose $1 \in A$ is a unital Banach Algebra. Consider

$\sigma(A) = \{\phi : A \to \mathcal{C} |$ algebra homomorphism $\}$. Then, the following conditions are satisfied

$$\begin{aligned} \phi(1) &= 1 \\ \phi(x+y) &= \phi(x+y) \\ \phi(\lambda x) &= \lambda\phi(x) \\ \phi(xy) = \phi(x)\phi(y) \end{aligned}$$

Observation 1 $\forall x \in A, \phi(x) \in \sigma(x)\ \forall \phi \in sigma(A)$.

Note that $\phi(x)\cdot 1 - x \in \ker(\phi)$.

$$\begin{aligned} &\Rightarrow \phi(\phi(x)1 \cdot x) = \phi(x)\phi(x) - \phi(x) = 0 \\ &\phi(x)\cdot 1 - x \in \ker(\phi). \end{aligned}$$

Recall that $\ker(\phi)$ is an ideal (two-sided) of A.

$\phi(x) = 0 \Rightarrow \phi(ax) = \phi(a)\phi(x) = 0\ \forall x \in A$

$\ker(\phi) \cap G(A) = \emptyset \iff \phi(x)1 - x \notin G(A) \Rightarrow \phi(x) \in \sigma(x)$.

Then, $||\phi(x)|| \leq r(x) \leq ||x||\ \forall x \in A \Rightarrow ||\phi|| \leq 1 \Rightarrow ||\phi|| = 1$. (since it is unital).

$\Rightarrow \sigma(A) \leq \{\phi \in A^* | ||\phi|| =\}$.

Theorem 21 (Spectral Algebra Theorem). There is a correspondence (bijection) between $\phi \in \phi(A) \to \ker(\phi) \in M(A)$ (co-dimension 1 $\iff$ maximal ideal)

Proof. Fix $\phi \in \sigma(A)$. Take $I = \ker(\phi)$.

Fix $I \not\leq J \leq A, \Rightarrow \exists x \in J - I$.

Consider $\phi(x) \neq 0$ and $1 - \dfrac{1}{\phi(x)} \cdot x \in \ker(\phi) = I$. □

Obs 1

$$1 = \underbrace{1 - \frac{1}{\phi(x)}x}_{\in J} + \underbrace{\frac{1}{\phi(x)}}_{\in J}$$

$\Rightarrow 1 \in J \Rightarrow J = A$.

Obs 2 If $I \in M(A)$ (maximal ideal) $\Rightarrow I = \bar{I}$.

Maximal ideal are closed. We need to show $I \subset \bar{I} \subset A$.

We want to exclude $\bar{I} = A$.

Proof. Suppose not.

Suppose $\bar{I} = A$ (unital) $\Rightarrow \exists y \in I$ such that $||1-y|| \leq 1 \Rightarrow y \in G(A) \Rightarrow I = A$, contradiction. □

Also note that for a fixed $I \in M(A) \to A/I$ (field since I is maximal).

We know that $A/I \equiv \mathbb{C}$.

$$\begin{tikzcd} \alpha \arrow[r, "\rho"] \arrow[dr, "\pi\circ\rho"'] & A \arrow[d, "\pi"] \\ & \mathbb{C} \end{tikzcd}$$

5.1 Weak-* Topology

Let $A^* = \{\phi : A \to \mathbb{C}, \text{linear, continuous}\}$ be any normed space.

This topology is defined as $\phi_i = \phi$ iff $\phi_i(x) \mapsto \phi(x)$ pointwise $\forall x \in A$. [Note that the unit ball is weak-* compact]

Theorem 22 (Banach-Alaoglu). $(A^*)_1$ is compact in weak*-topology.

Proof. (Proof Idea)

$f \in A^*, f : A \to \mathcal{C}$. Then, we have

$$\begin{aligned} & ||f|| \leq 1 \\ \iff & |f(x)| \leq ||x|| \ \forall x \\ f(x) \in D_{||x||}(0) & \end{aligned}$$

By Tychonhoff's Theorem, product of compact spaces is compact $(f(x))_{x\in A} \in \Pi_{x\in A} D_{||x||}(0)$ (x-tuple of closed unit discs).

Hence, we have $f_n(x) \to f(x) \iff$ each coordinate converges. □

Theorem 23. $\sigma(A) \subset (A^*)_1$ is a nonempty, compac set in weak-* topology.

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Theorem 24. $1 \in A$ (Banach Algebra) $\Rightarrow \sigma(A) \neq \emptyset$ is weak*-compact.

Proof. $x \notin G(A) \Rightarrow \exists I \subset A$ maximal ideal.
$x \in I \Rightarrow \exists \phi \in \sigma(A)$ by correspondence theorem. $I = \ker(\phi) \Rightarrow \phi(x) = 0$.
$\sigma(A) \neq \emptyset \subset A^*$ compact by **Banach-Alaoglu**. □

It is sufficient to show that $\sigma(A)$ is a weak*-closed.

Proof. $\phi \in \overline{\sigma(A)}^{W*} \Rightarrow \exists \phi_i \in \sigma(A) \to \phi W^*$
$\Rightarrow \phi_i(x) \to \phi(x)\ \forall x \in A$.
Fix $x, y \in A$. $\phi_i(x)\phi(y) = \phi_i(xy)\ \forall i \Rightarrow \phi(x) \cdot \phi(y) = \phi(xy)$ since $\phi_i(x) \to \phi(x), \phi_i(y) \to \phi(y)$ and $\phi_i(xy) \to \phi(xy)$.
Since the space is Hausdorff. limit is unique (similarly, this holds for addition, identity, etc.).
$\therefore \phi \in \sigma(A)$. □

Theorem 25. Suppose K is Hausdorff compact and $C(K) : \{f : K \to \mathbb{C}$ continuous $\}$. Then, $\exists$ a natural homomorphism

$$\begin{aligned} K &\to \sigma(C(K)) \quad \text{weak}^* \text{ topology} \\ k &\mapsto \phi_k \end{aligned}$$

defined by $\phi_k(f) = f(k)\ \forall k \in C(K)$.
This map is a homeomorphism.

Proof.

1. Show injective.

Suppose k_1 and k_2 such that

$$\begin{aligned} \phi_{k_1} &= \phi_{k_2} \\ \phi_{k_1}(f) &= \phi_{k_2}(f)\ \forall f \\ f(k_1) &= f(k_2)\ \forall \end{aligned}$$

Continuous function on a Hausdorff space is compact. Hence, because continuous functions separate points, $k_1 = k_2 \iff$ injective. □

Proof.

2. Show continuity i.e. $k_i \to k \Rightarrow \phi_{k_i} \xrightarrow{\text{weak}} \phi_k$.

Fix $f \in C(K) \Rightarrow \phi(k_i) \to \phi(k) \Rightarrow \phi_{k_i}(f) \to \phi(k(f))$ □

Exercise $M(C(K)) = \{f \in C(K) | f(K) = 0\}$. Show this is maximal.

Lemma 26. For any ideal $I \subset C(K)$, $\exists B \in K = 1$.

Definition 5.1 (Gelfand Transform)**.** $1 \in A$, take $C(\sigma(A))$ where $x \to (\Gamma(x))(\phi) = \phi(x)$.

Theorem 27. Γ is a homomorphism, contraction.
$A \in G(A) \iff \Gamma(a) \in G(C(\sigma(A))$.

Proof.

$$\begin{aligned}
&(\Gamma(x) \cdot \Gamma(y))(\phi) \text{ for } x, y \in A \\
&= (\Gamma(x)(\phi))\Gamma(y)(\phi)) \\
&= \phi(x)\phi(y) \\
&= \phi(xy) \\
&= (\Gamma(xy))(\phi)
\end{aligned}$$

□

Contraction $||\Gamma(x)||_\phi = \sup \Gamma(x)(\phi) = \sup_{\phi \in \sigma(A)} |\phi(x)|$

$a \in G(A) = 1 - \Gamma(1)\Gamma(a) = \Gamma(aa^{-1}) = \Gamma(a) \cdot \Gamma(a^{-1})$.

$\Leftarrow$ Assume $\sigma \neq G(A) \Rightarrow\ \exists \in G(A)(x)$ such that $\phi(a) = 0 \Rightarrow \rho(a)(\phi) \Rightarrow \rho(a) \notin G(\sigma(A))$.

Corollary 27.1. $\sigma(x) = \sigma(x^*)$

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Recall, **Gelfand Transform**:

$$\begin{aligned}\Gamma : A &\to C(\sigma(A)) \\ x &\mapsto \Gamma(x)(\phi) = \phi(x).\end{aligned}$$

Theorem 28. $a \in G(A) \iff \Gamma(a) \in C(G(A))$.

Corollary 28.1. $\sigma(a) = \sigma(\Gamma a)$.

$||a|| \geq r(a) = r(\Gamma(a)) = ||\Gamma(a)||$.

Theorem 29. Let $1 \in A, C^*$ algebra, **commutative**. Then, $\Gamma A \mapsto C(\sigma(A))$ is an **isometric $*$-isomorphism**.

(We need to upgrade homomorphism to $*$-homomorphism).

Proof.

(a) Show $\Gamma(x^* = \overline{\Gamma(x)}$.

(i) **Case 1:** (Self-adjoint) i.e. $x = x^*$.

$$\begin{aligned}x &= x^* \\ &\Rightarrow \sigma(x) \subset \mathbb{R} \\ &= \sigma(\Gamma(x)) \subset \mathbb{R}\end{aligned}$$

(Image is closed under $\mathbb{R}$, space where conjugation is the same).
Therefore, $\Gamma(x^*) = \Gamma(x) = \overline{\Gamma(x)}$.

(ii) **General Case** Let $x = a + ib$, where a, b are self-adjoint. Then,

$$\begin{aligned}x^* &= (a+ib)^* \\ &= a^* + (ib)^* \\ &= a - ib \\ \Rightarrow x + x^* &= (a+ib) + (a-ib) \\ &= 2a \\ a &= \frac{x+x^*}{2}\end{aligned}$$

Similarly, $x - x^* = (a+ib) - (a-ib) = 2ib \Rightarrow b = \dfrac{x-x^*}{2i} = \dfrac{i(x^*-x)}{2}$

$$\begin{aligned}\Gamma(x^*) &= \Gamma((a+ib)^*) \\ &= \Gamma(a-ib) \\ &= \Gamma(a) - i\Gamma(b) \\ &= \overline{\Gamma(a)}0i\overline{\Gamma(b)} \\ &= \overline{\Gamma(a) + i\Gamma(b)} \\ &= \overline{\Gamma(a + i\Gamma(b)} \\ &= \Gamma(x)\end{aligned}$$

(b) Now show isometry.

$$\Gamma(x)|| = r(\Gamma(x)) = r(x) = ||x||.$$

In addition, since x commutes with $x^* \iff$ Normal. This is injective since the kernel is trivial.

(c) Now show surjectivity.

$1 \in \Gamma(A) \subset C(\sigma(A))$

Note Contain constants, normed closed, and separate points.

Take $y \in \overline{\Gamma(A)} \Rightarrow \exists (y_n)_n \subset \Gamma(A), ||y_n - y|| \to C$.

$\Rightarrow (y_n)_n$ is a Cauchy sequence. Therefore, $||y_n - y_m|| \to \infty$ as $n, m \to \infty$.

$\Rightarrow y_n - \Gamma(x_n)$ for some $x_n \in A$. Then, we have:

$$\begin{aligned} ||y_n - y_m|| = ||\Gamma(x_n) - \Gamma(x_m)|| && = ||x_n - x_m|| \\ \to 0 \text{ as } n, m \to \infty \end{aligned}$$

$\Rightarrow$ Cauchy sequence $\Rightarrow \exists x \in A$ such that $x_n \Rightarrow x \Rightarrow \Gamma(x_n) \Rightarrow \Gamma(x) \Rightarrow y_n \to y$.

Therefore $\Gamma(A)$ separates the points of $\sigma(A)$.

(d) Fix $\phi_1 \neq \phi_2 \in \sigma(A)$. Then, **does there exist** $a \in A$ such that $\Gamma(a(\phi_1)) \neq F(a)(\phi_2) \iff \phi_1(a) \neq \phi_2(a)$.

(e) By **Stone-Weirstrauss Theorem**, $\Gamma(A) = C(\sigma(A))$. $\square$

6 Continuous Functional Calculus

6.1 Notation

1. $1 \in A = C^*$ algebra.
   $p \in A, p^2 = p$ (idempotent).
   $p^2 = p = p^*$ (projection).
2. $h \in A$, self- adjoint, $h = h^*$.
3. $u \in A$ is called **isometry** if $u^*u = 1$ (preserves norm).
4. $u \in A$ is called **partial isometry** if $u^*u =$ projection.

4b. $u \in A$ is **unitary** if $u^*u = uu^*$.

5. $x \in A$ is called **positive** if $x = y^*y$ for some $y \in A$.
   Take $A = \{\geq 0 | x \in A1\}$. $x = y$ for some $y \in A$.
   Let $A_+ = \{x \geq 0 | x \in A\}$.
   $\forall x \in A, xA_+x^*, x^*A_+x \subset A_+$.
   If you pick $z \in A_+, z = y^*y_- \Rightarrow xzx^* = xy^*yx^* = (yx^*)^*yx^*$.
6. **Partial order** $x \geq y \iff x - y \in A^+$.

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$$\begin{aligned} A &\to C(\sigma(A)) \\ x &\mapsto \Gamma(x)(\phi) = \phi(x) \end{aligned}$$

is an isometric isomorphism.

6.2 Continuous Functional Calculus

$$\sigma(A) \cong \sigma(x) \qquad \sigma(x) = \sigma(\Gamma(x))$$

Consider

$$\begin{aligned} \Gamma^{-1} : C(\sigma(x)) &\to A \\ f &\mapsto f(x) \in C^*(x, x^*, 1) \end{aligned}$$

Theorem 30. Let A, B, C^*-algebras with units. Let $x \in A$ such that $xx^* = x^*x$. Then, the functional calculus satisfies:

1. $$\begin{aligned} C(\sigma(A)) &\to A \\ f &\mapsto f(x) \end{aligned}$$
   is a $*$- homomorphism $\forall f = \sum_{k,\ell} a_{k\ell} z^k z^\ell$, $f(x) = \sum_{k,\ell} a_{k\ell} x^k x^{*\ell}$.
2. $\forall f \in C(\sigma(x))$, $\sigma(f(x)) = f(\sigma(x))$
3. If $\Phi : A \to B, \Phi(f(x)) = f(\Phi(x))$.
4. If $x_n \subset A$ normal, $x_n \xrightarrow{||\cdot||} x \Rightarrow$ Then $\forall \Omega \supset \sigma(x)$ compact neighborhood $\exists n > 0$ such that $\sigma(x_n) \subset \Omega$ and $\forall f \in C(\Omega)$,
   $$f(x_n) \xrightarrow{||\cdot||} f(x).$$

Proof. (1), (2) $\Rightarrow$ (3)
$f = \sum_{ak\ell} z^k \bar{z}^\ell$.
$\Phi(f(x)) = \Phi(\sum a_{k\ell} x^\ell x^{*k}) = \sum a_{k\ell} \Phi(x)^k \Phi(x)^{*\ell} = f(\Phi(x))$
$\forall f,\ \exists f_n, f_n \xrightarrow{\text{uniformly}} f_{\sigma(x)}$.
$||f_n - f||_\infty \to 0$

$$\begin{aligned}
||\Phi(f(x)) - f(\Phi(x))|| &= ||\Phi(f(x)) - \Phi(f_n(x)) + \Phi(f_n(x)) - f_n(\Phi(x)) + f_n(\Phi(x)) - f(\Phi(x))|| \\
&\leq ||\Phi(f(x) - f_n(x))|| + ||f_n(\Phi(x)) - f(\Phi(x))|| \\
&\leq ||f(x) - f_n(x)|| + ||f_n(\Phi(x)) - f(\Phi(x))|| \\
&\leq 2||(f - f_n)(x)||\ \forall n \\
&\to 0
\end{aligned}$$

4. $x \to x^{-1}$ is continuous and $||x_n - x|| \to 0$ gives the first part.
$c = \sup_n ||x_n|| < \infty$
$f \in C(\Omega),\ \exists g$ polynomial, $||f - g||_\infty < \epsilon$

$$\begin{aligned}
||f(x_n) - f(x)|| &= ||f(x_n) - g(x_n) + g(x_n) - g(x) + g(x) - f(x)|| \\
&\leq ||f(x_n) - g(x_n)|| + ||f(x) - g(x)|| + ||g(x_n) - g(x)|| \\
&\leq ||f - g||_\infty + ||f - g||_\infty + ||g(x_n) - g(x)|| \\
< 2\epsilon + \underbrace{||g(x_n) - g(x)||}_{\leq C||x_n - x||}
\end{aligned}$$

□

Theorem 31. $1 \in A,\ \forall x \in A,\ \exists u_1, u_2, u_3, u_4, \lambda_i \in \mathbb{C}$,

$$x = \sum_{i=1}^{4} \lambda_i u_i.$$

Proof. $x = \mathrm{Re}(x) + i\mathrm{Im}(x), \mathrm{Re}(x) = \dfrac{x + x^*}{2}, \mathrm{Im}(x) = \dfrac{x - x^*}{2i}$.
Let $u = x + i\sqrt{1 - x^2}$. Then,
$x = x^*, ||x|| \leq 1 \Rightarrow \sigma(x) \in [-1, 1]$.
$\sigma(x^2) \subset [0, 1]$.
$\sigma(1 - x^2) \subset [0, 1]$.
$t \to \sqrt{t}$
$\sqrt{\ } \xrightarrow{\Gamma^{-1}} \sqrt{1 - x^2}$ (Push forward).
$uu^* = (x + i\sqrt{1 - x^2})(x - i\sqrt{1 - x^2}) = 1$ □

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22 September 2022

6.3 Functional Analysis Application

$1 \in A,\ C^*$ Algebra.
Proposition. $\forall x \in A^* = \{x \in A | x = x^*\}\ \exists! x_+, x_- \in A$ such that $x = x_+ - x_-, x_+ \cdot x_- = 0$.
Jordan Decomposition $\sigma(x_+) = \sigma(x_-) \subset [0, \infty), f = f_+ - f_-$, where $f_+ = \sup\{f, 0\}, f_- = \sup\{0, -f\}$.
Note that $\vee$ symbolizes the supremum.

Proof. $t = (0 \vee t) - (0 \vee (-t))$

By Functional Calculus, we have $x = \underbrace{(0 \vee t)(x)}_{x_+} - \underbrace{(0 \vee (-t))(x)}_{x_-}$. Note that $x_+ \cdot x_- = 0$.

Then we have by $\sigma(f(x)) = f(\sigma(x))$ (push forward), $\sigma(x_+) = \sigma(0 \vee t)(x) = 0 \vee t(\sigma(x)) \subset [0, \infty)$. ☐

Proposition. Let $1 \in A, C^*$-algebra, assume $x \in A$ is normal. Then TFAE

1. x is self-adjoint $\iff \sigma(x) \in \mathbb{R}$.
2. x is positive $\iff \sigma(x) \subset [0, \mathbb{R})$.
3. x is unitary $\iff \sigma(x) \subset \Pi$ (Unit Circle)
4. x is projecitve $\iff \sigma(x) \subset \{0, 1\}$.

Note we have done $\Rightarrow$ before. We will be proving $\Leftarrow$. Also $\sigma(x) \subset \mathbb{R}, t \to t, t \to \bar{t} \Rightarrow x = x^* \Rightarrow t(x) = \bar{t}(x)$.

Proof.

4. $\Leftarrow \sigma(x) \subset \{0, 1\}$.

We see that $t = t^2$ when taking the max. Hence, by Functional Calculus, $x^* = x = x^2 \Rightarrow$ its is a projection. ☐

Now we move on to 2.

2. $\Leftarrow$ x is positive $\Rightarrow \exists y \in A$ such that $x = y^* y$.

If we know that $yy = normal$, then $y^* y = |t^2|(y)$.

Lemma 32. Let $x, y \in A^*$ such that $\sigma(x)\sigma(y) \subset [0, \infty)$. Then, $\sigma(x + y) \subset [0, \infty)$.

Proof. Observation 1: $|||x|| - x|| \leq ||x||$.

$$\begin{aligned} |\ ||x|| - x|| &= r(||x|| - x||) \\ &= \sup_{\lambda \in \sigma(x)} ||x|| - \lambda|| \leq ||x|| \end{aligned}$$

(The last inequality can be visualized on a number line. Write λ is between 0 and $||x||$. Then the inequality follows).

Similarly, $|||y|| - y|| \leq ||y||$. Then, we have:

$$\begin{aligned} ||x|| + ||y|| &\geq ||\ ||x|| - x|| + ||y|| - y|| \\ \Rightarrow ||(||x|| + ||y|| - (x + y)|| &= \sup_{\lambda \in \sigma(x+y)} (||x|| + ||y|| - \lambda| \\ &= r(||x|| + ||y|| - (x + y)|| \\ \iff \sup_{\lambda \in (x+y)} (||x|| + ||y|| - \lambda) &\leq ||x|| + ||y|| \\ &\Rightarrow 0 \leq \lambda \ \forall \in \sigma(x + y) \end{aligned}$$

The last line follows since $x + y$ is self-adjoint. ☐

Proof.

$\Leftarrow \sigma(x) \subset [0, \infty)$.

x is normal. Then, $t = \sqrt{t}\sqrt{t}\ \forall t \in \sigma(x) \Rightarrow x = \sqrt{x}\sqrt{x} = (\sqrt{x})^*(\sqrt{x})$.

Let $y = \sqrt{x}$. Then, we are done.

$\Rightarrow$ Assume x is positive $\Rightarrow x = y^*y$. Show $\sigma(x) \subset [0,\infty)$.

By Jordan decomposition, $x = x_+ + x_-$. (Both spectrums are positive). We need to show that $x_- = 0$.

Consider $a = y \cdot x_-$. Then,

$$\begin{aligned} a^*a &= (yx_-)^*.yx \\ &= x_-y^*yx_- \\ &= x_-xx^* \\ &= x_-(x_+ - x_-)x_- \\ &= -(x_{-)}^2 \\ \Rightarrow \sigma(aa^*) &\subset \sigma(a^*a) \cup \{0\} \\ &\subset (-\infty, 0] \\ \Rightarrow -\sigma(a,a^*) \subset (0,\infty) \end{aligned}$$

Let $a = z + it$, where z, t are self-adjoint. Then, we look at

$$\begin{aligned} a^*a + aa^* &= (z - it)(z + it) + (z + it)(z - it) \\ &= (z^2 + t^2) + itz - izt + (z^2 + t^2) \\ &= 2z^2 + 2t^2 \\ \Rightarrow \sigma(a^*a) &= \sigma(\underbrace{(2z^2 + 2t^2)}_{\subset[0,\infty)} + \underbrace{(-aa^*)}_{\subset[0,\infty)}) \end{aligned}$$

Hence, by the lemma, $\sigma(a^*a) \subset [0,\infty)$.

$$\begin{aligned} \sigma(a^*a) &= (-\infty, 0) \cap [0,\infty) = \{0\} \\ a^*a &= 0 \qquad \text{(By Functional Calculus)} \\ \iff -(x_-)^3 &= 0 \\ \iff -x_- &= 0 \\ \therefore x = x_+ \end{aligned}$$

□

Corollary 32.1. x is a partial isometry $\Rightarrow x^*$is a projection.

$$\begin{aligned} x^*x &= p \\ \Rightarrow \sigma(x^*x) &\subset \{0,1\} \\ \Rightarrow \sigma(xx^*) \subset \sigma(x^*x) \cup \{0\} &\subset \{0,1\} \end{aligned}$$

By (4.), xx^* is a projection as well.

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Corollary 32.2. $1inAC^*$ algebra, $a \in A, a \geq 0 \Rightarrow 0 \leq a \leq ||a|| \cdot 1$

Proof. $||a|| \cdot 1 - a \geq 0$

$[0,\infty) \supset \sigma(||a||1 - a) = \{||a|| - \lambda, \lambda \in \sigma(a)\}$ (This follows from (2.) by Proposition above). □

Proposition. $1 \in A$ C^* algebra. Suppose $x, y \in A$. Then

1. $0 \leq x \leq y \Rightarrow \sqrt{x} \leq \sqrt{y}$
2. Moreover if x, y are invertibles, then $y^{-1} \leq x^{-1}$.

2. $a \geq 0,\ \forall b, b^*ab \leq 0$.

   Let $a = z^*z \Rightarrow b^*z^*zb = (zb^*)^*(zb)$.

   Consider $y^{-1} \leq x^{-1}$. Let $b = x^{1/2} = \sqrt{x}$ (this is defined) and $x^{1/2} = (x^{1/2})^*$. Then, $(x^{1/2})^*y^{-1}x^{1/2} < (x^{1/2})^*x^{-1}x^{1/2}$. Also, note that

   $y^{-1} \leq x^{-1} \iff x^{1/2}y^{-1/2}y^{-1/2}x^{1/2} \iff x^{1/2}y^{-1}x^{1/2} \leq 1\ \ (b^*b)$.

   Now, we proceed to the proof (Note that spectral radius can compute).

Proof.

(1) $\Rightarrow$ (2)

$$\begin{aligned}
x &\leq y \\
\Rightarrow y^{-1/2}xy^{-1/2} &\leq y^{-1/2}y^{1/2} = 1 \\
\Rightarrow y^{-1/2}x^{1/2}x^{1/2}y^{-1/2} &\leq 1 \\
\iff x^{1/2}y^{-1}x^{1/2} &\leq ||x^{1/2}y^{-1}x^{1/2}|| \\
&= r(x^{1/2}y^{-1}x^{1/2})\ \ r(b^*b) \\
&= r(x^{1/2}y^{-1/2}y^{-1/2}x^{1/2}) \\
&= r(y^{-1/2}x^{1/2}x^{1/2}y^{-1/2} \\
&= r(y^{-1/2}xy^{-1/2}) \\
&= ||y^{-1/2}xy^{-1/2}|| \\
&\leq 1
\end{aligned}$$

$x^{1/2}y^{1/2} \iff y^{-1/4}x^{1/2}x^{-1/4} \leq 1$.

Finally, we have that:

$$\begin{aligned}
y^{-1/4}x^{1/2}y^{-1/4} &\leq ||y^{-1/4}x^{1/2}y^{1/4}|| \\
&= ||(y^{-1/4}x^{1/4})(x^{1/4}y^{-1/4})||\ \ (||b^*b|| \\
&= ||x^{1/4}y^{-1/4}||^2 \\
&= r(y^{-1/4}x^{1/2}y^{-1/4} \\
&= r(x^{1/2}y^{-1/2}) \\
&\leq ||x^{1/2}y^{-1/2}|| \\
&\leq 1
\end{aligned}$$

We can see this since $b^*b = (y^{-1/2}x^{1/2})(x^{1/2}y^{-1/2}) \leq 1 \Rightarrow ||b|| \leq 1$.

If $0 < a \leq 1 \Rightarrow 0 \leq ||a|| \leq 1 \Rightarrow \sigma(a) \subset [0,1]$.

Finally, $||a|| = r(a) = \sup_{\lambda \in A} |\lambda| \leq 1$.

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xis set of invertibles, $0 \leq x \leq y$.

Pick $\epsilon > 0$.

Consider $x + \epsilon 1, y + \epsilon 1$. Then,

$0 \leq x \Rightarrow \sigma(x) \subset [0,\infty) \Rightarrow \sigma(x + \epsilon 1) \subset [\epsilon, \infty) \Rightarrow x + \epsilon 1, y1 + \epsilon \in G(A)$.

Note that $0 \notin \sigma \Rightarrow$ invertible. Then,

$x + \epsilon 1 \leq y + \epsilon 1 \Rightarrow \sqrt{x + \epsilon 1} \leq \sqrt{y + \epsilon 1}$ (by 2 from previous part).

Hence, $0 < \epsilon \to 0 \Rightarrow \sqrt{\leq} x \iff \sqrt{x} \leq \sqrt{y}$ as a limit of $\sqrt{x + \epsilon 1} \leq \sqrt{y + \epsilon 1}$. □

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27 September 2021

HW

1. A C^* algebra the extreme points of (A_+) are the projections.
2. Extreme points of (A^k) are the self-adjoint unitaries.
3. Extreme points of (A) are the partial isometries.

Definition 6.1 (Extremal Points)**.** $X_0 \subset X$ convex then $x \in X_0$ is an extremal point if whenver $x = \dfrac{x_0 + x_1}{2}$ for some $x_0, x_1 \in X_0 \Rightarrow x = x_0 = x_1$.

Proof.

1. $\Rightarrow$ Pick $x \in Ext((A_+)_1)$, the set of extremal points.
   $x \in (A_+)_1 = \{x \geq 0, ||x|| \leq 1, 1 \geq x \geq 0\}$.

$$\begin{aligned}
0 \leq x \leq 1 \Rightarrow (x^{1/2})^* x x^{1/2} &\leq (x^{1/2})^* x^{1/2} = x \\
x^2 &= x^{1/2} x x^{1/2} \\
\Rightarrow x^2 &\leq x < 2x \\
x &= \frac{1}{2}(2x - x^2 + x^2) \\
&= \frac{2x - x^2 + x^2}{2}
\end{aligned}$$

Because $1 \geq x \geq 0, 0 \leq x^2 \leq x \leq 1 \Rightarrow x^2 \in (A_+)_2$.
Also

$$\begin{aligned}
&0 \leq 2x - x^2 \leq 1 \leq 1 \\
&\iff 1 - 2x + x^2 \geq 0 \\
&\iff (x-1)^2 \geq 0 \\
&\iff 2x - x^2 \in A_+
\end{aligned}$$

Hence, since x is an extremal point, $2x - x^2 = x = x^2 \Rightarrow \therefore x = x^2 \Rightarrow$ Projection. □

1. $\Leftarrow$ We need to use commutativity here. We will need to:
   (a) Show Abelian case [characteristic function of a clopen set]
   (b) Show other cases can reduce to the Abelian case.

Proof.

$$\begin{aligned}
\text{Let } p &= \frac{a+b}{2}, \text{ where } a, b \in (A_+)_1 \\
p &= \frac{a}{2} + \frac{b}{2} \\
\frac{b}{2} &= p - \frac{a}{2} \leq p \\
\Rightarrow \frac{b}{2} &\leq p, \frac{a}{2} \leq p
\end{aligned}$$

We will set this aside for a second and come back.

Claim Now suppose we have proven

$$bp = pb = b$$
$$ap = pa = a$$

Then,

$$\begin{aligned}
p &= \frac{a}{2} + \frac{b}{2} \\
(p - \frac{a}{2} &= \frac{b}{2})b \quad \text{by right multiplication} \\
\Rightarrow pb - \frac{ab}{2} &= \frac{b^2}{2} \\
-(bp - \frac{ba}{2} &= \frac{b^2}{2}) \qquad \text{(by left multiplication)} \\
\iff ab &= ba
\end{aligned}$$

Hence, it commutes.

Now, we shall prove the claim.

$$\begin{aligned}
p &= \frac{1}{2}(a+b) \\
\frac{1}{2}a &= p - \frac{1}{2}b \leq p \\
\Rightarrow 0 \leq a &\leq 2p
\end{aligned}$$

(First note that $0 \leq p \leq 1$ is self adjoint) Conjugating both sides, we get:

$$\begin{aligned}
0 \leq (1-p)^*a(1-p) &\leq (1-p)^*(2p)(1-p) \\
0 \leq (1-p)a(1-p) &\leq (1-p)(2p)(1-p) \\
(1-p)a(1-p) &\leq 2 \qquad \underbrace{(p-p^2)}_{=0\text{since this is a projection}} \qquad (1-p) \\
\therefore (1-p)a(1-p) &= 0 \\
\Rightarrow (a-pa)(1-p) &= 0 \\
\Rightarrow a - ap - pa + pap &= 0 \\
\Rightarrow (1-p)a^{1/2}a^{1//2}(1-p) &= 0 \\
((a^{1/2})(1-p))^*a^{1/2}(1-p) &= 0 \\
\iff y*y & \qquad\qquad\qquad\qquad\qquad = 0 \\
\iff y &= 0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow a^{1/2}(a^{1/2})^*(1-p) &= 0 \\
a(1-p) &= 0 \\
a - ap &= 0 \\
\Rightarrow (a &= ap)^* \\
a^* &= (ap)^* \\
a &= pa = ap
\end{aligned}$$

Similarly for $bp = pb = a$. Hence the claim is satisfied and the proof is complete. $\square$

7 GNS Representation

Abbreviation For Gelfand-Neumann-Segal

7.1 States on C^*- algebras

Let A be a C^* algebra.

Definition 7.1 (Dual)**.** $A^* = \{\phi : A \to \mathbb{C}$ continuous, linear $\}$ forms a bimodule structure on A.

In particular, A^* has $A - A$ bimodule structure (both left/right action) defined by

$$a \cdot \phi \cdot b(x) = \phi(bxa)$$

One can check that $(a_1a_2)\phi(x) = \phi(x(a_1a_2))$ and $(\phi b_1)(b_2)(x) = \phi(b_1b_2)(x)$.

Properties

1. Linear
2. Normal

$$\begin{aligned}||a \cdot \phi \cdot b|| &= \sup_{||x|| \leq 1} |a \cdot \phi \cdot b)(x)| \\ &= \sup_{||x|| \leq 1} |\phi(bxa)| \\ &\leq ||\phi|| \sup_{|x| \leq 1} ||b \cdot x \cdot a|| \leq ||b|| \cdot ||a|| \cdot ||x||\end{aligned}$$

We also need to preserve the **Hermitian**.

Definition 7.2. $\phi : A \to \mathbb{C}$ is **positive functional** $\iff \phi(x) \geq 0, x \leq 0$.

Definition 7.3. $\phi : A \mapsto \mathbb{C}$ is **faithful functional** $\iff \phi(x) \neq 0 \ \forall x \geq 0, x \neq 0$.

Definition 7.4. $\phi : A \mapsto \mathbb{C}$ is **state** $\iff \phi \geq 0$ and $||\phi|| = 1$.

$(A) \supset S(A) = \{\phi : A \to \mathbb{C} | \phi \text{ is a state}\}$ is a compact subset in weak*-topology (by Alaoglu).

29 September 2021

Let A be a C^* algebra. Assume $1 \in A$. Recall that $S(A) = \{\phi : A \to \mathbb{C}$ linear functional $\phi(x) \geq 0 \ \forall x \in A_+\}$, where A_+ is set of positive elements (preserves positivity).

$S(A) \subset (A^*)_1$ weak*-compact.

$(x_n)_m \subset (A)_1, x_n \geq 0$.

$\phi \in S(A)$. Pick $a_n \in \ell^1(\mathbb{N}), a_n \geq 0$.

Schur Product $\sum_n a_n\phi(x_n)]leq\phi(\sum a_nx_n) < \infty$

$\Rightarrow (\phi(x_n))_1 \in \ell^\infty(\mathbb{N})$. Then,

$(a_n)(b_n) \geq 0, (a_n) \in \ell^1\mathbb{N})$ [summable] and

$\sum a_nb_n < \infty \Rightarrow \sup_n ||b_n|| < \infty$

$\therefore \sup_{x \in (A_+)} \phi(x) < \infty \Rightarrow ||\phi|| < \infty$

In C^* algebra, people do not care about duals (unlike Banach algebra), they care about **state space** instead.

(Note that pure states generate state space)

Proposition. If $\phi \in S(A)$ then $|\phi(y^*(x)| \leq (\phi(x^*x)^{1/2}(\phi(y^*y)^{1/2} \ \forall x, y \in A$.

Proof.

$$\begin{aligned} < x, y >_\phi &= \phi(y^* x) \\ < x_1 + x_2, y >_\phi &= \phi(y^*(x_1 + x_2)) \\ &= \phi(y^* x_1 + y^* x_2)) \\ &= \phi(y^* x_1) + \phi(y^* x_2) \\ &=< x_1, \phi >_\phi + < x_2, \phi >_\phi \end{aligned}$$

$||x||_\phi^2 =< x, x >_\phi = \underbrace{\phi(x^* x)}_{\geq 0 \text{ (by positivity)}} \geq 0$

By Cauchy-Schwarz, $| < x, y >_\phi | \leq ||x||_\phi \cdot ||y||_\phi$. □

Proposition. Suppose $1 \in A$ C^* algebra, $\phi : A \to \mathbb{C}$ is a positive functional $\iff$ $||\phi|| = \phi(1)$.

Hermitian Note that $\phi(x^*) = \overline{\phi(x)}$ is the Hermitian (self-adjoint elements).

Proof.

$$\begin{aligned} \phi^*(x) &= \phi^*(x) \\ &= \overline{\phi(x)} \\ &= \phi_1 + i\phi_2 \end{aligned}$$

Note that a is self-adjoint i.e. $a \leq ||a|| \cdot 1$ $\forall a = a^*$. Then, we have

$$\begin{aligned} x + x^* &\leq ||x + x^*|| \cdot 1 \\ \Rightarrow ||x + x^*||1 - (x + x^*|| &\geq 0 \\ \Rightarrow \phi(||x + x||1 - (x + x^*) &\geq 0 \quad \text{(by positivity)} \\ \Rightarrow ||x + x^*||\phi(1) - \phi(x + x^* &\geq 0 \\ \Rightarrow |x + x^*|| \cdot \phi(1) &\geq \phi(x + x^*) \\ \Rightarrow ||x + x^*|| \cdot \phi(1) &\geq \phi(x + x^*) \\ \Rightarrow \frac{1}{2}(||x + x^*|| \cdot \phi(1) &\geq \frac{1}{2}(x + x^*) \\ \Rightarrow |\frac{x + x^*}{2}| \cdot \phi(1) &\geq \phi(\frac{x + x^*}{2}) \end{aligned}$$

□

Goal $||x||\phi(1) \geq |\phi(x)|$ $\forall x$

$$||\frac{x + x^*}{2}||^2 \cdot \phi(1) \geq (\frac{\phi(x + x^*)}{2})$$

After some simplification, we get that $||x|| \geq \phi(1)$.

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1 October 2021

Proposition. $1 \in A$ C^* algebra, $\phi : A \to \mathbb{C}$ line, $\phi \geq 0 \iff ||\phi|| = \phi(1)$.

Last time, we proved $\Rightarrow$. Now, we proof the other direction.

Proof.

$\Leftarrow$ Suppose $||\phi|| = \phi(1)$.

Pick $x \in A_+ = \{x \in A, x \geq 0\}$. Let $\phi(x) = a + ib, a, b \in \mathbb{R}$.

Consider $x + it \in A, t \in \mathbb{R}$. Then, we have

$$\begin{aligned} \phi(x+it) &= \phi(x) + \phi(it \cdot 1) \\ &= \phi(x) + it\phi(1) \\ &= a + ib + it||\phi|| \\ &= a + i(b + t||\phi||) \end{aligned}$$

Taking the norm and squaring both sides, we get the following:

$$\begin{aligned} |a^2 + (b + t||\phi||^2) &\leq ||x + it||^2||\phi||^2 \\ a^2 + (b + t||\phi||^2) &\leq (||x^2|| + t^2)||\phi||^2 \\ \Rightarrow a^2 + b^2 + 2bt||\phi|| + t^2||\phi||^2 &\leq ||x^2||||\phi||^2 + t^2||\phi||^2 \\ \Rightarrow a^2 + b^2 + 2bt||\phi|| &\leq ||x||^2 \cdot ||\phi||^2 \quad \forall t \in \mathbb{R} \\ \Rightarrow 2b &\equiv 0 \\ \Rightarrow b &= 0 \\ \Rightarrow \phi(x) &= a \in \mathbb{R} \end{aligned}$$

Then, we have

$$\begin{aligned} ||x|| &\geq x \\ \Rightarrow ||x|| - x &\geq 0 \\ 0 &\leq \phi(||x|| - x||) \\ &\leq ||||x - x|| \cdot ||\phi|| \\ &\leq ||x|| \cdot 1||\phi|| \\ \Rightarrow \phi(||x|| \cdot 1) - \phi(x) &= ||x \cdot \phi(1) - \phi(x) \\ \Rightarrow ||x|| \cdot ||\phi|| - \phi(x) & \\ \therefore ||x||||\phi|| - \phi(x) &\leq ||x|| \cdot ||\phi|| \\ \Rightarrow -\phi(x) &\leq 0 \\ \phi(x) &\geq 0 \end{aligned}$$

□

Proposition. Every C^* algebra has an overabundance of states.

Let A is a C^* algebra (may contain 1). Fix $x \in A$. Then, $\lambda \in \sigma(x)$, $\exists \phi \in \rho(A)$ such that $\phi(x) = x$.

Proof. We will use the Hahn Banach Theorem.

Take the linear span linspan $(x, 1) - \mathbb{C}x + \mathbb{C}1 \leq A$ (this is closed).

Define $\phi_0 : \mathbb{C}x + \mathbb{C}1 \to \mathbb{C}$ such that

$\phi_0(ax + b \cdot 1) = a \cdot x + b, ||\phi_0|| = 1$.

By the Hahn Banach Theorem, $\exists \phi : A \to \mathbb{C}$ linear, bound $||\phi|| = ||\phi_0|| = \phi(1)$.

From prior theorem, we have that $\phi \in S(A)$. □

How to use this theorem?

1. $x \in A$, then $x = 0 \iff \phi(x) = 0 \ \forall x \in \rho(A)$.
2. $x = x^* \iff \phi(x) = \overline{\phi(0)} \ \forall x \in \phi$.

---

4 October **7.2 GNS Construction**

Suppose $1 \in A, C^*$-algebra, $\phi in S(A) \Rightarrow \ \exists \pi L^2(A,\phi)$ - Hilbert space where $\pi : A \to B(L^{@}(A,\phi))$, $*$-represtations such that $\phi(x) =< \pi(x), 1 >$.
$\exists 1_\phi \in L^2(A,\phi)$ cyclic vectors such that

$$\phi(x) =< \pi(x)1_\phi, 1_\phi > \ \forall x \in A$$

Up to unitary equivalence, we have $p : A \to B(H),\ \exists \xi \in H$ cyclic such that $\phi(x) =< p(x)\xi, \xi > \Rightarrow p \sim \pi$
Consider $I_\phi = \{x \in A - (x^{-1}x) = 0\}$

Observe $\tilde{I_\phi} = \{x \in A | \phi(x^*y) = 0 \ y \in A\}$

Why? $|\phi(y^*x)| \leq \phi(x^*x)^{1/2}\phi(y^*y)^{1/2}$

$\therefore I_p = \tilde{I_p}$. Clearly, I_P is a linear subspace.

Suppose $x \in I_\phi, y \in A$. Show $yx \in I_\phi$.

Proof.

$$\begin{aligned}
\phi((yx)^*(yx)) &= \phi(x^*y^*yx) \\
\phi(x^*y^*yx) &\leq \phi(x^*||y^*y|||x|) \\
&= \phi(||y^*y|x^*x) \\
&= ||y^*y||\phi(x^*x) \\
&= 0 \\
\Rightarrow yx \in I_\phi &
\end{aligned}$$

□

Now consider the quotient of a linear space.
$[x] + I_\phi \in A/I_\phi$.

Definition 7.5. $< [x], [y] >_\phi = \phi(y^*x)$

This is well defined since

$$\begin{aligned}
[x] &= [x_1] \\
[y] &= [y_1] \\
\Rightarrow \phi(y_1^*x_1) &= \phi(y^*x)
\end{aligned}$$

Completion $L^2(A,\phi) = \overline{A/I_\phi} \supset A/I_\phi$ as a dense set.
Now "extend action class y." For $x \in A$, we have

$$\begin{aligned}
L^2(A,\phi) &\to L^2(A,\phi) \\
\pi(x)([y]) &= [xy]
\end{aligned}$$

We now prove that this well defined.

Proof.

$$\begin{aligned}
\Pi(x)[y_1] &= \Pi(x)[y_2] \\
[xy_1] &= [xy_2] \\
\Rightarrow xy_1 - xy_2 &\in I_\phi \\
x(y_1 - y_2) &\in I_\phi
\end{aligned}$$

The last line follows since $(y_1 - y_2) \in I_\phi$ is an ideal. □

Let $\Pi(x) : A/I_\phi \to A/I_\phi$. Then,

$$\begin{aligned}
||\Pi(x)[y]||_{2,\phi} &= ||(xy)_2||_{2,\phi} \\
&= \phi((xy)^*(xy))^{1/2} \\
&= \phi(y^*x^*xy)^{1/2} \\
&= \phi(x^*|x^*x||y)^{1/2} \\
&= ||x^*x||_\infty^{1/2}\phi(y^*y)^{1/2} \\
&= ||x||||[y]||_{2,\phi}
\end{aligned}$$

Idea $\tilde{\tau} : H \to H$ where $T : K \to K$ linear, $\bar{K}$ Hilbert space.

If $||T(\xi)|| \leq C||\xi||\ \forall \xi \in K \Rightarrow \exists\ \tilde{\tau} : H \to H$ such that $\tilde{\tau}|_K = \tau$.

Furthermore, $||\tilde{\tau}(\xi)|| \leq C||\xi||\ \forall \xi \in K$.

Proof. Suppose $\xi_n \in K \to \xi \in H$.

First we show that $(\tilde{\xi_n})_n \subset K,\ \exists \eta \in H$ such that $T(\xi_n) \to \eta$.

$$\begin{aligned}
||T\xi_n - T\xi_m|| &= ||T(\xi_n - \xi_m)|| \\
&\leq C||\xi_n - \xi_m|| \\
&\Rightarrow T\xi_n \to \eta \quad \text{(since in a Hilbert space)} \\
&\Rightarrow T\xi_n = \eta
\end{aligned}$$

Let $\xi_n', \xi_n \to \xi$ and $||\xi_n' - \xi_n|| \to 0$. Then,

$$\begin{aligned}
|T(\xi_n') - T(\xi_n)|| &= ||T(\xi_n - \xi_n')|| \\
&\leq C||\xi_n - \xi_n'|| \Rightarrow ||\tilde{T}(\xi)|| \qquad\qquad = \eta \\
&= \lim_{n\to\infty} T(\xi_n) \\
&\leq \lim_{n\to\infty} ||T||_\infty ||\xi_n|| \\
&= ||T||_\infty ||\xi||
\end{aligned}$$

□

---

6 October Last time we proved existence. Now uniqueness.

Proof. $\exists\ \sigma : A \to B(H)*$-representation, $\ \xi \in H; A_\xi = H$ and $\phi(x) =< \sigma(x)\xi, \xi >$.

Let $U : L^2(A, \phi) \to H$ be unitary with $U(\Pi_p a 1_\phi) = \sigma(a0\xi)$.

Now we show that this is an isometric map (takes care of well- defined).

$$\begin{aligned}
< U(\Pi_p a1_\phi), U(\Pi_p b1_\phi) >_H &=< \Pi_\phi(a)1_\phi, \Pi_\phi(b)1_\phi > \\
< \sigma(a)\xi, \sigma(b)\xi) >_H &=< \Pi_p(b)^*\Pi_p(a), 1_\phi > \\
< \sigma(b)^*\sigma(a)\xi, \xi > &=< \Pi_\phi(\phi^*a)1_\phi, 1_\phi > \\
< \sigma(b^*a\xi, \xi > &= \phi(b^*a)
\end{aligned}$$

For $b = a$, we have

$$||U(\Pi_p(a)1_\phi|| = ||\Pi_p(a)1_\phi||$$

$\therefore \Pi_\phi(a)1_\phi = \Pi_\phi(b) \iff ||\phi(a-b)|| = 0.$ □

(Can lift to whole space). For anyting in the closure, $< U(\eta_1), U(\eta_2) >=< \eta_1, \eta_2 >$.

U is an isometry from $L^2(A, \phi) \to H$. Now show unitary (show bijection).

$\therefore U(L^2(1\phi) = H, \eta(\Pi_p H), 1\phi)$.

Corollary 32.3. $1 \in C^*$- algebra admits only **faithful** representations i.e. $\Pi(x) = 0 \iff x = 0$.

Proof. $\phi \in S(A) \rightsquigarrow \Pi_\phi : A \to B(L^2(A, \phi))$.

$\oplus_{\phi \in S(A)} \Pi_p = \Pi : A \to B(\oplus_{\phi \in S(A)} L^2(A, \phi))$

Π is faithful.

$\pi(x) = 0 \Rightarrow \Pi_p(x) = 0 \ \forall \phi$.

$\phi(x) =< \Pi_\phi(x) 2\phi, 1\phi >= 0$.

Note that $\phi, \Psi \in SA, \phi \leq \Psi \Rightarrow \Psi(a) \geq \Phi(a) \ \forall a \in A$. □

Theorem 33. Suppose $\Psi = \phi \geq 0 \Rightarrow \phi_1 \Psi \geq 0$ on C^* algebra $A, \phi \leq$

$\iff \exists!\ y \in \Pi_\Psi(A)$ such that $0 < y \leq 1$ and $\phi(a) =< \Pi_\Psi(a) y 1_\Psi, 1_\Psi >$ for all $a \in A$.

Proof. GNS wrt Ψ :

$\Pi_\Psi : A \to B(L^2(A, \Psi))$, $*$-representation

$\Psi(a) =< \Pi_{\Psi(a)} 1_\Psi, 1_\Psi), \overline{A 1_\Psi} = L^2(A, \Psi)$ (completion)

Pick $0 \leq a \in A_+ \Rightarrow \Pi_{\Psi(a)} \in (B(L^2(A, \Psi))$. Then,

$$\begin{aligned} 0 \leq y \leq 1 \iff & \Pi_{\Psi(a)} \cdot y(\Pi_{\Psi(a)} 1/2) y \\ & = (\Pi^{1/2}_{\Psi(a)}((\Pi_{\Psi(a)})^{1/2} \\ & = (\Pi_{\Psi(a)} \ \forall a \geq 0 \end{aligned}$$

$T \geq 0 \Rightarrow < T\xi, \xi > \geq 0$.

$\phi(a) =< \Pi_{\Psi(a)} y 1_\phi, 1_\Psi > \leq < (\Pi_{\Psi(a)} 1_\Psi, 1_\Psi ? = \Psi(a)$. □

Proof. $(\Rightarrow)$ Now suppose $\phi \leq \Psi$. Then, for $a, b \in A, L^2(A, \phi) = K < [a], [b] >_\phi = |\phi(b^* a)|$

$$\begin{aligned} & \leq \phi(b^* b)^{1/2} \phi(a^* a)^{1/2} \\ & \leq \psi(b^* b)^{1/2} \phi(a^* a)^{1/2} \\ & = ||b||_\Psi ||a|| \end{aligned}$$

□

$(\cdot) : L^2(A, \Psi) \times l^2(A, \mathbb{C}) \to \mathbb{C}$. Then, $([a], [b]) = \phi(b^* a)$.

Note that **sesquilinear** means linear in 1st component and antilinear in the second component. Since $\overline{A} = L^2(A, \phi)$, we have $|[a][b]|_\phi \leq ||b||_\Psi ||a||_\Psi$

Let $(\cdot) : H \times H \to \mathbb{C}$ be bounded $\Rightarrow \exists T : H \to$ linear space (ξ, η).

$$\begin{aligned} \phi(b^* a(= (a, b)_\phi &=< y(\Pi_{\Psi(a)} 1_\Psi, (\Pi_{\Psi(b)} 1_\Psi) > \\ \Psi(a^* a) &\geq < y(\Pi_{\Psi(a)} 1_\Psi, (\Pi_{\Psi(a)} 1_\Psi > \\ < (\Pi_{\Psi(a)} 1_\Psi 1_\Psi, (\Pi_{\Psi(a)} 1_\Psi > &\geq < y(\Pi_{\Psi(a)} 1_\Psi, (\Pi_{\Psi(a)} 1_\Psi > \\ \Rightarrow < (1-y)(\Pi_{\Psi(a)} 1_\Psi, (\Pi_{\Psi(a)} 1_\Psi > &\geq 0 \ \forall a \\ < (1-y)\xi, \xi > &\geq 0 \ \forall \xi \in L^2(A) \\ \Rightarrow (1-y) &\geq 0 \\ 1 \geq y &\geq 0 \end{aligned}$$

Now we check that $(\Pi_{\Psi(a)} y = y(\Pi_{\Psi(a)}$.

It is enough to check in a dense set $TU : H \to K$ where $< T\xi, \eta >=< U\xi, \eta >, \quad \xi, \eta \in H$.

Proof.

$$\begin{aligned}
(\Pi_{\Psi(a)}y &= y(\Pi_{\Psi(a)} \\
&=< (\Pi_{\Psi(a)}y \cdot (\Pi_{\Psi(b)}1_\Psi), (\Pi_{\Psi(b)}1_\Psi) > \\
&=< y(\Pi_{\Psi(a)}(\Pi_{\Psi(b)}1_\Psi, (\Pi_{\Psi(c)}1_\Psi) > \\
&=< y(\Pi_{\Psi(b)}1_\Psi, (\Pi_{\Psi(a)} * (\Pi_{\Psi(c)}1_\Psi > \\
&=< y(\Pi_{\Psi(b)}1_\Psi, (\Pi_{\Psi(a*c)}1_\Psi >= \phi(a^*c^*)b \\
&= \phi(c^*ab)
\end{aligned}$$

□

Definition 7.6. $\phi \in S(A)$ is a pure state $\iff$ ϕ is an extremal point of $S(A)$.

Exercise

Proposition. $\phi \in S(A)$, AC^*- algebra, ϕ is a pure state $\iff$ the corresponding GNS construction $\Pi_p : A \to B(L^2(A,\phi))$ with corresponding cyclic vector 1_ϕ irreducible.

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11 October

Corollary 33.1. Fix $1 \in A, C^*$-algebra, $x \in A$.

If $x \neq 0$, $\exists \Pi : A \to B(H)$ irreducible $*$-representation such that $\Pi(x) \neq 0$.

Next we present a preliminary theorem that will be used to prove other theorems.

Theorem 34 (Krien-Milman). A compact convex set of a Hausdorff locally convex topological vector space is equal to the closed convex hull of its extremal points, denoted by $C_0(\text{ext}(K))$ for compact Hausdorff space $K \subset X$.

Corollary 34.1. AC^*-algebra.

Let $C_0(S_p(A))$ weak$*$-topology on $S(A)$.

We can see this from the Krien-Milman theorem, where $X = (S(A)$, weak$*$-topology), where $S(A)$ is convex and compact from Alaoglu.

Recall proposition:

$\phi \in S_p(A) \rightsquigarrow$ GNS construction $\Pi_\phi : A \to B(L^2(A,\phi))$ given by

$\phi(x) =< \Pi_p(x), 1_\phi)_2 >$ and $\overline{\Pi_\phi(A)1_\phi} = \overline{L^2(A,\Psi)}$.

Putting these two together, we get the corollary.

Proof. $x \neq 0$, $\exists \phi \in S(A), \phi(x) \neq 0$ suc h that $|\phi(x) - \sum \alpha_i(\phi_i)(x)| < \epsilon$.

Here, $\exists i \in N$ such that $\phi_i(x0 \neq 0$, perform the GNS construction from the porposition $\Rightarrow$ $\Pi_{\phi_i}(x) \neq 0$. □

8 Jordon Decomposition

AC^*-algebra, $\phi \in A^*$ such that $\phi = \phi^*$ (Hermitian).

$\exists! \phi_+, \phi_- \geq 0$ such that

$\phi = \phi_+ - \phi_-, ||\phi|| = ||\phi_+|| + ||\phi_-||$

Proof Idea Can define continuous $\phi : A \to C(\sigma)$.

Can take all positive linear functionals (and just like Gelfand Transform), apply the separation argument and Radon measure argument.

Corollary 34.2. Every functional can be decomposed in a linear combination of positive functions.

9 Intro to Von Neumann Algebra

$B(H, ||\cdot||_\infty$. Let $x \in B(H)$ be operators.

$||x||_\infty = \sup_{||\xi||\leq 1} ||x\xi||$

Note that $< x\xi, eta >=< \xi, x^*\eta >$.

Lemma 35. $x \in B(H), \ker(x) = \mathrm{Ran}(x^*)^\perp, \ker(x)^\perp = \overline{\mathrm{Ran}(x^*}$

Proof. $\xi \in \ker(x) \Rightarrow x\xi = 0$.

$(\Rightarrow)$

$\forall \eta \in H, 0 =< 0, \eta >=< x\xi, \eta >=< \xi, x^*\eta >$

$\Rightarrow \xi \perp \mathrm{Ran}(x^*) \Rightarrow \xi \in \mathrm{Ran}(x^*)^\perp$

$\Rightarrow \ker(x) \subset \mathrm{Ran}(x^*)^\perp$ □

Proof. $(\Leftarrow)$

$\xi \in \mathrm{Ran}(x^*)^\perp \Rightarrow < \xi, x^*\eta >= 0\ \forall \eta \in H \Rightarrow < x\xi, \eta >= 0 \Rightarrow x\xi \perp \eta\ \forall \eta$.

Pick $\eta = x\xi$. Then, $< x\xi, x\xi >= 0 \Rightarrow ||x\xi||^2 = 0 \iff x\xi = 0 \iff \xi \in \ker(x)$.

$\Rightarrow \mathrm{Ran}(x^*)^\perp \subset \ker(x)$

$\therefore \ker(x) = \mathrm{Ran}(x^*)^\perp$

□

9.1 Point Spectrum

$x \in B(H)$.

$\sigma(x) = \{\lambda \in \mathbb{C} | x - \lambda 1 \text{ not invertible}\}$

(One way fails invertibility if kernel is non-trivial i.e. $\sigma_p(x) - \{\lambda \in \mathbb{C} | \ker(x - \lambda 1) \neq 0\}$.

Definition 9.1 (Approximate Kernel)**.** Let $xa \in B(H)$. The **approximate kernel** is $(\xi_n)_n \subset H$ such that

$$\xi\eta \subset H \text{ such that } ||\xi\eta|| = 1 \text{ and } ||x\xi\eta|| \to 0$$

as $n \to \infty$.

$\sigma_{ap}(x) = \{\lambda \in \mathbb{C}x - \lambda \text{ has an approximate kernel}\}$

$\sigma_p(x) \subset \sigma_{ap}(x) \subset \sigma(x)$.

Lemma 36. $x \in B(H)$ normal operator. Then, $\sigma_p(x)^* = \overline{\sigma_p(x)}$

Proof. x is normal $\iff xx^* = x^*x, \lambda \in \mathbb{C} \Rightarrow (x + \lambda 1)$ is normal as well.

Now we need to show that the operator is normal, which means

$$\begin{aligned} x &= x^* \\ ||x\xi|| &= ||x^*\xi||\ \forall \xi \in H \\ < x\xi, x\xi > &=< x^*\xi, x^*\xi > \\ < x^*x\xi, \xi > &=< xx^*\xi, \xi > \end{aligned}$$

Then we have $||(x - \lambda)\xi|| = ||(x^* - \bar{\lambda})\xi||$ (essentially an isometry).

Suppose $\xi, \eta \in H$ eignevectors for x $(\lambda \neq \mu)$. Then,

$(x - \lambda 1) = 0 \Rightarrow x\xi = \lambda\xi$. On the other hand

$(x - \mu 1)\eta = 0 \Rightarrow x\xi = \mu\eta \Rightarrow x^*\eta = \bar{\mu}\eta$.

Then, for $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\lambda < \xi, \eta > &=< \lambda\xi, \eta > \\ &=< x\xi, \eta > \\ &=< \eta, \bar{\mu}\eta > \\ &=< \mu\xi, \eta > \\ &= \mu < \xi, \eta >\end{aligned}$$

$\therefore < \xi, \eta >= 0.$ □

Proposition. $x \in B(H)$. Then, $\partial\sigma(x) \subset \sigma_{ap}(x)$.

Proof. Suppose $\lambda \in \partial\sigma(x)$ (That means we can approximate from outside the boundary).

$\Rightarrow \exists \lambda_n \ in \mathbb{C} - \sigma(x) = P(x)$ with $\lambda_n \to \lambda$.

$\Rightarrow x - \lambda_n \to x - \lambda$ (not invertible, limit point outside, blows up).

Using a prior lemma we have $||(x - \lambda_n)^{-1}|| \to \infty \Rightarrow \exists \xi_n \in H$ such that $||\xi_n|| \to 0$ such that $||x - \lambda_n)^{-1}\xi_n|| = 1$. □

Normally, $||\xi_n|| = 1$ such that $||(x - \lambda_n)^{-1}\xi_n|| = c_n \to \infty$ (after rescaling/normalizing).

$||(x - \lambda)(x - \lambda_n)^{-1}\xi_n| \leq |\lambda - \lambda_n|||(x - \lambda_n)\xi_n|| + ||\xi_n|| \to 0$.

Therefore, $(x - \lambda_n^{-1}(\xi_n))$ is the approximate kernel.

14 October Recall $partial(\sigma(x)) \subset \sigma_{ap}(x)$

Lemma 37. $x \in B(H)$

x invertible $\iff$ neither x nor x^* has an approximate kernel.

$\exists \xi_n \in ||, ||\xi_n|| = 1$.

$||x\xi + n|| \to 0$ as $n \to \infty$.

Consequently, $\sigma(x) = \sigma_{ap}(x) \cup \overline{\sigma_{ap}(x^*} \ \forall x \in B(H)$

Proof. $(\Rightarrow)$

Suppose x invertible, $\exists x^{-1} \in B(H)$ such that

$$\begin{aligned}||x^{-1}||_\infty ||x\xi|| \geq ||x^{-1}(x\xi)|| = ||\xi|| \ \forall \xi \in H \\ ||T\xi|| \leq ||T||_\infty ||\xi|| \quad T \in B(H)\end{aligned}$$

$||x|| = 1 \Rightarrow ||x^{-1}||||x\xi|| \geq 1 \Rightarrow ||x\xi|| \geq \dfrac{1}{||x||} \geq 0.$

$\therefore x$ does not have an approximate kernel for x^*. □

Proof. $(\Leftarrow)$

Suppose that neither x nor x^* has approximate kernel. Show that x has dense range.

Recall,

$\ker(x) = \mathrm{Ran}(x^*)^\perp, \ker(x)^\perp = \overline{\mathrm{Ran}(x)^*}$

Pick $(x\xi_n) \subset \mathrm{Ran}(x)$.

This is injective and dense (almost surjective). Need to show it is fully onto.

$(x\xi_n) \subset \mathrm{Ran}(x) - \{x\xi, \xi \in H\} \in H$.

Assume Cauchy sequence.

$\Rightarrow ||x\xi_n - x\xi_m| \to n, m \to \infty = ||x(\xi_n - \xi_m)|| \to 0$. (operator does not have approximate kernel).

Claim $||\xi_n - \xi_m|| \to 0$.

If not, $||\xi_n - \xi_m|| \geq C_0 > 0 \ \forall n, m$.

$||\dfrac{X(\xi_n - \xi_m)}{(||\xi_n - \xi_m)} \to 0$

$\Rightarrow X$ will have non-trivial approximate kernel.

H Hilbert space $\Rightarrow$ complete $\Rightarrow \xi_n \to \xi \Rightarrow x\xi_n \to x\xi \in$ Range

Everything in closure is in Range $\Rightarrow$ closed.

$\square$

Summary $\eta \in \overline{\text{Ran}}(x), ||\eta - x\xi_n|| \to 0$.

By triangle inequality, $||x\xi_n - \eta + \eta - x\xi_n|| \to 0$.

This is Cauchy. x is a linear bijective map. Now we apply the Open Mapping Theorem.

$x : H \to H$ bijective $\Rightarrow$(open $\exists x^{-1} : H \to H$ exists.

$\underline{\{< x\xi, \xi|||\xi = 1 >, \xi \in H.\text{ Records all the complex numbers. (This is where all complex numbers lie).}}$

Lemma 38. $x \in B(H)$ Hilbert space. Then, $\sigma(x) \subset W(x)$.

Proof. Fix $\lambda \in \sigma(x) \Rightarrow (x - \lambda 1)$ not invertible.

Either $x - \lambda$ or $(x - \lambda)^*$ has an approximate kernel.

$\Rightarrow \ \exists \xi_n \in H, ||\xi_n|| = 1, ||(x - \lambda)\xi_n|| \to 0$ or $||(x - \lambda)^*\xi_n|| \to 0$

$$\begin{aligned}
| < x\xi_n, \xi_n > -\lambda 1|| &= | < x\xi_n, \xi)n > -\lambda||\xi_n||^2| \\
&= | < x\xi_n, \xi_n > -\lambda < \xi_n, \xi_n > | \\
&= | < x\xi_n - \lambda\xi_n, \xi_n > | \\
&\leq ||x\xi_n - \lambda\xi_n|||\xi_n|| \\
&= ||x\xi_n - \xi_n| \to 0
\end{aligned}$$

$\therefore < x\xi_n, \xi_n > \to \lambda \Rightarrow \lambda \in W(X)$.

(Repeat with $< \xi_n, x^*\xi_n >$ - same story). $\square$

Theorem 39. $\lambda \in W(X), x \in B(H)$. TFAE:
(1) x normal $\iff ||x\xi|| = ||x^*\xi|| \ \forall \xi \in H$.
(2) $x = x^* \iff < x\xi, x > \in P$ (closed subspace of $\mathbb{C}\ \forall \xi \in H$
(3) $x \geq 0 \iff < x\xi, \xi > \geq 0\ \forall \xi \in H$
(4) x isometry $(x^*x = 1) \iff ||x\xi|| = ||\xi||\ \forall \xi \in H$.
(5) x is a projective $\iff x = P_k$ (orthogonal projection on $\bar{K} = K \subset H$).
(6) x is a partial isometry $\iff \exists K = \bar{K} \leq H$ such that $x|_K$ is an isometry $\iff x|_{\bar{K}} = 0$.

Proof. (1)

$$\begin{aligned} ||x\xi||^2 &= ||x^*\xi||^2 \\ < x\xi, x\xi > &= < x^*\xi, x^*\xi > \\ < x^*x\xi, \xi > &= < xx^*\xi, \xi > \\ < x^*x - xx^*\xi, \xi > &= 0\ \forall \xi \in H \end{aligned}$$

□

Proof. (2)

$$\begin{aligned} < x\xi, \xi > &= \overline{< x\xi, \xi >} \\ &= < \xi, x\xi > \\ &= < x^*\xi, x > \\ &= < (x - x^*\xi, \xi > = 0 \\ \iff x &= x^* \end{aligned}$$

□

Proof. (3)
If $x = y^*y$ for some $y \in B(H)$, then $< y^*y\xi, \xi > = < y\xi, y\xi > \geq 0$ for all $\xi \in H$.
Conversely, if $< x\xi, \xi > \geq 0$ for all $\xi \in H$ then $x = x^*$ by (ii) and by the previous lemma $\sigma(x) \subset W(x) \subset [0, \infty)$, so x is positive. □

Proof. (4)
If x is an isometry, then $x^*x = 1$ and so $||x\xi||^2 = < x^*x\xi, \xi > = ||\xi||^2$ for all $\xi \in H$.
Conversely, assuming $< x^*x\xi, \xi > = 0$ for all $\xi \in H$ and applying the polarization identity, we have $x^*x = 1$. □

Proof. (5)
Suppose x is a projection and let $K = \overline{R(x)} = (\text{Ran} x)^\perp$. Notice for all $\xi \in K, \eta \in \ker x$ and $x\zeta \in R(x)$, we have $< x\xi, \eta + x\zeta > = < \xi, x\xi >$.
So, $x\xi \in K$ and $x\xi = \xi$.Therefore, x is the orthogonal projection onto K. □

Proposition (Polar Decomposition). Let H be a Hilbert space and $x \in B(H)$, then there exists a partial isometry v such that $x = v|x|$ and $\ker v = \ker |x| = \ker x$. Moreover, this decomposition is unique in that if $x = wy$ where $y \geq 0$ and w is a partial isometry with $\ker w = \ker y$, then $y = |x|$ and $v = w$.
(Moreover, $v^*x = x|$ as $< v^*x\xi, |x|\eta. = < x\xi, x\eta > = < |x|^2\xi, \eta >$).

Proof. Define a linear operator $v_0 : R(|x|) \to R(x)$ by $v_0(|x|\xi) = x\xi$ for any $\xi \in H$.
Notice, $|||x|\xi|| = ||x\xi||$ is v_0 is well defined and bounded, so it extends to a partial isometry $v : \overline{R(|x|} \to \overline{R(x)}$ and we have $v|x| = x$.
We also have $\ker v = R(|x|)^\perp = \ker(|x|) = \ker(x)$. Now suppose $x = wy$ with $y \geq 0$ and w is a partial isometry with $\ker w = \ker y$.
Then $|x|^2 = x^*x = yw^*wy = y^2$ and hence $|x| = (|x|^2)^{1/2}y$.
Then, $\ker w = \overline{R(|x|)}^\perp$ and $||w(x)\xi|| = ||x\xi||$ for all $\xi \in H$ and $w = v$. □

10 Trace Class of Operators

20 October

Definition 10.1. $x \in B(H), x > 0$. Then,
$\mathrm{Tr}(x) = \sum_{i \in I} < x\xi_i, \xi_i >$ where $(\xi_i)_{i\in I}$ is an orthonormal basis of $H \Rightarrow \xi_i \perp \xi_j\ \forall i, j$ and $||\xi_i|| = 1\ \forall i$.
Lemma 40. Suppose $x \in B(H), (\xi)_i \subset H$ ONB. Define Trace as $\mathrm{Tr}(xx^*) = \mathrm{Tr}(x^*x)$
Theorem 41. If $x \in B(H), x \geq 0$, $\mathrm{Tr}(x)$ is independent on the choice of $(\xi_i)_{i\in I} \subset H$ ONB.

Proof. Let $(\xi_i)_{i\in I}, (\eta_j)_{j\in I} \subset H$ ONB.
Let $u : H \to H$ be a unitary operator with $u^*(\xi_i) = \eta_i\ \forall i$. Then, we have the following:

$$\begin{aligned}
\mathrm{Tr}(x) &= \sum < x\xi_i, \xi_i > \qquad (x = y^*y)\\
\mathrm{Tr}(y^*y) &= \mathrm{Tr}(y^*u^*uy)\\
&= \mathrm{Tr}((uy)^*uy))\\
&= \mathrm{Tr}((uy)(uy)^*)\\
&= \mathrm{Tr}(uyy^*u^*) \qquad \text{(by lemma)}\\
&= \sum_i < uyy^*u^*\xi_i, \xi_i >\\
&= \sum_i < yy^*u^*\xi_i, u^*\xi_i >\\
&= \sum_i < yy^*\eta_i, \eta_i >\\
&= \sum_i < y^*y\eta_i, \eta_i >\\
&= \sum_i < x\eta_i, \eta_i >
\end{aligned}$$

(Trace with respect to η_i) ☐

Proof. Let $(\xi_i)_{i\in I} \subset H$ ONB. Then,

$$\begin{aligned}
\mathrm{Tr}(x^*x) &= \sum_{i\in I} < x^*x\xi_i, \xi_i >\\
&= \sum_{i\in I} < x\xi_i, x\xi_i >= \sum_{i\in I} ||x\xi_i||^2\\
&= \sum_i (\sum_j | < x\xi_i, \xi_j|^2)\\
&= \sum_i \sum_j < x\xi_i, \xi_j >< \overline{x\xi_i, \xi_j} >\\
&= \sum_j \sum_i < \xi, x^*\xi_j >< \overline{\xi_i, x^*\xi_j} >\\
&= \sum_j \sum_i | < \xi_i, x^*\xi_j > |^2\\
&= \sum_j ||x^*\xi_j||^2\\
&= \sum_j < x^*\xi_i, x^*\xi_j >\\
&= \sum_j < xx^*\xi_i, \xi_j >\\
&= \mathrm{Tr}(xx^*)
\end{aligned}$$

$\square$

Definition 10.2. $x \in B(H), ||x||_1 = \text{Tr}(|x|)$
x is **trace class** $\iff ||x||_1 < \infty (L^1(B(H))$.

Trace Class Operators
$x \in L^1(B(H)), \text{Tr} = \sum_{i \in I} < x\xi_i, \xi_i >$
Lemma 42. $2|\text{Tr}| \leq \text{Tr}(|x|) + \text{Tr}(v|x|v^*) \leq 2||x||_1$

$x = v|x|, x \in B(H), x \in v|x|, v\xi \in H$. Then,
$2| < x\xi, \xi > | \leq < |x|\xi, \xi > + < |x|v^*\xi, v^*\xi >$

Proof. Take $||(x - |x|^{1/2} + v^*)\xi||$. Then,

$$\begin{aligned}
||(x - |x|^{1/2} + v^*)\xi|| &\leq < (|x|^{1/2} - |x|^{1/2} + v^*)\xi, |x|^{1/2} - |x|^{1/2}v^* \\
&= < |x|^{1/2}\xi, |x^{1/2}(\xi) + |t|^2 < |x|^2 v^*\xi > -2\text{Re}(< |x|^{1/2}\xi, ||x||_t^{1/2} v^*\xi >) \\
\Rightarrow 2\text{Re}(t < |x|\xi, v^*\xi > &\leq < |x|\xi, \xi > + t^2 < |x|v^*\xi, v^*\xi >
\end{aligned}$$

(Here $t \in \mathbb{C}$). $\square$

22 October $\ell^1(B(H)) = \{x \in B(H) : \text{Tr}(|x|) = ||x||_1 < \infty\}$

Theorem 43. $\mathcal{L}^1(B(H))$ is a two sided ideal in $B(H)$, elements of $\mathcal{L}^1(B(H))$ appears as finite linear combination of positive operators of finite trace.

Proof. Show $|| \cdot ||_1$ is a norm $\in \mathcal{L}^1(B(H))$
$x, y \in \mathcal{L}^1(B(H)) \to x + y \in \mathcal{L}^1(B(H))$.
Using the polar decomposition theorem ($x = w|x| \Rightarrow w^*w = r(x)$ i.e. $w^*w|x| = x$), $\exists w \in B(H)$ partial isometry such that

$$\begin{aligned}
x + y &= w|x + y| \\
\iff w^*w|x + y| &= |x + y| \\
\Rightarrow w^*x + w^*y &= w^*(x + y) \\
&= w^*w|x + y| \\
&= |x + y|
\end{aligned}$$

$x, y \in \mathcal{L}^1(B(H)) \Rightarrow w^*x, w^*y \in \mathcal{L}^1(B(H))$.
$x \in \mathcal{L}^1(B(H))$,

Know w is a partial isometry $\Rightarrow w^*w$ is a projection $\Rightarrow ww^* = 1$.

$$\begin{aligned}
ww^* &\leq 1 \\
x^*ww^*x &\leq x^*1x \\
&= x^*x \\
\Rightarrow (w^*x)^*w^*x &= x^*x \\
|w^*x|^2 &\leq |x|^2 \\
\Rightarrow |w^*x| &\leq |x|
\end{aligned}$$

For $x \geq 0$, we have
$\text{Tr} = \sum_i < x\xi_i, \xi_i > \geq 0 \Rightarrow \text{Tr}(|w^*x|) \leq \text{Tr}(|x|) < \infty$.
(This way, we can show this is a left ideal). Next,

$$
\begin{aligned}
||x+y||_1 &= \mathrm{Tr}(|x+y|) \\
&= \sum_{i\in I} < |x+y|\xi_i, \xi_i > \qquad (\xi_i)_{i\in I} \in H \text{ ONB} \\
&= \sum_{i=1} < (w^*x + w^*y)\xi_i, \xi_i > \\
&= \sum_{i=1} < w^*x\xi_i, \xi_i > + \sum_{i} < w^*x\xi_i, \xi_i >< \infty
\end{aligned}
$$

$\Rightarrow x+y \in \mathcal{L}^1(B(H))$. Therefore, it is a vector linear subspace. Furthermore,

$$
\begin{aligned}
||x+y||_1 &= \mathrm{Tr}(w^*x) + \mathrm{Tr}(w^*y) \\
&\leq ||w^*x||_1 + ||w^*y||_1 \\
&\leq ||x||_1 + ||y||_1
\end{aligned}
$$

Also note that $||cx||_1 = |c|||x||_1$ since $|cx| = |c||x|$. So far, we have shown this is a semi norm. Finally, we have

$$
\begin{aligned}
||x||_1 &= 0 \\
\Rightarrow \mathrm{Tr}(|x|) &= 0 \\
\sum_{i=1} < |x|\xi_i, \xi_i > &\geq 0 \qquad (\text{ since it is a positive operator }) \\
\Rightarrow < |x|\xi_i, \xi_i > &= 0 \\
< |x|^{1/2}|x|^{1/2}\xi_i, \xi_i > &= 0 \\
< |x|^{1/2}\xi_i, |x|^{1/2}\xi_i > &= 0 \\
||x||^{1/2}\xi_i|| &= 0 \\
|x^{1/2}\xi_i| &= 0 \\
|x|\xi_i &= 0 \ \forall i \\
\Rightarrow |x| &= 0 \\
\therefore x &= 0
\end{aligned}
$$

Hence we have shown that this is a norm. □

An element $x = w|x|$ (by polar decomposition).

$y \in B(H) \Rightarrow y|x| = \dfrac{1}{4}\sum_{k=0}^{3}(y + i^k \cdot 1)|x|(y + i^* 1)^*$

After some computation, we come to the conclusion that $\mathrm{Tr}(|x|^{1/2}(y+i^k)^*(y+i^k y)(|x|^{1/2}) = ||y + i^k||\mathrm{Tr}(|x|) < \infty$

Theorem 44. $x \in \mathcal{L}^1(B(H)), a, b \in B(H)$. Then TFH:

(1)$||x||_\infty \leq ||x||_1$

(2)$||ax+b||_1 \leq ||a||_\infty ||b||_\infty ||x||_1$

(3)$\mathrm{Tr}(ax) = \mathrm{Tr}(xa)$

Proof. $||x||_\infty = \sup_{||\xi||\leq 1} ||x\xi||$

Fix $\xi \in H, ||\xi|| = 1$. Then, $\exists (\xi_i)_{i\in I} \subset H$ ONB. Then,

$$\begin{aligned}
||x\xi||^2 &= < x^*x\xi, \xi_i > \\
&\leq \sum_{i=1} < x^*x\xi_i, \xi_i > \\
&= \mathrm{Tr}(x^*x) \\
&= \mathrm{Tr}(|x|^2) \\
&\leq ||x||_\infty \mathrm{Tr}(|x|) \\
&= ||x||_\infty ||x||_1
\end{aligned}$$

Now consider $||ax||_1$. Then,

$$\begin{aligned}
a^*a &\leq ||a||^2 1 \\
x^*a^*ax &\leq ||a||_\infty^2 x^*x \\
(ax)^*ax &\leq ||a||_\infty^2 \cdot x^*x \\
|ax|^2 &\leq ||a||_\infty^2 |x|^2 \\
\Rightarrow |ax|| &\leq ||a||_\infty |x| \\
\mathrm{Tr}((ax)) &\leq \mathrm{Tr}(||a||_\infty \cdot |x|) \\
\therefore ||ax||_1 &\leq ||a||_\infty ||x||_1
\end{aligned}$$

$u \in U(B(H))$
$\mathrm{Tr}(xu) = (\xi_i)_{i\in I}$ ONB. Then,

$$\begin{aligned}
\mathrm{Tr}(xu) &= \sum_i < xu\xi_i, \xi_i > \\
&= \sum_i < xu\xi_i, u^*u\xi_i > \\
&= \sum_i < uxu\xi_i, u\xi_i > \\
&= \mathrm{Tr}(ux)
\end{aligned}$$

□

25 October

Exercise Show that the finite rank operator $FR(H) \subset L^1(B(H))$ and moreover $FR(H) = L^1(B(H))$.

Hint $\mathrm{Tr}(\xi \otimes \bar{\eta}) =< \xi, \eta >$ (finite linear combination). Show dense.

Apply definition. Let $(b_i)_{i\in I}$ be an ONB. Then,

$$\begin{aligned}
\mathrm{Tr}(\xi \otimes \bar{\eta}) &= \sum_{i=1}^{\infty} \xi \otimes \bar{\eta} < b_i, b_i > \\
&= \sum_{i\in I} << b_i, \eta > \xi, b_i > \\
&= \sum_{i\in I} < b_i, \eta >< \xi, b_i > \qquad \text{Prove this is the same as dot product} \\
&= \sum_{i\in I} < \xi_i, < \xi_i, \eta >, \xi_i > \\
&=< \xi_i, \sum_i \overline{< b_i, \eta >} b_i > \\
&=< \xi_i, \sum < \eta, b_i > b_i > \\
&=< \xi, \eta >, \quad i \in I
\end{aligned}$$

Theorem 45. $(L^1(B(H)), ||\cdot||_1)$ is a Banach space.

Proof. (From previous lectures), $(L^1(B(H)), ||\cdot||_1)$ is a normed space.

Check Completeness Fix $(x_i)_n subset L^1(B(H))$ is a Cauchy sequence.

$\forall n, m \in \mathbb{N}, ||x_n - x_m||_\infty \leq ||x_n - x_m||_1 < \epsilon$.

$\Rightarrow (x_n)_n \subset B(H)$ is a $||\cdot||_\infty$ -Cauchy sequence.

$\Rightarrow \exists x \in B(H)$ such that $||x_n - x||_\infty \to 0$.

$||x_n - x_m||_\infty \to 0$ as $n, m \to \infty$.

$\Rightarrow ||x_n - x||_\infty \to 0$ as $n \to \infty$

$\Rightarrow |||x_n| - x||_\infty \to 0$ as $n \to \infty$ (application of continuous functions).

Now suppose $\xi_1, \xi_2, ..., \xi_n \subset H$ is orthonormal system. Then, we need to analyze the trace as follows:

$$\begin{aligned}
\sum_{i=1}^{k} < |x|\xi_i, \xi_i > &= \lim_{n\to\infty} \sum_{i=1}^{k} < |x_n|\xi_i, \xi_i > \\
&\leq \mathrm{Tr}(x_n) \\
&= ||x_n||_1 \\
&\leq C \text{ for every } n \in N
\end{aligned}$$

This is bounded by constant for all k.

$||x_|||_1 = \lim_{k\to\infty} \sum_{i=1}^{k} < |x|\xi_i, \xi_i >= C \Rightarrow x \in L^1(B(H))$.

Fix $\epsilon > 0$. $\exists N \geq > 0$, $\forall n \geq N$ such that $|x_n - x_N||_1 < \epsilon$.

Let $H_0 \leq H$ be a finite dimensional subspace. Then $||x_N P_{H_0^\perp}|| < \epsilon$ (by theorem above).

Also assume that $||x P_{H_0^\perp}||_1 < \epsilon$. Then,

$$
\begin{aligned}
||x - x_n||_1 &= ||(x - x_n) \circ (P_{H_0} + P_{H^\perp})||_1 \\
&\leq ||(x - x_n) \circ P_{H_0}||_1 + ||(x - x_n) \circ P_{H_0^\perp}||_1 \\
&\leq ||(x - x_n) \circ P_{H_0}||_1 + ||(x - x_N) \circ P_{H_0^\perp}||_1 + ||(x_N - x_n)P_{H_0^\perp}||_1 \\
&\leq ||(x - x_N)P_{H_0}||_1 + 3\epsilon \\
&= \sum_{i=1}^{k} < (x - x_n)\xi_i, \xi_i > +3\epsilon \\
&\leq k||x - x_n||_\infty + 3\epsilon
\end{aligned}
$$

□

Theorem 46 (Pre-Dual Property).

$$
\begin{aligned}
\exists \text{map} \Psi : B(H) &\to (L^1(B(H)), || \cdot ||)^* \\
a &\mapsto \Psi_a(x) = Tr(ax), x \in L^1(B(H))
\end{aligned}
$$

is a Banach space since this is an isomorphism.

Proof. Show surjectivity.

Fix $\phi \in L^1(B)H))^*$.

Take $(\xi, \eta) \in H \times H$. Then, $(\xi, \eta) \to \phi(\xi \otimes \bar{\eta})$, (rank 1) is a bounded, sesquilinear form y the Riesz Representation Theorem.

$\exists a \in B(H))$ such that

$\mathrm{Tr}(a \otimes \bar{\eta}) =< a\xi, \eta >= \phi(\xi \otimes \bar{\eta})\ \forall \xi, \eta \Rightarrow \phi = \Psi_a$.

Since $\mathrm{Tr}(ax) = \phi(x)$, this is onto. □

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27 October

11 Hilbert Schmidt Operators

Recall $L^1(B(H)) = \{x \in B(H), ||x||_1 = \mathrm{Tr}(|x|) < \infty\}$.

$L^2(B(H)) = \{x \in B(H)|x|^2 \in L^1(B(H)) = Tr(x^*x) = Tr(|x|^2) < \infty\}$

Here are some properties for $L^1(B(H)); \mathrm{Tr})$.

Lemma 47.

1. $L^2(B(H)) \subset B(H)$ is a 2-sided ideal.
2. $\forall xy \in L^2(B(H)) \Rightarrow xy, yx \in L^1(B(H))$
3. $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$

Proof. $x, y \in L^2 \Rightarrow x + y \in L^2$. Then,

$$
\begin{aligned}
|x + y|^2 &\leq |x + y|^2 + |x - y|^2 \\
&= (x + y)^*(x + y) + (x - y)^*(x - y) \\
&= (x^* + y^*)(x + y) + (x^* - y^*)(x - y) \\
&= x^*x + x^*y + y^*x + y^*y + (x^*x - x^*y - x^*x + y^*y) \\
&= 2x^*x + 2y^*y \\
&= 2|x^2| + 2|y^2| \\
\mathrm{Tr}(|x + y|^2) &\leq 2\mathrm{Tr}(|x|^2) + 2\mathrm{Tr}(|y|^2) < \infty
\end{aligned}
$$

Therefore, it is a linear subspace.

$$\begin{aligned}
|ax|^2 &= (ax)^*(ax)\\
&= x^*aax\\
&= ||x^*a|| - x^*x\\
&= k||a||^2 \cdot |x|^2
\end{aligned}$$

for $a \in B(H), x \in L^2(B(H))$.

We will now use Polarization.

$x, y \in L^2 \Rightarrow xy$.

$$\begin{aligned}
y^*x &= \frac{1}{4}\sum_{p=0}^{3} i^k(y+i^k)^*(y+i_x^k)\\
\mathrm{Tr}(y^*x) &= \mathrm{Tr}(\frac{1}{4}\sum_{p=0}^{3} y+i_x^k)^*(y+i_x^k)\\
&= \frac{1}{4}\sum_{i=0}^{3} \mathrm{Tr}((y+i_x^k)^*) * (y+i_x^k)\\
&= \frac{1}{4}\sum \mathrm{Tr}((y+i_x^k)(y+i_x^k)\\
&= \mathrm{Tr}(xy^*)
\end{aligned}$$

□

11.1 Hilbet Schmidt Operators

$x, y \in L^2(B(H), \mathrm{Tr})$.

Define $< x, y >_2 = \mathrm{Tr}(y^*x)$. Then,

$$\begin{aligned}
0 \leq < x, x >_2 &= \mathrm{Tr}(|x|)^2 = ||x||_2^2\\
||x_{ii}||_2^2 &= < |x|^2\xi_i, \xi_i > = 0
\end{aligned}$$

$||axb||_2 \leq ||a||_\infty \cdot ||b||_\infty \cdot ||x||_2 \ \forall a, \in B(H), x \in L^2(B(H))$.

$$\begin{aligned}
||x||_2 = \sup_{||y||_2 \leq 1, y \in L^2(B(H))} &= \sup_{||y||_2 \leq 1} |\mathrm{Tr}(y^*x)|\\
&\leq \sup_{||y||_2} ||y^*||_\infty ||x||_1\\
&\leq ||x||_1
\end{aligned}$$

Therefore $||x||_\infty \leq ||x||_2 \leq ||x||_1 \ \forall x$.

$$\begin{aligned}
||x||_2^2 &= \mathrm{Tr} < |x|^2\xi_i, \xi_i > \geq < |x|\xi_i, \xi_i >\\
&= |||x|\xi_i||^2\\
&= ||x||_\infty^2
\end{aligned}$$

$x, y \in L^2(B(H)) \Rightarrow ||xy||_1 \leq ||x||_2||y||_2$.

Proof.

$$\begin{aligned}
||xy||_1 &= \mathrm{Tr}(|xy|) \\
&= \mathrm{Tr}(x^*v)^*y) \\
&= | < y, x^*v >_2 | \\
&\leq ||y||_2 \cdot ||x^*v||_2 \\
&\leq ||y||_2 \cdot ||x^*||_2 ||v||_\infty \\
&= ||y||_2 ||x||_2
\end{aligned}$$

Here v is the projection. □

---

29 October H, K Hilbert spaces. Consider $H\bar{\otimes}K$, which is new Hilbert space.

$HS(H,K) = \{x : H \to K \text{linear, bounded} | \tilde{x} \in L^2(B(H \oplus K)\}$

$x : H \to K$.

Definition 11.1.

$$\begin{aligned}
\tilde{x} : H \oplus K \to H \oplus K \\
\tilde{x}(\xi \oplus \eta) = O \oplus x(\xi)
\end{aligned}$$

(bounded operator $\Rightarrow \in L^2(B(H \oplus K))$.

Lemma 48. *$HS(H,K)$ form a closed subspace $L^2(B(H \oplus K))$*

Proof. Exercise. □

Observe (When $K = \mathbb{C}$), $HS(H, \mathbb{C})$ is the dual of H (by Riesz Representation $H^* = \bar{H}$).

By Riesz Representation Theorem, this is naturally anti-isomorphic to H.

11.2 Lifting Procedure

$\bar{H}$ = conjugate of H.

$$\begin{aligned}
B(H) \to B(\bar{H}) \\
x \mapsto \bar{x}, \quad \bar{x}(\bar{\xi}) = \overline{x\xi}
\end{aligned}$$

Consider $H\bar{\otimes}H$ (completion of algebraic structure), where$(\xi_i)_i \subset H$ ONB and $(\eta_j)_j \subset K$ ONB. Then, $(\xi_i \otimes \eta_j)_{i \in I, j \in J}$ ONB for $H \otimes k$.

Note that the span of $\overline{\xi_i \otimes \eta_j} = H\overline{\otimes}K$.

Also note that $H\overline{\otimes}K \supset H \otimes_{\mathbb{C}} K = \{\sum_{\text{finite}} \xi \otimes \eta | \xi \in H, \eta \in K\}$.

Then, we apply tensor to a vector.

$x \in B(H), y \in B(K)$, then we can define a new tensor $x \otimes y \in B(H\bar{\otimes}K)$.

Define $(x \otimes y)(h) = xhy^*$. Then,

$$\begin{aligned}
||(x \otimes y)(\xi_i \otimes \eta)| &= ||(x\xi) \otimes (y\eta)|| \\
&\leq ||x\eta|| ||\eta h|| \\
&\leq ||x||_\infty ||\xi|| + ||hy||_\infty \\
\Rightarrow ||x \otimes y||_\infty &\leq ||x||_\infty ||y||_\infty \\
&= ||x||_\infty ||y||_\infty ||\xi \otimes \eta||
\end{aligned}$$

This is bounded on finite numbers.

$$(x \otimes y)^* =< x^*(xy)^*, \eta x^* \eta >$$

1 November 2021

Theorem 49. $L^2(X \times X, \eta \times \eta) \ni k \to T_k \in \ell^2(B(L^2(X)))$

Then $T_k(\xi)(x) = \int_X k(x,y)\xi(y)\ d\mu(y)$ is a unitary, $T_k^* = T_{\bar{k}}$,

Q. Why is this an isometry?

$$||T_k|| \leq ||k||_2 \text{ (Want to show equality)}$$

Suppose $k = \sum_{i,j=1}^n c_{ij}\xi_i \otimes \xi_j$.

$\xi_i, \xi_j \in L^2(X), c_{i,j} \in \mathbb{C}$. For $\eta \in L^2(X)$, we have

$$\begin{aligned} T_k(\eta)(x) &= \int_X k(x,y)\eta\ d\mu(y) \\ &= \int \sum_{i=1}^n c_{ij}\xi_i(x)\xi_j(y)\eta(y)\ d\mu(y) \\ &- \sum_{i=1}^n c_{ij}\xi_i(x) \int_X \bar{\xi}_i(y)\eta(y)\ d\mu(y) \\ &= \sum_{i,j}^n c_{ij}\xi_i < \xi_j, \bar{\eta} > \\ &= c_{ij} x i_i \otimes \bar{\xi_j}(\eta) \end{aligned}$$

$T_k = \sum_{i,j} c_{ij}\xi_i \otimes \xi_j$ (finite rank operator)

$||T_k|| = ||\sum c_{ij}\xi_i \otimes \xi_j|| = ||k||_2$. (If the sum is finite, we have equality).

1. $||T_k|| \leq ||k||_2$
2. $k = \sum_{i=1,j} c_j\xi_i \otimes \xi_j) \Rightarrow ||T_k|| = ||k||_2$

   Fix $k \in L^2(X \times X), \epsilon > 0$. Then,

   $\exists k_\epsilon = \sum_{i,j=1}^n c_{i,j}\xi_i \otimes \xi_j$ such that $||k_\epsilon - k||_2 < \epsilon \Rightarrow ||k_\epsilon|| \geq ||k|| - \epsilon$ (by reverse triangle inequality).

$$\begin{aligned} ||T_k|| &= ||T_k - T_{k_\epsilon} + T_{k_\epsilon}|| \qquad\qquad = ||T_{k_\epsilon} + T_k k_\epsilon|| \\ &\geq -||T_{k-k_\epsilon}|| + ||T_{k_\epsilon}|| \\ &\geq ||T_{k_\epsilon}||_2 - ||T_k - k_\epsilon||_2 \\ &= ||k_\epsilon||_2 - ||k - k_\epsilon||_2 \\ &\geq ||k_\epsilon|| - \epsilon \\ &\geq ||k||_2 - \epsilon - \epsilon \\ \therefore ||T_k|| &\geq ||k|| - 2\epsilon\ \forall \epsilon \end{aligned}$$

As $\epsilon \to 0, ||T_k|| \geq ||k||_2$ (due to continuity).

Theorem 50. $x \in B(H)$. Then FAE:

(1) $x \in \overline{FR(H)}$

(2) $(\xi_\alpha)_\alpha \subset (H)_1 \Rightarrow \xi_\alpha \xrightarrow{\text{weakly}} \xi \Rightarrow x(\xi_\alpha) \to x(\xi)$ (follows from weak topology continuity for x to norm topology of H)

(3) $x(H)_1$ is compact in norm topology

(4) $x(H)_1$ has compact closure in norm topology.

Definition 11.2. $\forall x \in B(H)$ satisfying one of the previous conditions is called a compact operator is

$$\overline{FR(H)}^{||\cdot||_\infty} = K(H)$$

Proof. (1) $\Rightarrow$ (2)

Fix $(\xi_\alpha)_{\alpha\in I} \subset H$.

$\xi_\alpha \xrightarrow{weakly} \xi$. Then, $<\xi_\alpha, \eta> \to <\xi, \eta> \ \forall \eta$

Fix $\epsilon > 0$. Then $\exists y \in FR(H)$ such that $||x - y||_\infty < \epsilon$.

Whenver $\xi_\alpha \xrightarrow{\text{weakly}} \xi \Rightarrow ||x\xi\alpha - x\xi|| \to 0$. Then we apply the triangle inequality:

$$\begin{aligned} ||x\xi_\alpha - x\xi|| &\leq ||x\xi_\alpha - y\xi_\alpha + y\xi_\alpha - x\xi|| \\ &\leq ||x\xi_\alpha - y\xi_\alpha|| + ||y\xi_\alpha - y\xi|| + ||(x-y)\xi_\alpha|| \\ &\leq ||x-y||_\infty ||\xi_\alpha|| \\ &< 2\epsilon + ||y\xi_\alpha - y\xi|| \to 0 \end{aligned}$$

i.e. $y \in FR \Rightarrow \dim(\mathrm{Ran}(y)) < \infty$.

$\therefore$, we have prove (1) $\Rightarrow$ (2). $\square$

Proof. (2) $\Rightarrow$ (3)

Idea $x(H_1)$ will move compact sets to compact sets.

Since H_1 is weakly compact, then by (2), $x(H_1)$ is noncompact.

$\square$

Proof. (3) $\Rightarrow$ (4)

Trivial. $\square$

Proof. (4) $\Rightarrow$ (1)

Suppose $(P_\alpha)_{\alpha\in X} \in B(H)$ orthogonal projections. Then,

$||P_\alpha(\xi) - \xi|| \to 0 \ \forall \xi \in H$.

Note that $\dim_{\mathbb{C}}(P_\alpha) < \infty$.

$P_n = \overline{\mathrm{span}\{\xi_1, ..., \xi_n\}}$

By composition of operators, $||P_\alpha \circ x - x||_\infty \to 0$.

By way of contradiction, suppose this does not converge uniformly $\to 0$.

$\Rightarrow \ \exists(\xi_\alpha) \in H_1$ such that $|P_\alpha \circ (\xi_\alpha) - \xi_\alpha||| \geq \xi_0 > 0$

By (4), $x\xi_\alpha$ has a limit point $\Rightarrow \ \exists x \in H$ such that $x\xi_\alpha \to x$. Then

$$\begin{aligned} \xi_0 &\leq ||x\xi_\alpha - Px\xi_\alpha|| \\ &\leq ||\xi - P_\alpha\xi|| + ||(1 - P_\alpha(x\xi_\alpha - \xi)|| \\ &\leq ||\xi - P_\alpha\xi|| + 2||x\xi_\alpha 0\epsilon|| \\ &\to 0 \end{aligned}$$

$\square$

---

3 November 2021 Recall, last time we had a theorem that said $x \in B(H)$ is compact iff $x \in \overline{FR(H)}$

Lemma 51. Suppose $x \in K(H), \sigma_{ap}(x) - \{0\} \leq \sigma_p(x)$.

Proof. Pick $\lambda \in \sigma_{ap}(x) - \{0\}$. Then,

$\exists \ \{\xi\}_i \subset (H)_1 = \{\xi \in H | ||\xi|| = 1\}$ such that $||(x-\lambda)\xi_n|| \to 0$ as $n \to \infty$.

$||x\xi_n - \lambda\xi_n|| \to 0$. $\square$

$x \in K(H)$, $\{x\xi_n\}_n$ is pre-compact.

$\exists \xi \in H$ such that $x\xi_n \to \xi$ as $n \to \infty$ after passing to a subsequence.

Then, by continuity we have $x(x\xi_n) \to x\xi$. Next,

$$\begin{aligned}
||x\xi_n - \lambda\xi_n|| &\to 0 \\
||x(x\xi_n - \lambda\xi_n|| \le |x||_\infty \cdot ||x\xi_n - \lambda\xi_n|| &\to 0 \\
||x(x\xi_n) - \lambda x\xi_n|| &\to ||x\xi - \lambda\xi|| \\
\Rightarrow ||x\xi - \lambda\xi|| &= 0 \\
\Rightarrow x\xi &= \lambda\xi \quad \lambda \in \sigma_{ap}
\end{aligned}$$

Lemma 52. $x \in K(H)$ $\forall$ point in $\sigma(x) - \{0\}$ is an isolated point.

Proof. Let $(\lambda_n)_n \subset \sigma(x) - \{0\}$ such that $\lambda_n \to \lambda$.

$\sigma(x) = \sigma_{ap}(x) = \overline{\sigma_{ap}(x^*}$.

We can assume without loss of generality that passing to a subsequence $(\lambda_n)_n \subset \sigma_{ap}(x) - \{0\}$ □

Note that λ_n are distinct eigenvalues $\iff$ will have nonzero eigenvectors.

$\therefore \exists \eta_k \neq 0$ such that $\eta_n in H$ such that

$$x\xi_n = \lambda_n \eta_n \ \forall n$$

.

$\therefore \{\eta_n | n \in \mathbb{N}\}$ are linearly independent.

5 November 2021 Last time, we discussed a lemma that if $x \in K(H)$ (compact operator), then $\sigma(x) - \{0\}$ are isolated (no limit points).

Theorem 53 (Fredholm Alternative). Let $x \in K(H)$. Then $\sigma(x) - \{0\} = \sigma_p(x) - \{0\}$.

Proof. $\sigma(x) - \{0\} \subset \partial(\sigma(x)) \subset \sigma_{ap}(x) - \{0\} \subset \sigma_p(x)$ □

Theorem 54. Suppose $x \in K(H)$ normal.

$\forall \lambda \in \sigma_p(x), E_\lambda$ the eigenspace (subspace of Hilbert space).

$x = \sum_{\lambda \in \sigma(x) - \{0\}} \lambda P_{E_\lambda}$ (orthogonal projection) , where the sum is in the topology induced by $|| \cdot ||_\infty$.

12 Locally Convex Topologies on $B(H)$

8 November 2021 WOT, SOT is generated by family of semi-norms.

(1) Weak Operator Topology

$$B(H) \ni x \mapsto |< x\xi, \eta >| = |P_{\xi,\eta}(x)| \in \mathbb{R}_+ \ \forall \xi, \eta \in H$$

This is the samllest topology on $B(H)$ so that $P_{\xi,\eta}$ are continuous for $\xi, \eta \in H$.

(2) Strong Operator Topology (finer topology - WOT < SOT < Norm)

$$B(H) \ni x \mapsto ||x\xi|| = P\xi(x)$$

This is the smallest topology on $B(H)$ such that P_ξ is continuous.

Note that $P_{\xi,\eta}(x) = |< x\xi, \eta >| \le ||x\xi|| ||\eta|| = P_\xi ||\eta||$

Basis $V_{\text{WOT}}(x_0, .., \xi_1, .., \xi_n, \eta_1, ..., \eta_n) = \{x \in B(H) | P_{\xi,\eta_i}(x - x_0) < \epsilon, i > \bar{i}, \eta\}$

Theorem 55. WOT, SOT have the same dual i.e.
$\phi : B(H) \to \mathbb{C}$ linear functional. TFAE
(1) $\exists \xi_1, \xi_2, ..., \xi_n \in H$ such that $\phi(x) = \sum_{i=1}^{n} < x\xi_i, \eta_i >$.
(2) ϕ is WOT-continuous
(3) ϕ is SOT-continuous

Consequence Two different duals if duals same, the closure on convex sets is the same.

Proof. (1) $\Rightarrow$ (2)
For $x_i \xrightarrow{\text{WOT}} x$, show that $\phi(x_i) \to \phi(x)$.

$$|\phi(x_i)| = |\sum_{i=1}^{n} < x\xi_k, \eta_k > \\ \leq \sum_{k=1}^{n} | < x\xi_i \eta_k > \to 0$$

□

Proof. (2) $\Rightarrow$ (3)
$x_i \xrightarrow{\text{SOT}} x \Rightarrow x \xrightarrow{\text{WOT}} x \Rightarrow \phi(x_i) \to \phi(x)$. □

Proof. (3) $\Rightarrow$ (1) (This is hard)
Assume ϕ is SOT-continuous.
$\exists\ k > 0, \xi_1, .\xi_2, ..., \xi_n \in H$ such that
$|\phi(x)| \leq k \sum_{i=1}^{n} ||x\xi_i||^2 \leq k(\eta \sum_{i=1}^{n} ||x\xi||^2)^{1/2}$
$\phi^{-1}(D(0,1)) \subset B(H)^{\text{SOT}}$ open set.
$\exists \xi_0 > 0, \xi_1, ..., \xi_n \in H$ such that

$$V(0, \xi_1, .., \xi_n, \xi_i > 0) \subset \phi^{-1}(D(0,1))$$

$\forall x \in V, |\phi(x)| \leq 1$. $||x\xi_i|| < \xi_0 \forall i = \overline{1,n}$. Then,

$$||(\frac{\xi_0}{\sum_{i=1}^{n} ||x\xi_i} \cdot x)\xi k||_2 \leq ||\frac{\xi_0/2||x\xi_k||}{\sum_{i=1}^{n} ||x\xi_k||} \\ \leq |\frac{\xi_0}{2(\sum ||x\xi_i||)}|\phi(x_i)| \qquad < 1$$

$$|\phi(x) \leq k_0^{1/2} (\sum_{i=1}^{n} |x\xi_i||^2)^{1/2} \\ = \oplus_{i=1}^{n} x\xi_i||^2$$

Consider $H_0 = \{(x\xi_1, x\xi_2, .., .x\xi_n) = \oplus_{i=1}^{n} x\xi_i\} \leq \oplus H = H^n$
$\tilde{\phi}(\oplus_{i=1}^{n} x\xi_i) = \phi(x)$
We need to make sure it is well efined.
$K_0|| \oplus_{i=1}^{n} x\xi_i|| \geq |\phi(x)| = \tilde{\phi}(\oplus_{i=1}^{n} x\xi_i)$
We note that a subspace of a Hilbert space is Hilbert space. We can also apply Hahn-Banach to extent $(\tilde{\phi})$.

By application of H-B and Riesz-Representation Theorem, we get that

$$||\tilde{\phi}(\oplus_{i=1}^{n}\phi_i|| = \sum_{i=1}^{n} < h_i, k_i >$$
$$\Rightarrow \phi(x) = \sum_{i=1}^{n} < x\xi_i, \overline{\xi_i} >$$
$$\therefore \tilde{\xi_i} = \xi_i$$

$\square$

Corollary 55.1. $K \subset B(H)$ is a convex set and $\overline{K}^{\text{WOT}} = \overline{K}^{\text{SOT}}$.

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Lemma 56. $\phi : B(H) \to \mathbb{C}$ linear, then TFAE:

1. $\exists\ \xi_1, \xi_2, ..., \xi_n, \eta_1, ...\eta_n \in H$ such that

$$\phi(x) = \sum_{i=1}^{n} < x\xi_i, \eta_i > \tag{3}$$

2. ϕ is WOT-continuous

3. ϕ is SOT-continuous

Corollary 56.1. $K \subset B(H)$, K convex.

$\overline{K}^{\text{WOT}} = \overline{K}^{\text{SOT}}$

$\Rightarrow$ We first show $\overline{K}^{\text{WOT}} \supset \overline{K}^{\text{SOT}}$.

$\exists x \in \overline{K}^{\text{WOT}}$ such that $x \notin \overline{K}^{\text{SOT}}$.

Then, we have $\phi \in (B(H))^*, \text{WOT})$. We can use this to come up with the conclusion that

Re $(\phi(x)) > \epsilon_0$+Re$(\phi(y)), y \in \overline{K}^{\text{SOT}}$.

Contradiction.

13 σ-WOT, σ-SOT

Consider the 'inflated space' $B(H\overline{\otimes}\ell^2(\mathbb{N}))$.

There is a natural map

$$B(H) \to B(H\overline{\otimes}\ell^2(\mathbb{N})^{\text{WOT}}$$
$$x \mapsto x \otimes 1$$

Now we can think of the pullback topology i.e.

σ-WOT = pullback of $B(H \otimes \ell^2(\mathbb{N})$ under this map.

σ-SOT = pullback of $(B \otimes \ell^n(\mathbb{N})$ under this map.

$B(H) \ni T \mapsto |\text{Tr}(aT)|, a \in \ell^1(B(H))$.

$B(H) = L^{\perp}(B(H))^*, \Psi(a) \in \text{Tr}(a)$.

σ-WOT on $B(H)$ agrees with the weak*-topology of $B(H)$ (with this, we have compactness properties on the unit ball).

Lemma 57. $\phi : B(H) \to \mathbb{C}$ linear. Then TFAE:

1. $\exists a \in \ell^1(B(H))$ such that $\phi(x) = \text{Tr}(ax)\ \forall x \in B(H))$.

2. ϕ is σ-WOT.

3. ϕ is σ-SOT.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

$3 \Rightarrow 1$ Consider $\phi : B(H) \to C$, where $B(H) \subset B(H \otimes \ell^2(\mathbb{N})$.

(By the Hahn-Banach Theorem, we can extend this i.e.)

$\exists\ \tilde{\phi} : B(H \otimes \ell^2(\mathbb{N}) \to \mathbb{C}$ SOT continuous.

By the first lemma above, $\exists \xi_1, ...\xi_n, \eta_1, ..., \eta)n \in H \otimes \ell^2(\mathbb{N})$ such that $\phi(x) = \sum_{i=1}^{n} < x\xi_i, \eta_i >$.

Now, let

$$H \otimes \ell^2(\mathbb{N}) = \text{HS}(H, \ell^2(\mathbb{N})$$
$$\xi_i\eta_i \to a_i b_i$$

(where HS stands for the Hilbert-Schmidt operator).

Let $a = \sum_{i=1}^{n} b_i^* a_i \in \ell^1(B(H))$. Then,

Tr $(ax) = \text{Tr}(xa) = \sum < a_i x, b_i >_2$. Since Trace is linear, we have the following

$$\begin{aligned} \text{Tr}(xa) &= \text{Tr}(x \sum_{i=1}^{n} b^i a_i) \\ &= \sum_{i=1}^{n} \text{Tr}(x b^i a_i) \\ &= \sum_{i=1}^{n} < \text{id} \otimes 1(x)\xi_i, \eta_i > \\ &= \phi(x) \end{aligned}$$

□

Corollary 57.1. $((B(H))_1$ (unit ball) is σ-WOT compact.

(i.e. agress with the weak* topology and Alaoglu's theorem gets us htere).

Corollary 57.2. WOT, σ-WOT agree on bounded sets.

Homework Proof of the corollary above.

14 Von Neumann Algebras

Definition 14.1 (Von Neumann Algebra)**.** $M \subset B(H)$ is called a VN algebra if $1 \in M$ and $M = \overline{M}^{\text{WOT}} = (\overline{M}^{\text{SOT}}) = M''$.

Definition 14.2. $1 \in M \subset B(H)$ self-adjoint algebra is called a VN algebra iff $\overline{M}^{SOT} = \overline{M}^{WOT} = M$.

Notation $A \subset B(H)$. Then, $W^*(A) =$ VN algebra generated by $A = \cap_{A \subset M} M$.

Corollary 57.3. (von Neumann bicommutant) Let $a \subset B(H)$, $*$-subalgebra, then $A'' = \overline{A}^{\text{SOT}}$.

Definition 14.3. $S \subset B(H)$. Then, (commutant) $S' = \{T \in B(H) | Tx = xT \ \forall x \in S\}$. We will denote this by $\star$.

1. $\star$ is a subalgebra of $B(H)$.

2. If S is self-adjoint then $\star$ is a $*$- sublagebra of $B(H)$.

Lemma 58. $S \subset B(H)$ is a slef adjoint st $(S = S^*)$, then $S' \subset B(H)$ is a VN algebra.

Proof. S' is WOT-closed.

Fix $x_\alpha \to x$ WOT i.e. $< x_\alpha \xi_\alpha, \eta_\alpha > \to < x \xi_\alpha, \eta_\alpha > \ \forall \xi_\alpha, \eta_\alpha \in H$.

(Here $x_\alpha, x \in S'$.

Show $[x, a] = xa - ax = 0 \forall a \in S$ (commutator). Fix $\xi_n \in H$.

Then, we have $< [x, a]\xi, \eta >= 0$. Then,

$$\begin{aligned} < [x, a]\xi, \eta > &=< (xa - ax)\xi, \eta > \\ &=< xa\xi, \eta > - < ax\xi, \eta > \\ &=< xa\xi, \eta > - < x\xi, a^*\eta > \\ &= \lim_\alpha < x_\alpha a\xi, \eta > - \lim_\alpha < x_\alpha \xi, a^*\eta > \\ &= \lim_\alpha < x_\alpha a\xi, \eta > - \lim_\alpha < x_\alpha \xi, a^*\eta >) \\ &= \lim_\alpha (< x_\alpha a\xi, \eta > - < a x_\alpha \xi, \eta >) \\ &= \lim_\alpha < x_\alpha a - a x_\alpha \xi, \eta > \\ &= 0 \end{aligned}$$

□

Lemma 59. $1 \in A \subset B(H)$ self -adjoint ($*$-subalgebra)
$\forall \xi \in H, \forall x \in A'', \exists (x_\alpha)_\alpha \subset A$ such that

$$||x_\alpha \epsilon - x\epsilon|| \to 0.$$

Show $x_\epsilon \in \overline{A_\epsilon}$.

Proof. $\xi \in H$. Take $\overline{A\xi} \leq H$.
$H_0 = \overline{A\xi} \leq H$.
$p' = P_{H_0}$ is the orthogonal projection (Remarkable: Projection lines in bicommutant).
Let $p \in A$. Then,
$a(\overline{A\xi}) \subset \overline{A\xi}$.
$aH_0 \subset H_0$. (H_0 is an invariant space for a).
Pick $x \in H_0$

$$\begin{aligned} p(a\eta) &= a\eta \\ pap(\eta_0) &= ap(\eta_0) \quad \forall \eta_0 \in H \\ pap &= ap \quad forall a \in A \\ \Rightarrow (pa^*p)^* &= (a^*p)^* \\ pap^* &= pa \qquad \text{since } p = p^* \end{aligned}$$

$\therefore pa = pap = ap \Rightarrow pa = ap$.
Let $x \in A'', p \in A'$. Then,

$$\begin{aligned} xp &= px \\ xp'(\eta) &= px(\eta) \in \overline{A\xi}, \text{ where } \eta \in H \\ p(H) &= \overline{A\xi} \\ x\xi &\in \overline{A\xi} \text{ since } A \text{ has a unit} \end{aligned}$$

$\Rightarrow (x_\alpha)_a \subset A$ such that $||x\xi - x_\alpha \xi|| \to 0$. □

Claim $A'' \supset A \Rightarrow \overline{A''}^{\text{SOT}} \supset \overline{A}^{\text{SOT}}$.

Show $A'' \subset \overline{A}^{\text{SOT}}$.
Fix $x \in A'' \Rightarrow x \in \overline{A}^{\text{SOT}}$.
$\forall \xi_1, ..., \xi_n \in H,\ \exists (x_\alpha)_\alpha \subset A$ such that $||x_\alpha \xi - x\xi_i|| \to 0, p_i(x_\alpha - x) - \eta_i)$.
Let $H^n = \underbrace{H \oplus H ... \oplus H}_{n \text{ times}}$. Then,
$B(H^n) \cong M_n(B(H))$.

$$\tilde{A} \subset B(H^n) = \{ \begin{pmatrix} a & ..... & 0 \\ 0 & a.... & 0 \\ . & .... & .. \\ 0 & .... & a \end{pmatrix} | a \in A \}$$

$$x \in \{ \begin{pmatrix} x & ..... & 0 \\ 0 & x.... & 0 \\ . & .... & .. \\ 0 & .... & x \end{pmatrix}, x \in A.$$

$\tilde{\lambda} \in \tilde{A}''$.

$||\tilde{x}\tilde{\xi} - \tilde{x_\alpha}\tilde{\xi}|| \to 0$, where $\tilde{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}$

3 December 2021

15 Borelian Functional Calculus

$x \in B(H), xx^* = x^*x.$

$A = C^*(xx^*, 1) = C^*(x, 1) \subset B(H)$ (Abelian C^*- algebra).

Recall $\pi : A \to B(H)$ $*$-representation

$\exists!$ spectral measure $\sigma(A)$ such that $\pi(x) = \int \Gamma(x)\ dE$

$\sigma(A)$ $\sigma(x)$, we obtain an isomorphism

$$C(\sigma(X)) \mapsto B(H)$$
$$f \to \int_{\sigma(X)} f\ dE$$

$B^\infty(\sigma(x)) \ni f \to \int_{\sigma(x)} f dE \in B(H)$, where $f\ dE = f(x)$.

$\forall f \in B^\infty(|x|)$. Let $f(x) = \int_{\sigma(x)} f\ dE$. Here,

$f(z) = z, x = \int_{\sigma(x)} t\ dE$

Theorem 60 (Borelian Functional Calculus). Let $A \subset B(H)$ be a VN Algebra and let $x \in A$ normal. Then, the functional calculus defined $f \mapsto f(x$ satisfies the following proposition:

(i) $f \mapsto f(x)$ is a continuous unital x-homomorphism

(ii) $\forall f \in B^\infty(\sigma(x)), \sigma(f(x)) \subset \overline{f(\sigma(x))}$

(iii) If $f \in C(\sigma(x))$, Borel calculus agrees in the continuous one.

$X_{(\xi,\infty)}(t) \cdot t \geq \epsilon X_{(\xi,\infty)}(t)$

$X_{(\xi,\infty)}(x) \cdot x \geq \epsilon X_{(\xi,\infty)}(x)$

Call $X_{(\xi,\infty)}(x) = e_\xi$. Then,

$xe_\xi \xi e_\xi \Rightarrow x \geq e_\xi$. On this algebra, this is invertible.

15.1 Take Home Exam

Let H be a separable Hilbert space and $M \subset B(H)$ is an abelian VN algebra. Then show that M is $*$-isomorphic to $L^\infty(X, \mu), X$ is a compact, metric space and μ is a Borel regular measure on X.

---

16 Decomposition Into Types for VN Algebras

$M \subset B(H), P(M) = \{p \in M | p = p^2 = p^*\}, U(M) = \{u \in M | uu^* = u^*u = 1\}$

$(p_i)_i \in P(B(H))$.

Definition 16.1 ("Smallest projection that dominates everything")**.** $\vee_{i \in I} p_i$ = smallest $p \in P(H)$ such that $p \geq p_i \; \forall i$

Equivalently, this is the Proj $\overline{\text{lin span}}\{\text{p}_i(H) | i \in I\}$

Definition 16.2 ("Largest projection that is smaller than everything")**.** $\wedge_{i \in I} p_i$ = largest $p \in P(B(H))$ such that $p \leq p_i \; \forall i$

$(=$ Proj $\cap_{i \in I} p_i H)$

Proposition. If $(p_i)_i \subset P(M) \Rightarrow \wedge_i p_i \in M, \vee_i p_i \in M$.

$p \in P(B(H)), p \in M = (M')$

$[p, y] = 0 \; forally \in M' \cap B(H))$

$B(H)$ is invariant $\forall y \in M$.

$Z(M) = M \cap M'$

M is a factor $Z(M) = \mathbb{C}1$

$p \in P(M)$ is called a central $p \in Z(M)$

Central support of p (that is in center such that when you multiply, it does not change) is the smallest $z(p) \in Z(M)$ such that $0 \subset Z(p)$

Theorem 61. $p \in M \subset B(H)$ $z(p) = \overline{MpH}$

$z(p) =$Proj $\overline{MpH} \in B(H)$

$pH \subset MpH$

$\Rightarrow pH \subset \overline{MpH} \to b \leq z$ (Range of 1 contained in range of z)

$x(\overline{MpH}) \subset \overline{MpH}$. Since it is continuous,

$y(MpH) \Rightarrow (mpyH) \subset mPH$ (can move to closure)

---

6 December 2021 Recall, $p \in P(M), z(p) =$central support and $z(p) \in Z(M) = M \cap M' \ni \mathbb{C}1$ is the smallest project $z \in Z(M)$ such that $p \leq z$.

Theorem 62. $z(p) = P_{\overline{MpH}}$.

$z = \overline{MpH} \leq H \in M \cap M$.

$z(p), 1 \in M$. Then,

$$\begin{aligned} 1 \cdot \overline{pH} &= \overline{MpH} \\ p(H) &= \overline{MpH} \\ \Rightarrow p \leq z(p) &\iff z(p) \cdot p = p \\ &= pz(p) \end{aligned}$$

Note that the range of a projection is closed. Then,

$f \leq e \iff fH \subset eH \iff fe = ef = f$

Proof. (Proof Idea) $(\Rightarrow)$

$$\begin{aligned} e\xi &= xi \\ e(f\eta) &= f\eta \; \forall \eta \in H \\ \Rightarrow ef(\eta) &= f(\eta) \Rightarrow ef = f \end{aligned}$$

($\Leftarrow$)

$$\begin{aligned}(ef = f)^* \\ f &= f^* e^* \\ &= (ef)^* \\ &= f^* \\ &= f\end{aligned}$$

$\square$

Show $z \leq z(p)$ (By minimality, they have to be equal)

Proof. $\overline{MpH} = Mz(p) \cdot H \subset \overline{Mz(p)H} = z(p) \therefore z \leq z(p)$ (smallest project) $\square$

Exercise

$$\vee_{\sup\ \eta \in B(H)} upu^* z(p)$$

Theorem 63. Suppose $M \subset B(H), p \in P(M), p' \in P(M')$. Then,

$$pMp = \{pxp | x \in M\}$$

This is an algebra.

Proof. (Proof Idea)
$pxp + pyp' = p(xppy)p'$. Hence, still an algebra. $\square$

Now why is this a VN algebra?
$Mp' = \{xp' | x \in M\} \subset B(pH)$
$pxp\xi = pxp(p\xi)$
Theorem 64. The TFH:
$(a)Mp' \subset B(p'(H))$ is a VN algebra.
Compute commutant $(Mp')' = p'M'p'$
$(b)pMp \subset B(pH)$ is a VN algebra
$(pMp)' = M'p$
Corollary 64.1. $Z(pMp) = Z(M)p, Z(Mp') = Z(M)p$
(i) $pMp \cap (pMp)' = Mp \cap Mp'$ (elements in form $Z(m)p$)
(ii) $M'p \subset (pMp)'$ For $m' in M'$

$$\begin{aligned} m'ppp &= ppxpm' \\ &= pxpmp \end{aligned}$$

Therefore, commutes.

Show $(pMp)' \subset M'p$.
Since every element is a linear combination of 4 unitaries of $(pMp)'$, then it suffices to prove the following:
$\forall u \in U((pMp')), \exists u' \in U(M')$ such that
$u = \tilde{u}p, pu = up$.

Proof. Suppose $x_1, x_2, .., x_n \in M, \xi_1, \xi_2, .., \xi_u \in pH$.
$U^* p x_j^* x) jpu = U^* p x_j^* x_i p U$.
Here $U : pH \to pH$ given by
$Up : Up(p\xi) = U(p\xi) Up = p$.
Define $\tilde{u} : H \to H$. Then,

$$\begin{aligned}
\tilde{U}(\xi) &= \sum_{i=1}^{n} x_i U \xi_i, \qquad \xi = \sum_{i=1}^{n} x_i \xi_i \in MpH \\
\Rightarrow ||\tilde{U}(\xi)||^2 &= ||\sum_{i=1}^{n} x_i U \xi_i||^2 \\
&= \sum_{i,j} < x_i U \xi_i, x_j U \xi_i > \\
&= \sum_{i,j} < U^* x) j^* x_i, U \xi_i \xi_i > \\
&= \sum_{i,j} < U^* p x_j^* x_i p U \xi_i, \xi_j > \\
&= ||\sum x_j \xi_i||
\end{aligned}$$

$p \in P(M), M = M''$ □

[8 December 2021]

Proof. (2)$pMP \subset B(pH)$ VN algebra, $(pMp)' = M'p$.
We need to show that $U \in (pMp)'$ unitary, $\exists \tilde{U} \in M'$ such that $U = \tilde{U}p$
Let $x_1, .., x_n \in M, \xi_1, .., \xi_n \in pH$.
$\tilde{U} : H \to H$.

$$\tilde{U}(\xi) = \sum_{i=1}^{n} x_i U \xi_i \quad \forall \xi = \sum_{i=1}^{n} x_i \xi_i \in MpH$$

Last time, $||\tilde{\eta}(\xi)|| = ||\xi|| \forall \xi \in \overline{MpH} = \mathrm{Ran}(Z(p))$.
$\tilde{U}(0) = 0 \ \forall 1_\sigma \in \overline{MpH}^{\perp}$
Initial support of $\tilde{\mu} =$ final support of $\tilde{u} = \overline{MpH}$.
$\tilde{U}p = U, \tilde{U} \in M'$.

$$\begin{aligned} M &= M \cdot 1 \\ &= M(z + (1 - z)) \\ &= Mz + M(1 - z) \end{aligned}$$

If $y \in M(1 - z), \tilde{U} \in y = \tilde{U}(1 - z)y$
$z\tilde{U} = \tilde{U} = \tilde{U}z = \tilde{U}z(1 - z)y = 0 \Rightarrow \tilde{U}y = 0$.
Similarly, $y\tilde{U} = 0$.

Let $y \in Mz; x \in M, x\xi \in MpH$. Then,

$$\begin{aligned} \tilde{U}y(x\xi) &= \tilde{U}(yx\xi) \\ &= y(xu\xi) \\ &= y\tilde{U}(x\xi) \\ \iff U\tilde{y}z &= \tilde{y}Uz \\ U\tilde{y} &= \tilde{y}u \end{aligned}$$

$\square$

Show VN Algebra
$(pMp)'' = pMp$.
$x \in (pMp)''$. If $y \in M'$, then $py = yp \in M'p = (pMp)'$.
$xy = x(py) = (py)x \Rightarrow (yp)x = y(px) = yx$.
$(M')' = M = pMp \Rightarrow x \in (M')' = M$.

$p, \xi \in P(M), p \prec q$ (subequivalent).

Definition 16.3. $\exists\, v \in M$ partial isometry such that $v^*v = p$ and $vv^* \prec q$

Two projections equivalent if partial isometry.
Theorem 65. (1)$p \sim p$
(2)$p \prec q, q \prec r \Rightarrow p \prec r$
(3)$p \prec q, q \prec p = 1, p \sim q$
(Cantor-Bernstein Theorem)

[10 December 2021] $p, q \in P(M), p \sim q$ (von Neumann equivlance)
$\Rightarrow \exists v \in M$ such that $v^*v = p, vv^* = q$
Theorem 66. $\sim$ is an equivalence relation

Proof. $p \sim q \Rightarrow v^*v = p, vv^* = q$
$\Rightarrow v^*(v*)^* \Rightarrow = q \sim p = (v^*)^*(v^*)$ □

$p \prec q$ if $p \sim_0$, where $q_0 \leq q$.
Theorem 67. This is $\propto$.
$p \propto q,\ q \propto p \Rightarrow p \sim q$ (Generalization of C-B).

17 Connection to Cantor-Bernstein

Let S_1, S_2 be two sets.
Form $\ell^2(S_1), \ell^2(S_2)$, each Hilbert space.
$\phi_1 : S_1 \to S_2$ injection $\rightsquigarrow P_{\ell^2 S^1} \propto P_{\ell^2 S^2}$
$\phi_2 : S_2 \to S_1$ injection $\rightsquigarrow P_{\ell^2 S^2} \propto P_{\ell^2 S_1}$
By generalization, $P_{\ell^2 S_1} \sim P_{\ell^2 S^2} \iff \dim(\ell^2 S_1) = \dim(\ell^2 S_2)) \iff |S_1| = |S_2|$.

Lemma 68. Let M VN algebra.
(1) If $p, q \in P(M)$.
Then, $p \sim q \rightsquigarrow \Psi : pMp \to \xi M \xi$ $*$- isomorphic given by $\Psi(x) = vxv^*$.
$v^*v = p, vv^* = \xi$. Then,
$\xi(p) = vpv^* = (vv) \propto (vv)^* = q\eta = \eta$.

Proof. (1) $xy \in pMp$.

$$\begin{aligned}\Psi(x) \cdot \Psi(y) &= vxv^*vyv^* \\ &= vxpyv^* \\ &= vxyv^* \\ &= vxyv^* \\ &= \Psi(xy)\end{aligned}$$

□

Proof. (2)
Suppose $\{p_i\}, \{q_i\} \subset P(M)$
$p_i \sim q_i\ \forall i$
$p_i \cdot p_j = 0(\perp) \Rightarrow q_i q_j = 0$ for $i, j \neq 0$.
$\Rightarrow \sum p_i \sim \sum q_J$. □

Proof. $(3) p \sim q, z \in P(Z(M)) \iff pz \sim qz$
$(vz)^*(vz) = zv^*vz = v^*vz = vvz = Pz$ □

Lemma 69. $p, q \in P(M), M$ VN algebra. Then TFAE
$(1) \exists pMq \neq \{0\}$
$(2) \exists 0 \neq p_1 \leq p, 0 \neq q_1 \leq 1$ such that $p_1 \sim z_1$
$\in pZ(\xi) \neq 0$.

Proof. (2) $\Rightarrow$ (1)
$p1 \leq p$
$\exists 0 \neq v \in M$ partial isometry of $v^*v = p_1 = p, vv^* = q \leq q$.
$pv^* = pv^*(vv^*) = pv^*vv^*q = pv^*q = \subset pMq$ □

Proof. (1) $\Rightarrow$ (2)
$\exists\ x \in pMq \Rightarrow$

$$\begin{aligned} x &= pyq \\ pxq = p^2yq^2 \\ &= pyq \\ \therefore pxq &= x \end{aligned}$$

□

Now let us look at (1) and (3)
$pMq \neq \{0\}, x \in M, \xi \in H$.
$p(xq\xi) = 0$. Take $Mqh = H$ (Hilbert subspace).
$pzq \neq 0$.

Proof. Suppose $pMq = 0$. Then,

$$\begin{aligned} p\eta = 0\ \forall \eta \in \overline{MqH} \leq H\ pz(q) & & = 0 \\ x(p)\xi &= 0 \\ z(q)(x(p)(\xi))) &= 0 \\ \Rightarrow z(q)z(p) &= 0 \end{aligned}$$

($\Leftarrow$)
$z(p) \cdot z(q) = 0 \Rightarrow pxq = pz(p)xq = pxz(p)z(q) = px = 0$ □

Theorem 70 (Comparison Theorem). $p, q \in M_2, M$ VN algebra.
$\exists z \in Z(M)$ such that $pz \prec qz$ and $q(1-z) \prec p(1-z)$
$\therefore p \prec q$ of $q \prec p$.

Proof. (Maximality argument)
$\{p_i\}_{i \in I}, \{q_i\}$ such that
$p_ip_j = 0, q_iq_j = 0$. Then,
$p_1 \geq p \sim q_1 \leq q$.
$p1 = \sum_i p_i' \sim q_1 \sum_i q_i' \leq q$
$\Rightarrow p_2 = p = p1, q = q_1 = q_2$.
$pz = (p_1 + p_2)z = p_1z + p_2z \sim q_1z \leq q_1z + q_2z = qz$ □

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18 Type Classification

2nd Semester 19 January 2022

18.1 Type of projections

Definition 18.1. $p \in P(M) = \{p \in M | p = p^2 = p^*\}$

1. p is minimal $iff \forall q \leq p \Rightarrow q = 0 \iff pMp \cong \mathbb{C}p$ (minimal projection)
2. p is abelian $\iff pMp$ is abelian vN algebra.
3. p is finite $\iff \forall q \leq p$ if $q \sim p \Rightarrow q = p$. (Cannot be subequivalent)
4. p is semifinite $\iff p = \sum_\alpha p_\alpha, p_\alpha$ is finite (Sum of projections. if sum finite p is finite, but could also be infinite)
5. $0 \neq p$ is purely infinite $\iff \nexists q \leq p$ such that q is finite.
6. $0 \neq p$ is properly infinite $\iff \forall 0 \neq z \in Z(M), pz$ is not finite.

18.2 Observations

Minimal $\Rightarrow$ abelian (since if pMp is trivial, then it is abelian).

($q \in p$. Then, $q \in pqp$. There is a partial isometry $w \in pMp$ uch that $w^*w \in p, ww^* = p$.)

Summary 1. minimal $\Rightarrow$ abelian $\Rightarrow$ finite $\Rightarrow$ semifiite $\Rightarrow$ not purely infinite.

2. infinite $\Rightarrow$ properly infinite.

   Now we want to move from projections to algebra (Ex: M is isometry $\Rightarrow 1 \in M$ is isometry).

3. M is finite $\Rightarrow (v^*v = 1 \iff vv^* = 1)$ (i.e. every isometry is a unitary).

4. $B(H)$ (algebra) is finite $\iff H$ is finite dimensional (i.e. it is a matrix algebra). It is also Type I as we will see.

4b. Otherwise $B(H)$ is semifinite if $1 = \sum_\alpha [\mathbb{C}\xi_\alpha], (\xi_\alpha)_{\alpha \in I} \subset H$ ONB.

   (Prototype of minimal)

   [] is orthogonal projection on this space. If $H_0 \leq H$, then $[H] = P_{H_0}$.

18.3 Lemmas

Proposition. M vN algebra, $\{p_\alpha\}_\alpha \subset P(M)$ centrally orthogonal ($z(p_\alpha)z(p_\beta) = 0 \ \forall \alpha \neq \beta$)
Let $p = \sum_\alpha p_\alpha$
If p_α are abelian for $\forall \alpha$ (resp. finite) $\Rightarrow p$ is abelian (resp. finite)

(If you take supremum, this is not generally true. Counter example -¿ infinite dimensional matrix).

Proof. (Abelian Case)
p_α abelian, then pMp_α is an abelian algebra $\forall \alpha$. Show pMp is abelian.
For $x, y \in M$, show that $(pxp)(pyp) = (pyp)(pxp)$.

$$
\begin{aligned}
(pxp)(pyp) &= \sum_\alpha p_\alpha x p_\alpha y p_\alpha \\
p_\alpha xppyp_\beta &= p_\alpha z(p_\alpha) p \cdot pyz(p_\beta) p_\beta \\
&= p_\alpha \underbrace{z(p_\alpha z(p_\beta)}_{=0 \text{ when } \alpha \neq \beta} xppyp_\beta
\end{aligned}
$$

Note $p_\alpha = p_\alpha \cdot z(p_\alpha)$ (since it is central and you multiply by something bigger, it does not change). Hence we have

$$
\begin{aligned}
(pxp)(pyp) &= \sum_\alpha p_\alpha xpyp_{alpha} \\
&= \sum_\alpha p_\alpha x p_\alpha y p_\alpha \\
&= \sum_\alpha p_\alpha x p_\alpha^2 y p_\alpha \\
&= \sum_\alpha p_\alpha x p_\alpha p_\alpha y p_\alpha \\
&= \sum_\alpha p_\alpha y p_\alpha p_\alpha x p_\alpha \\
&= \sum p_\alpha y p_\alpha x p_\alpha \\
&= pyp \cdot pxp
\end{aligned}
$$

□

Proof. (Finite Case)

Assume p_α is finite $\forall \alpha \Rightarrow p$ is finite.

Assume p is subequivalent i.e. $\exists p \sim z \leq p$ (Note,we have a partial isometry $uu^* = p, u^*u = p, uu^* \leq p$). Show that $uu^* = q$.

Look at $u^*u = p$. Fix $\alpha \in I$. Multiply on left/right by central support.

$$
\begin{aligned}
u^*u &= p \\
\Rightarrow z(p_\alpha)u^*uz(p_\alpha) &= z(p_\alpha)pz(p_\alpha) \\
&= z(p_\alpha) \cdot (\sum_\beta p_\beta z(p_\alpha) \\
&= z(p_\alpha)p_\alpha z(p_\alpha) \\
&= p_\alpha \\
&= \underbrace{z(p_\alpha)}_{w^*} \underbrace{u^*uz(p_\alpha)}_{w} \\
\Rightarrow uz(p_\alpha)z(p_\alpha)u^* &= z(p_\alpha)uu^*z(p_\alpha) \\
&\leq z(p_\alpha)pz(p_\alpha) \\
&= p_\alpha \\
&= \underbrace{uz(p_\alpha)}_{w} \underbrace{z(p_\alpha)u^*}_{w^*} \\
&\leq p_\alpha
\end{aligned}
$$

Hence we have $uz(p_\alpha)u^* = p_\alpha \ \forall \alpha$ (since finite) □

Consequences

$$uu^* = uz(p)u^* = u\sum_\alpha z(p_\alpha)u^* = \sum_\alpha uz(p_\alpha)u^* = \sum_\alpha p_\alpha = p \quad \text{(from above)}$$

$$\begin{aligned} u^*u = p &\iff \mathbf{up = u = p\ u} \\ uu^* \le p &\iff uu^*p = uu^* = puu^* \end{aligned}$$

$$\begin{aligned} u &= p \cdot |z(p) \\ uz(p) &= upz(p) = up = u \end{aligned}$$

Proposition. Let M be a vN algebra and let $p \in P(M)$ is finite. Then, every $q \prec p$ is also finite (purely infinite).

Proof. (2 Cases)

$q \prec p \iff \exists p_0 \le p$ such that $q \sim p_0$.

(i) Assume $p_0 \le p$ and $\exists u \in M$ such that $u^*u = p_0, uu^* \le p_0 \Rightarrow ?uu^* = p$ (equivalent to a subprojection).

consider $w = u + p = p_0$ (Not immediately obvious that this is a partial isometry, so we need to check)

$w^*w = p_0 + (p - p_0) = p$.

Note $up = u = pu$ still holds true. Here $up_0 = u = p_0u$.

$$\begin{aligned} ww^* &= (u + p - p_0)(u + p - p_0)^* \\ &= (u + p - p_0)(u^* + p - p_0) \\ &= uu^* + u(p - p_0) + (p - p_0)u^* + (p - p_0)^2 \\ &\le p + up_0(p - p_0) + (p - p_0)p_0u^* + p - p_0 \\ &\le p_0p - p_0 = p \quad \text{(Equality since p finite)} \end{aligned}$$

The equality will happe iff $uu^* = p_0$, which is exactly what we needed.

The middle terms cancel because projections are orthogonal. $\square$

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18.4 Geometry of Projections

Type III is the building block for everything.

(Continued from last time).

Proposition. $M \subset B(H)$ vN algebra.

$0 \neq p, q \in P(M)$ such that $p \prec q$. If q is finite (respectively purely infinite), then p is also finite (respectively purely infinite)

Proof. (Left from Last time).

$p \sim q \Rightarrow p$ is finite.

Since $p \sim q \Rightarrow \exists v \in M$ partial isometry such that $v^*v = p, vv^* = q$.

Suppose $u^*u = p$ and $uu^* \leq p \Rightarrow$ Show $uu^* = p$.

(From last time, with this assumption, we have) $up = pu = u$.

Now look at

$$\begin{aligned}(vuv^*)^*(vuv^*) &= vu^*v^*vuv^* \\ &= vu^*uv^* \\ &= vpv^* \\ &= v(v^*v)v^* \\ &= q \cdot q \\ &= q\end{aligned}$$

On the other hand, look at

$$\begin{aligned}(vuv^*)(vuv^*)^* &= vuv^*vu^*v^* \\ vupu^*v^* & \\ &= vuu^*v^* \\ &\leq vpv^* \\ &= q\end{aligned}$$

Since q is assumed to be finite $\Rightarrow vxv^*(vuv^*)^* = q \Rightarrow vuu^*v^* = vpv^* \Rightarrow uu^* = p$. ☐

Proposition. Let $p \in P(M)$.

Then, p is semifinite $\iff$ $p = \vee_{i \in I} p_i$ when p_i are finite. In particular, if p_j are semifinite $\Rightarrow \vee_j p_j$ is semifinite.

$\Rightarrow$ *Proof.* Since p is semifinite $\Rightarrow p = \sum_{i \in I} p_i, p_i$ are finite, $p_i \perp p_j$ $\forall i \neq j$.

Then, $p = \vee_{i \in I} p_i$ (supremum) ☐

$\Leftarrow$ *Proof.* Assume that $p = \vee_{i \in I} p_i, p_i$ are finite (supremum) [i.e. at least one projection is finite $\Rightarrow$ nonempty for the set below $\Rightarrow$ by Zorn Lemma, can reorder and find a maximum]

(Maximality argument) Let $\{q_j\}_{j \in J}$ be a maximal family of finite pairwise orthogonal projections, $q_j \leq p$ (by Zorn's Lemma)

Consider the difference (show it is 0):

$q_0 = p - \sum_{j \in J} q_j$

$(\Rightarrow q_0 \perp q_j$ $\forall j \in J, q_0 \leq p.)$

If $q_0 \neq 0$, it follows that $\exists$ $j_0 \in J$ such that $p_{j_0} \cdot q_0 \neq 0$.

Then the central support is not perpendicular, i.e. $z(p_{j_0}) \cdot z(q_0) \neq 0 \Rightarrow$ $\exists \tilde{q_0} \leq q_0,$ $\tilde{q_0} \prec p_{j_0}$ (by Lemma)

However, we know that p_{j_0} is finite $\Rightarrow \tilde{q_0}$ is finite means that $\tilde{q_0} \leq q_0$ if $\tilde{q_0} \leq q_0 \leq p$ means you are perpendicular to all q_j's (mutually orthogonal) i.e.

Can construct the set $\{q_j\} \cup \{\tilde{q_0}\}$ forms a family of mutually orthogonal finite subprojections of p, which contradicts the maximality of $\{q_j\}_{j\in J}$ (since assume $q_0 \neq 0$).

Thus $q_0 = 0 \Rightarrow p = \sum_{j\in J} q_j$. ☐

Corollary 70.1. $M \subset B(H)$ vN algebra.

Let $p \in P(M)$. Then p is semifinite $\Rightarrow z(p)$ is semifinite.

Proof. (Central support)

$$\begin{aligned} z(p) &= P_{\overline{MpH}} = [MpH] \\ &= \vee_{q\sim p} q \end{aligned}$$

Homework:

Hint: $p \sim q \Rightarrow p = v'v, q = vv^*, (q \leq z(p))'$

$p \sim q \Rightarrow q \leq z(p) \Rightarrow \vee_{q\sim p} q \leq z(p)$

(The other way: subequivalent means it lives inside of it. Equivalent means lives under the support).

Take a partial isometry $p = v^*v, q = vv^*, qH = vv^*(H) \subset vH \subset \overline{MvH} = Mvv^*vH \subset Mv = \overline{MpH} = z(p)$

$z(p) = \vee_{q\sim p} q = \vee_{q\sim p}(\vee_{i\in I} q_i) = \vee_{q_i}$

$p =$semifinite $\Rightarrow p = \vee_i p_i$ (anything equivalent to it is semifinite as well) ☐

Proposition. $p, q \in P(M)$ such that $p \prec q$. If q is semifinite $\Rightarrow p$ is semifinite.

Proof. $p \prec q \Rightarrow z(p) \leq z(q)$. (HOMEWORK)

$p \leq z(p) \leq z(q) \Rightarrow p \leq z(q)$.

We only need to prove our statement for $p \leq q \in Z(M)$ (in the center).

$p \leq z \in Z(M)$.

(Maximality argument) Let $p_0 = \vee_{i\in I} p_i$ where $p_i \leq p, p_i$ is finite.

Since q is semifinite, then $q = \vee q_j$ where $q_i \leq q, q_i$ finite.

Take $p - p_0$ (subprojects of p that will not have any subprojections).

Then, its central support $z(p - p_0) \leq q = z(q)$ (since $p - p_0 \leq p \leq q$)

$\Rightarrow$If $p - p_0 \neq 0$, $\exists 0 \neq \tilde{q_0} \leq q, \tilde{q_0} \prec p - p_0$, which contradicts the definition of p_0.

Thus $p - p_0 = 0 \Rightarrow p = \vee_{i\in I, p_i \leq p, p_i \text{finite}} p_i \Rightarrow p$ semifinite. ☐

Theorem 71. $\forall\, p, q \in M$ finite projections $\Rightarrow p \vee q$ is finite.

(Kaplanksi Formula) - Relationship between supremum and infimum.

$p, q \in P(M) \Rightarrow p \vee q - p \sim q - p \wedge q$ (if Abelian, we have equality).

Analogy (like inclusion/exclusion principle for 2 sets or measure theory)

$m(A \cup B) = m(A) + m(B) - m(A \cap B)$

$m(A \cup B) - m(A) = m(B) - m(A \cap B)$

$m((A \cup B) - A) = m(B - (A \cap B))$

(Can also think combinatorially)

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Theorem 72. Le M be a vN algebra.

$e, f \in P(M)$. If e, f are finite $\Rightarrow e \vee f$ is finite (bunch of finite projections supremum here is finite. $\vee$ Smallest projection that dominates both of them i.e. $h \geq e, f$).

Lemma 73. Let $M \subset B(H)$ be a purely infinite vN algebra. Then $\exists\, p \in P(M)$ such that $p \sim 1-p \sim 1$.

(Comes from Physics)

Proof. 1 is purely infinite (i.e. does not have finite subprojections so there is a subprojection that is equivalent) $\Rightarrow$ $\exists u$, a partial isometry in M such that $u \in M, uu^* < 1, uu^* = 1$.

$$\begin{tikzcd}
\bullet \arrow[r, mapsto, "uu^*"] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r, "p_0"] & 1 \\
\bullet \arrow[r, mapsto] & \bullet \arrow[r, mapsto, "p_1"] & {}
\end{tikzcd}$$

In other words we have,

$$\begin{aligned}
p_0 &= 1 - uu^* \\
p_1 &= up_0u^* \\
p_2 &= up_1u^* \\
&= u^2p_0(u^*)^2 \\
\Rightarrow p_n &= u^np_0(u^*)^n, \qquad \{p_n | n \in \mathbb{N}\}
\end{aligned}$$

These are mutually orthogonal equivalent projections.

Let $\{q_i\}_i$ be a maximal family of pairwise orthogonal equivalent projections containing $\{p_n | n \in \mathbb{N}\}$

(such a family does exist by Zorn's Lemma)

We now try to use comparison as follows (up to something in the center, we can compare them).

$$q_0 = -1 - \sum_{i \in I} q_i \qquad \text{(There can still be complement)}$$

Fix q_0, q_{i_0}. Then by Comparison theorem, $\exists z \in Z(M)$ such that $q_0 z \prec q_{i_0} z$ and $q_{i_0}(1-z)- \preceq q_0(1-z)$.

If $z = 0 \Rightarrow q_{i_0} \prec q_0 \Rightarrow$ Would contradict maximality of $\{q_i\}_i$

Thus $z \neq 0$. Then,

$$\begin{aligned}
z &= 1z \\
&= (q_0 + \sum_{i \in I} q_i)z \\
&= q_0 z + \sum_{i \in I} q_i z \\
&\prec q_{i_0} z + \sum_{i \in I_{i_0}} q_i z \\
&= \sum_{i \in I} q_i z \\
&\leq z
\end{aligned}$$

(Infinite family of pairwise orthogonal equivalent projection. Pull out a term -¿ still infinite (still in bijection)).

(Inequality follows from) Consider $\phi : I \to I - \{i_0\}$ with the mapping $q_i z \mapsto q_{\phi_i} z$.

So we have shown that $q \preceq \sum_i q_i z, \sum q_i z$. Then, by Cantor-Bernstein, $\sum_{i \in I} qiz \sim z$. $\square$

Proof. Now part 2 of the proof (split the infinte set in two).
$\exists I = I_1 \sqcup I_2$ such that $|I| = |I_1| = |I_2|$.

$$\begin{aligned} p &= \sum_{i\in I_1} q_i z \\ \Rightarrow \sum_{i\in I_1} q_i z \sim \sum_{i\in I_2} q_i z &\sim \sum_{i\in I} q_i z \\ p \sim z - p &\sim z \end{aligned}$$

If $z = 1$, we are done.

(Fact that we use above: If $a_i \sim b_i$ and they are all mutually orthogonal then, $\sum a_i \sim \sum b_i$).

Let $\{r_j\}_j$ be a maximal family of centrally orthogonal projection (i.e. their centers are mutually orthogonal) such that $r_j \sim z(r_j) \sim z(r_j) - r_j$ (each equivalent to its central support and complement)

$$\begin{aligned} p &= \sum_j r_j \\ &\sim \sum_{i\in I} z(r_j) \sim (\sum (z(r_j)) - (\sum r_j)) \\ \therefore &\sim p \sim 1 \sim 1 - p \end{aligned}$$

(Note: IF $\sum_{i\in I} z(r_j) \neq 1$, then we can take the complement and repeat the process - "cut corners") □

Lemma 74 (Kaplanaski). Let $M \subset B(H)$ vN algebra where $p, q \in P(M)$. Then,

$$p \vee q - p \sim q - p \wedge q$$

In an abelian vN algebra, infinum $p \wedge q = p \cdot q$. Then,

$$\begin{aligned} p \vee q - q &= p - p \cdot q \\ p \vee q &= p + q - p \cdot \end{aligned}$$

(Cardinality of union is the cardinality of first plus cardinality of second minus cardinality of intersection. Similar to measure theory).

Notation Recall: $P_{\overline{x^* H}} = [x^* H]$

Proof.

$$\begin{aligned} p \vee q - q &\sim p - p \wedge q \\ P_{x^* H} &= [x^* H] \sim [xH], \end{aligned}$$

where $x = (1 - p) \cdot q$.

Can show that $\ker(x) = \ker(q) \oplus (qH \cap pH)$. Then, by Rank-Nullity Theorem,

$$\begin{aligned} P_{\overline{x^* H}} &= [x^* H] \\ &= 1 - (1 - q + q \wedge p) \\ &= q - q \wedge p \\ P_{\overline{xH}} &= (1 - p) - (1 - p) \wedge (1 - q) \\ &= 1 - p - (1 - p \vee q) \\ &= p \vee q - p \end{aligned}$$

□

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Theorem 75. $M \subset B(H)$ vN algebra.

Let $p, q \in P(M)$ such that p, q are finite . Then $p \vee q$ is finite.

Recall

Lemma 76. Let $M \subset B(H)$ vN algebra properly infinite.

$\exists\, r \in P(M)$ such that $r \sim 1 - r \sim 1$.

Kaplanski $M \subset B(H)$ vN algebra.

$\forall p, q \in P(M)$, we have $p \vee q - p \sim q - p \wedge q$.

Proof.

$$\begin{aligned} p \vee q &= p \vee q - p + p \\ &= q - p \wedge q \qquad \text{(by Kaplanski formula)} \\ &\leq q \end{aligned}$$

WLOG, we can assume $p \perp q$. (i.e. $p \cdot q = 0$).

$$\begin{aligned} p \vee q &= p + q \qquad \text{(finite)} \\ \iff &= (p+q)M(p+q) \end{aligned}$$

WLOG we can assume $p + q = 1$.

(Just showed that if we have two finite orthogonal projection, then the sum is finite).

(Now, ensure that infinite parts are prevented).

Let $z_0 = \bigvee_{z \in Z(M) \text{ finite}} z$ (supremum of centrally orthogonal projections) by prior lemma.

If $z_0 = 1$, we are done (maximal finite projection if this case happens).

If $z_0 \neq 1$, can excise as follows. Let $0 \neq 1 - z_0$ and consider $(1 - z_0)p$ and $(1 - z_0)q$. So thus $z_0 = 0$.

Assume by contradiction that $z_0 = 0$.

Proof Begins Here Assume by contradiction:

(a) $p \perp q$, (b) $p + q = 1$, (c) $z_0 = 0 \Rightarrow$ properly infinite, (d) p, qfinite.

By (c) and Lemma 1 today $\Rightarrow \exists r \in P(M)$ such that $r \sim (1 - r) \sim 1$.

Consider, $p \wedge r$ and $q \wedge (1 - r) \in P(M)$. Now by Comparison Theorem,

$\exists z \in Z(M)$ such that

$$\begin{aligned} z(p \wedge r) &\preceq z(q \wedge (1 - r)) \\ (1 - z)(q \wedge (1 - r)) &\preceq (1 - z)(p \wedge r) \end{aligned}$$

$$\begin{aligned} zr &\sim z(1 - r) \sim z \\ \Rightarrow (zp) \wedge (zr) &= z(p \wedge r) \\ &\preceq z(q \wedge (1 - r)) \\ &= (zq) \wedge (z(1 - r)) \end{aligned}$$

Now we use the Kaplanski formula on zr as follows:

$$\begin{aligned} zr &= z(r - p \wedge r + p \wedge r) \\ &= z(r - p \wedge r) + z(p \wedge r) \\ &\preceq z(p \vee r - p) + z(q \wedge (1 - r)) =?zq \\ &= z(p \vee r - p + q \wedge (1 - r)) \end{aligned}$$

We need to show the $= zq$ part. (Now here is the catch...)

Formula: $e \vee f = 1 - (1-e) \wedge (1-f)$. (algebraic trick to go between infimum and supremum). Proof continued:

$$\begin{aligned}
zr &= z(p \vee r - p + (1-p) \wedge (1-r)) \\
&= z(1 - (1-p) \wedge (1-r) - p + (1-p) \wedge (1-r)) \quad \text{(by formula above)} \\
&= z(1-p) \\
&= zq
\end{aligned}$$

$\therefore$ zr is finite $\Rightarrow zr \sim z \Rightarrow z$ is finite also. We have found a finite central projection but since M is properly infinite $\Rightarrow z = 0$.

$\Rightarrow q \wedge (1-r) \prec p \wedge r$ (plug in $z = 0$ in the first equation above).

(This will lead to subequivalent below, which will help us get the contradiction).

Now consider the following:

$$\begin{aligned}
1 - r &= (1 - r - (1-r) \wedge q) + (1-r) \wedge q \\
&\preceq (1-r) \vee q - q + p \wedge r \quad \text{(by Kaplanski)} \\
&= (1-r) \vee (1-p) - (1-p) + p \wedge r \\
&= 1 - r \wedge p - (1-p) + p \wedge r \quad \text{(by the infimum-supremum formula)} \\
&= p
\end{aligned}$$

Hence we have $1 - r \preceq p$.

p finite $\Rightarrow 1 - r$ finite but then $1 - r \sim r \sim 1$. Contradiction. □

Note Infinite supremum might not be finite (for finite set, can do an inductive argument, but not in infinite case).

Corollary 76.1. Suppose $M \subset B(H)$ vN algebra.

Let $p, q \in P(M)$ be finite projections. Then, if

$p \sim q[v] \Rightarrow (1-p)[w] \sim (1-q)$

(v, w) respective isometries.

In particular, $\exists\, u \in U(M)$ such that $p = uqu^*$.

$$\begin{aligned}
u &= v + w \\
uu^* &= (v+w)(v+w)^* \\
&= (v+w)(v^* + w^*)
\end{aligned}$$

$v^*v = p, vv^* = q, 1 - p = w^*w, 1 - q = ww^*$. Then,

$$\begin{aligned}
&= vv^* + vw^* + wv^* + ww^* \\
&= q + v \underbrace{v^*v}_{p} \underbrace{w^*w}_{1-p=0} w^* + w \underbrace{w^*w}_{1-p=0} \underbrace{v^*v}_{p} + 1 - q \\
&= 1
\end{aligned}$$

$$\begin{aligned}
uqu^* &= (v+w)q(v^* + w^*) \\
&= v^*qw + w^*qv + v^*qw + w^*qv \\
&= v^*vv^*v + w^*wwq \\
&= p + 1 - q + 0
\end{aligned}$$

$p \vee q M p \vee q$. M is finite (if not, $1 - p \vee q$)

$p \sim q \Rightarrow p \vee q - p \sim p \vee q - q$

28 Jan 2022 Recall from last time,

$\forall\, p, q \in P(M)$ finite and $M \subset B(H)$ vN algebra $\Rightarrow p \vee q$ is finite.

Corollary 76.2. $M \subset B(H)$ vN algebra. Then, $\forall p, q \in P(M)$ finite if $p \sim q \Rightarrow 1-p \sim 1-q$. In particular, $\exists u \in U(M)$ such that $upu^* = q$.

Generic idea: $\sum p_i = 1, \sum q_i = 1, p_i \sim q_i \Rightarrow \exists u \in M upu^* = \sum$ (add both the partial isometries).

Example 77. Consider $B(H)$ such that $\dim_{\mathbb{C}}(H) = \infty$.

$\{\xi_i\}_{i\in I} \subset H$ ONB.

span $\{\xi | i \neq j\} = K$.

$p = P_K$

$\{\xi_i : i \neq j\}, \{\xi_i | i \in I\}$ (There is a bjiection)

Example 78. (If infinite, corollary not true) Consider $B(\ell^2(\mathbb{N}))$.

Now look at the map:

$$V : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$$
$$V : (\delta_x) = \delta_{1+x}$$

$V^*V = 1, VV^* = \text{Proj}_{\ell^2(\mathbb{N}-\{1\})}$.

Then, $1 \sim \text{Proj}_{\ell^2(\mathbb{N}-1)}$

Proof. $p \sim q \Rightarrow 1-p \sim 1-q$.

By the theorem, p, q finite $\Rightarrow$ the supremum, $p \vee q$ is finite.

$$p \vee qMp \vee q = rMr \qquad \text{(finite } vN \text{ algebra)}$$
$$1 - p = 1 - r + r - p$$
$$1 - q = 1 - r + r - q$$

Lemma 79. If two families of projections equivalent i.e.

$\{p_i\}_i, \{q_i\} \subset P(M)$.

$p_i \perp p_j\ \forall i \neq j, \xi_i \perp \xi)i\ \forall i \neq j, \sum v_i$

If $p_i \sim q_i\ \forall i \Rightarrow \sum_{i\in I} p_i \sim \sum_{i \in I} q_i$

Based on the lemma, all we really need to show is that $r - p \sim r - q$.

Note that $p, q \in rMr$ (finite since r is a finite projection). $p \sim q$ (in rMr).

Show: $r - p \sim r - q$.

Therefore, WLOG assume $r = 1$.

Working Assumption: (i) $p, q \in M$ (finite), (ii) $p \sim q$. Now, by Comparison,

$\exists z \in Z(M)$ such that $(1-p)z \preceq (1-q)z$ and $(1-q)(1-z) \preceq (1-p)(1-z)$. Need to show that these subequivalences are actually equivalences.

$(1-p)z \sim p_1 \leq (1-q)z$ and $(1-q)(1-z) \sim p_2 \leq (1-p_(1-z)$

Note*: Discussions For the calculations below:

$$\begin{aligned} z &= (1-p)z + pz \\ &\sim p_1 + qz \\ &\preceq (1-q)z + qz \\ &= z \\ \Rightarrow z &\preceq z \end{aligned}$$

Now consider

$$\begin{aligned} 1 - z &= (1-q)(1-z) + q(1-z) \\ &\sim p_2 + p(1-z) \\ &\leq (1-p)(1-z) + p(1-z) \\ &= 1 - z \end{aligned}$$

$z, 1-z$ finite, then subequivalent needs to be equivalent.

Hence, the subequivalneces above become equivalences. □

Definition 18.2 (Countably Decomposable)**.** $\forall$ family of mutually orthogonal projections is at most countable.

Exercise $p \preceq q \Rightarrow z(p) \leq z(q)$.

Let $M \subset B(H)$ be countably decomposable. If p, q are properly infinite and $z(p) \leq z(q) \Rightarrow p \preceq q$.

Corollary 79.1. If M is countably decomposable properly infinite factors. Then, any two nonzero projections are equivalent.

19 Type Decomposition/Classification

$M \subset B(H)$ vN algebra is

Definition 19.1 (Type I)**.** if every nonzero projection has an abelian subprojection.

Example 80. $B(H)$ is Type I.

Subtypes (one finite $\mathrm{I}_{\mathrm{fin}}$, one infinite I_∞)

Definition 19.2 (Type II)**.** if it is **semifinite** and there are no nonzero abelian subprojections.

Definition 19.3 (Type II_1)**.** if M is finite

Definition 19.4. Type II_∞ if M is properly infinite

Definition 19.5 (Type III)**.** if M has no finite projections.

Theorem 81. Let $M \subset B(H)$ vN. Then $\exists!\, p_{\mathrm{I}}, p_{\mathrm{II}_1}, p_{\mathrm{II}_\infty}, p_{\mathrm{III}} \in Z(M)$ such that $p_{\mathrm{I}} + p_{\mathrm{II}_1} + p_{\mathrm{II}_\infty} + p_{\mathrm{III}} = 1$.

$M \cdot p_{\mathrm{I}}$ is type I, Mp_{II} is type II_∞.

Mp_{II_1} is type II_1, Mp_{III} is type III.

Proof. $p_I = \bigvee_{p \in P(M), p\text{abelian}} p = \bigvee_{p \in P(M), p \text{ abelian}} upu^* = u(\bigvee p)u^* = up_{\mathrm{I}}u^*$. Then,

$up_{\mathrm{I}} = p_{\mathrm{I}}u \Rightarrow xp_{\mathrm{I}} = p_{\mathrm{I}}x$ $\forall x \in M, p_{\mathrm{I}} \in Z(M)$.

$Mp_{\mathrm{I}} \ni p$.

$0 \neq p \leq p_{\mathrm{I}}$ $\exists p_0 \in M$ abelian such that $pp_0 \neq 0$

$\Rightarrow z(p)z(p_0) \neq 0$. Then, by comparison,

$\exists$ $q_0 \leq p, q_1 \leq p_0$ such that $q_0 \sim q_1$. Abelianess is preserved.

Hence, the algebra is abelian. □

31 Jan 2022

Theorem 82. Let $M \subset B(H)$ vN algebra. Then $\exists : p_{\mathrm{I}}, p_{\mathrm{II}_1}, p_{\mathrm{II}_\infty}, p_{\mathrm{III}} \in Z(M)$ such that

$p_{\mathrm{I}} + p_{\mathrm{II}_1} + p_{\mathrm{II}_\infty} + p_{\mathrm{III}} = 1$ and

Mp_{I} is type I.

Mp

Corollary 82.1. $\forall$ $M \subset B(H)$ a factor is either type I or type II_1 or type II_∞or type III.

Theorem 83. $M \subset B(H)$ type I factor. Then, $\exists$ K Hilbert space.

$M \cong B(K), \dim_\sigma(K) = n \Rightarrow M \cong M_n(\mathbb{C})$.

Proof. (Idea: Take supremum of all abelian projections)

$p_{\mathrm{I}} = \vee_{z \in P(M)} z \in Z(M)$, where z are abelian projections.

Now consider $1 - p_{\mathrm{I}}$ has no abelian subprojections.

$p_{\mathrm{II}_1} = \vee_{z \leq 1 - p_I} z \in Z(M)$, where z central, finite (if none exist, put 0).

Therefore, p_{II_1} finite projection.

Now let $p_{\mathrm{II}_\infty} \leq 1 - p_{\mathrm{I}} - p_{\mathrm{II}_1}$ be defined as $p_{\mathrm{II}_\infty} = \vee_{p \in P(M)} p$ finite such that $p \leq 1 - p_{\mathrm{I}} - p_{\mathrm{II}_1}$. (closed under conjugacy).

Since p is finite,

$$\begin{aligned}\Rightarrow \text{(finite) } upu^* &\leq u(1 - p_{\mathrm{I}} - p_{\mathrm{II}_1})u^* \\ &= (1 - p_{\mathrm{I}} - p_{\mathrm{II}_1})uu^* \\ &= 1 - p_{\mathrm{I}} - p_{\mathrm{II}_1}\end{aligned}$$

(Above follows since $a \leq b \Rightarrow xax^* \leq xbx^*$. Then, we have

$$\begin{aligned}p_{\mathrm{II}_\infty} &= \vee_{p \in P(M)} p \\ &= \vee upu \qquad\qquad * \\ &= \vee_{u \in u(M), p \in P(M)} u p_{\mathrm{II}_\infty} u^*\end{aligned}$$

(After some calculations, we get) $\Rightarrow 1 - p_{\mathrm{I}} - p_{\mathrm{II}_1)} - p_{\mathrm{II}_\infty} = p_{\mathrm{III}}$ ☐

$$\begin{aligned}1 - p_{\mathrm{I}} - p_{\mathrm{II}_1)} - p_{\mathrm{II}_\infty} &= p_{\mathrm{III}} \\ 1 &= p_{\mathrm{I}} + p_{\mathrm{II}_1)} + p_{\mathrm{II}_\infty} + p_{\mathrm{III}} \\ 1 &= q_I + q_{\mathrm{II}_1} + 1_{\mathrm{II}_\infty} + q_{p_{\mathrm{III}}}\end{aligned}$$

$Mp_{\mathrm{III}}, Mq_{\mathrm{III}}$ purely infinite. Then,

$$\begin{aligned}q_{\mathrm{III}} \cdot p_{\mathrm{I}} &= 0 \\ q_{\mathrm{III}} \cdot p_{\mathrm{II}_1} &= 0 \\ q_{\mathrm{III}} \cdot p_{\mathrm{II}_\infty} &= 0 \\ \Rightarrow q_{\mathrm{III}} &= q_{\mathrm{III}} \cdot 1 \\ &= q_{\mathrm{III}}(p_{\mathrm{I}} + p_{\mathrm{II}_1)} + p_{\mathrm{II}_\infty} + p_{\mathrm{III}}) \\ &= q_{\mathrm{III}} p_{\mathrm{III}}\end{aligned}$$

$p_{\mathrm{III}} \geq q_{\mathrm{III}}$, Reversing the roles of p_{III} and q_{III}, we get $p_{\mathrm{III}} = q_{\mathrm{III}}$

Now consider $(p_{\mathrm{II}_1} + p_{\mathrm{II}_\infty}) \cdot q_{\mathrm{I}} = 0$ (living under q_{I} means there is some abelian part).

$\Rightarrow p_{\mathrm{II}_1} + p_{\mathrm{II}_\infty} = q_{\mathrm{II}_1} + q_{\mathrm{II}_\infty} \Rightarrow p_{\mathrm{I}} = q_{\mathrm{I}}$.

Note*: Focus will be on Type II.

(Last semester, if you compresss a vN algebra, it is still a vN algebra).

Proposition. $M \subset B(H)$ vN algebra.

Let $p \in P(M), q \in P(M')$ (commutant).

$pMpq$ is a vN algebra. Then, consider its commutant:

$(pMpq)' \cap B(pqH) = qM'qp$

$Z(M) = \mathbb{C}$ so is $pMpq$.

Theorem 84. $M \subset B(H)$ vN.

$p \in P(M), q \in P(M')$.

If M is type I or has no nonzero abelian projections, then so is $pMpq$ is type I or has no nonzero abelian projection (i.e. type is preserved by the compression).

Similar for semifinite.

Theorem 85. $M \subset B(H)$ vN algebra.

M is of type I $\iff$ M' is type I. M is of type II $\iff$ M' is type II. M is type III $\iff$ M' is type III.

Definition 19.6. $M \subset B(H)$, $\xi \in H$ is called a cyclic vector $\Rightarrow \overline{M\xi} = H$ $\forall$ M $P(M') \supset p_\xi = P_{\overline{M\xi}} = [M\xi] \in P(M')$

$p'_\xi = P_{\overline{M'\xi}} = [M'\xi] \in P(M)$ (orthogonal projection)

(Notion of cyclic vectors and irreducibility in this theorem)

Theorem 86 (BT Theorem von Neumann). Let $M \subset B(H)$ vN algebra with cyclic vector $\xi \in H$. Then, for each element, $\forall$ $\eta \in H$, $\exists x, y \in M, x \geq 0$ and $1_{gamma} \in \overline{xH}$ (vector in range of x)

Such that $x1_\gamma = \xi, y1_\gamma = \eta$.

$x1_\gamma = \xi \Rightarrow 1_\gamma =$ "x^{-1}"ξ $y1_\gamma = \eta$

$\eta = yx^{-1}\xi$ (BT)

$\overline{M\xi} = H$

(*Note: There may be typos in the day above)

2 February 2022 Recall the BT Theorem.

Let $M \subset B(H)$ vN algebra, $\xi \in H$ such that $\overline{M\xi} = H$.

Then, $\forall \eta \in H$, there exist $x, y \in M, x \geq 0, \zeta \in \overline{x(H)}$ such that

$x\zeta = \xi, y\zeta = \eta$.

if $x^{-1} \exists\ \zeta = x^{-1}\xi$.

$yx^{-1}\xi = \eta$ (by BT)

(B - Bounded, T - affiliated - does not live in the algebra)

Proof. WLOG, assume $||\xi||, ||\eta|| \leq 1$. Because $\overline{M\xi} = H$ (can be approximated - have to work in a series).

$\eta \in H$.

$\forall x \in \mathbb{N}$, $\exists x_i$ operators inM, $i = 1, ..., n$ such that

$$||\eta - \sum_{i=1}^{n} x_i\xi|| < \frac{1}{4^n} \qquad \forall n$$

$$||\eta - x_1\xi|| < \frac{1}{4} \qquad \text{(do this repeatedly)}$$

$$||\eta - x_1\xi - x_2\xi|| < \frac{1}{4}$$

Define the following operator (sequence of increasing positive operators)

$$\begin{aligned} y_n &= 1 + \sum_{i=1}^{n} 4^n x_i^* x_i, \qquad 1 \leq y_n \leq y_{n+1} \forall n \\ z_n &= y_n^{1/2} \qquad \forall n \\ 0 &\leq z_{n+1}^{-1} \leq z_n^{-1} \leq 1\ \forall n \\ z_n^{-1} &\searrow x \qquad \text{(Weirstrauss Thm, decreasing SOT limit)} \end{aligned}$$

Note that this is because:

$$\begin{aligned} 1 &\leq y_n \leq y_{n+1} \\ \Rightarrow y_{n+1}^{-1} &\leq y_n^{-1} \leq 1 \\ 1 &\leq y_n^{1/2} \leq y_{n+1}^{1/2} \\ 1 &\leq z_n \leq z_{n+1} \\ 0 &\leq z_{n+1}^{-1} \leq z_n^{-1} \leq 1 \end{aligned}$$

$\{z_n\xi\}_{n\in\mathbb{N}} \subset H$ is bounded. Then,

$$\begin{aligned}
||z_n\xi||^2 &=< z_n\xi, z_n\xi > \\
&=< z_n^2\xi, \xi > \\
&=< y_n\xi, \xi > \\
&=< (1 + 4^n \sum_{i=1}^{n} x_i^* x_i)\xi, \xi > \\
&=< \xi, \xi > +4^n \sum_{i-=1}^{n} < x_i^* x_i \xi, \xi > \\
&= ||\xi||^2 + 4^n \sum_{i=1}^{n} ||x_i\xi||^2 \\
&\leq 1 + 4^n \sum_{i=1}^{n} ||x_i\xi||^2 \\
&\leq 1 + 4^n (\sum_{i=1}^{n} 4^{-i}) \\
&\leq 1 + 4^n \frac{1}{4^n} 3 \\
\Rightarrow ||z_n\xi|| \leq 2 \quad &\forall n
\end{aligned}$$

Note the second to last line follows because:

$$\begin{aligned}
||x_i\xi|| &\leq \frac{1}{2^n} \\
||\eta - \sum_{i=1}^{n} x_i\xi|| &\leq \frac{1}{4^n} \\
||\eta - 0\sum_{i=1}^{n+1} x_i\xi|| &\leq \frac{1}{4^{n+1}} \\
||x_{n+1}\xi|| &= ||(\eta - \sum_{i=1}^{4} x_i\xi) - (\eta - \sum_{i=1}^{n_1} x_i\xi)|| \\
&\leq \frac{1}{4^n} + \frac{1}{4^{n+1}} \leq \frac{1}{2^n} \\
\therefore ||x_n\xi|| &\leq \frac{1}{2^{n-1}}
\end{aligned}$$

$\exists \zeta \in H$ a cluster point (subsequence argument).

Assume that $z_n\xi \to \zeta$ (weak convergence).

Now we prove the main part:

$$\begin{aligned}
x\zeta &= \xi \\
\iff x\zeta - \xi &= 0 \\
\iff < x\zeta - \xi, \xi_0 > &= 0 \quad \forall \xi_0 \subset H
\end{aligned}$$

We need to prove this last line.

$\iff \forall\ \epsilon > 0$ such that $| < x\zeta - \xi, \xi_0 > | < \epsilon$.

$$
\begin{aligned}
| < x\zeta - \xi, \xi_0 > &= < x\zeta - xz_n\xi + xz_n\xi - \xi, \xi_0 > \\
&\leq | < x(\zeta - z_n\xi), \xi_0 > | + | < xz_n\xi - \xi, \xi_0 > \\
&= \underbrace{| < \zeta - x_n\xi, x\xi_0 |}_{<\epsilon} + | < xz_n\xi - \xi, \xi_0 > | \\
&< \epsilon + | < xz_n\xi - \xi, \xi_0 > \\
&= \epsilon + | < z_n\xi, x\xi_0 > - < \xi, \xi_0 > | \\
&= \epsilon + | < z_n\xi, (x\xi_0 - z_n^{-1}\xi_0 + z_n^{-1}\xi_0) > - < \xi, \xi_0 > | \\
&\leq \epsilon + |z_n\xi, x\xi_0 - z_n^{-1}\xi_0 > | + | < z_n\xi, z_n^{-1}\xi_0 > - < \xi, \xi_0 > | \qquad \text{(triangle inequality)} \\
&\leq \epsilon + ||z_n\epsilon|| ||x\xi_n - z_n^{-1}\xi_0|| + \underbrace{| < \xi, z_n z_0^{-1} z_0 > - M\xi, \xi_0 > |}_{=0} \\
&\leq \epsilon + 2\epsilon
\end{aligned}
$$

We have used the following facts:

$$1. < z_n\zeta - \xi, a > \to 0\ \forall a \in H \Rightarrow | < z_n\zeta - \xi, x\xi_0 > | < \epsilon, \quad \zeta \in H$$

$$2. z_n^{-1} \searrow x, \qquad ||z_n^{-1}\xi_0 - x\xi_0|| < \epsilon\ \forall n >>$$

$\therefore$, we have shown that $x\zeta = \xi$. Now we need to get to our y.
Let us look at the following:

$$
\begin{aligned}
0 \leq z_n^{-1} 4^i x_i^* x_i z_n^{-1} &\leq \underbrace{z_n^{-1}(1 + \sum_i 4^i x_i x_i) z_i^{-1}}_{z_n^{-1} z_n^2 z_n^{-1} = 1} \\
&< 1
\end{aligned}
$$

$\{z_n^{-1} 4^i x_i^* x_i z_n\}_{n \geq 1}$. As $n \to \infty$, we have $4^i x x_i x_1 x \Rightarrow ||4^i x x_i x_i x|| \leq 1$ (strong norm) $\Rightarrow 2^i ||x_i x_i||^2 \leq 1$. Then, we have

$$
\begin{aligned}
||2^i x_i x||_\infty &\leq 1 \\
||x_i x||_\infty &\leq \frac{1}{2^i}\ \forall i \\
\sum_{i=1}^n x_i x &= y > 0.
\end{aligned}
$$

is convergent.
$y\zeta = (\sum x_i x)\zeta = \sum x_i x\zeta = \sum_{i=1}^\infty x_i \xi = \eta$
Can check that $\ker(y) \leq \ker(x)$. Can replace $\zeta \iff \overline{xH\zeta}$ $\square$

4 February 2021

Recall $M \subset B(H)$ vN. $[M\xi] = 1) \iff \overline{M\xi} = H$ (cyclic).

$\forall\ \eta \in H,\ \exists x, y \in M, x \geq 0.$

$\zeta \in \overline{xH}$ such that $x\zeta = \xi, y\zeta = \eta$.

Theorem 87. $M \subset B(H)$ vN algebra.

$\forall\ \xi, \eta \in H$, the following holds:

$[M\xi] \preceq_{M'} [M\eta] \iff [M'\xi] \preceq_M [M'\eta]$.

$[M\xi] = P_{\overline{M\xi}}$

Lemma 88. Supose that $M \subset B(H)$ vN algebra. Suppose that we have two cyclic projections $\xi, \eta \in H$ i.e. $[M\xi] = [M\eta] = 1$.

$\Rightarrow [M'\xi] \sim_M [M'\eta]$

Proof. (Idea: Use BT Theorem to understand, can relate ξ and η)

(Part I of the Proof) Using BT Theorem (for ξ, η), one can find $\exists\ x, yinM, x \geq 0, \zeta \in \overline{xH}$ such that $x\zeta = \xi, y\zeta = \eta$.

Claim: $[M'\xi] \sim [M'\zeta] \sim [M'\eta]$

Let us call $[M'\zeta] = p$ (projection in M).

Know: (i) $\zeta \in \overline{xH}$., (ii) $p\zeta = \zeta$.

$\Rightarrow \zeta \in \overline{pxH}$ (since $p(\overline{xH}) \in \overline{pxH}$).

Therefore, $p \leq [M'pxH] \leq [pxH] \leq [pH] = p$ (absorbed in H).

$\Rightarrow p = [pxH]$ (Then we use ideas from Polar Decomposition Theorem)

$$\begin{aligned}
p &= [pxH] \\
&\sim [(px)^* H] \\
&= [x^* p^* H] \\
&= [xpH] \\
&= [xM'\zeta] \qquad \text{(they commute)} \\
&= [M'x\zeta] \\
&= [M'\xi]
\end{aligned}$$

$\therefore [M'\xi] \sim [M'\zeta]$.

(Part 2 of the Proof)

$$\begin{aligned}
[M'\eta] &= [M'y\zeta] \\
&= [yM'\zeta] \\
&= [ypH] \\
&\sim [(yp)^* H] \\
&= [pyH] \\
&\leq [pH] \\
&= p
\end{aligned}$$

At this point, we have subequivalence i.e. $[M'\eta] \prec p$ (we need equivalence).

We also have $[M\zeta \sim p] \Rightarrow [M'\eta] \preceq [M'\xi]$.

Reversing the role of ξ and η, we get $[M'\xi] \preceq [M'\eta]$. Hence the conclusion holds by C-B. $\square$

Lemma 89. $M \subset B(H)$ vN algebra. Assume $\forall\ \xi, \eta \in H$ vectors, we have that $[M'\xi] \sim [M'\eta] \iff [M\xi] \sim [M\eta]$.

Proof. $[M'\xi] \leq [M'\eta] \Rightarrow [M'\xi] \sim p \leq [M'\eta]$

Fact: $p = [M'p\eta]$ (sub-projection of a cyclic projection is cyclic)

$$\begin{aligned} &[M'\xi] \sim [M'p\eta] \iff [M\zeta] \sim [Mp\eta] \\ &\qquad\qquad [Mp\eta] \leq [M\eta] \qquad \text{(since p is an element of M } \Rightarrow Mp \text{ contained in} M) \\ &\Rightarrow [M\xi] \preceq [M\eta] \end{aligned}$$

(Real proof begins here) Assume

$$\begin{aligned} [M'\xi] &\sim_v [M'\eta] \qquad (v^*v = [M'\xi] = P_{\overline{M'\xi}}, vv^* = [M'\eta]) \\ [M'v\xi] &= [vM'\xi] = v[M'\xi] \\ \Rightarrow [Mv\xi] &\leq [M\xi] = [Mv^*v\xi] \leq [Mv\xi] \\ \Rightarrow [Mv\xi] &= [M\xi] \end{aligned}$$

Explanation for line 2: Since v partial isometry, $\overline{M'v\xi} = v(\overline{M\xi})$

Allows us to work with $v\xi$ instead of ξ (reduction). We can assume $[M'\xi] = [M'\eta] = p_0$. (It is better to cut something that is central).

Let $z = z([M'\xi]) \in Z(M)$ be the central support. Then,

$$\begin{aligned} z &= [MM'\eta] = [M'M\eta] = [MM'\xi] \\ &= [Mp_0H] \end{aligned}$$

In the Mz algebra, we can assume WLOG that $[M\xi], [M\eta]$ such that $[M'\xi] = [M'\eta] = p_0$ have central support z. Can assume central support is the unit, 1.

Want to show they are equivalent. In particular,

$$\begin{aligned} x &\mapsto xp_0 \\ M' &\to M'z \qquad \text{(isomorphism)} \end{aligned}$$

$$\begin{aligned} &[M\xi] \sim [M\eta] \\ \iff &[M\xi]p_0 \sim [M\eta]p_0 \\ &\quad p_0\xi \qquad p_0\eta \end{aligned}$$

(cyclic vectors $M'p_0$). ∎

Theorems coming up

7 Feb 2022

Recall

Proposition. Let $\xi, \eta \in H$. Then,

$[M\xi] \preceq [M\eta] \Rightarrow [M'\xi] \preceq [M'\eta]$.

Proposition. Let $M \subset B(H)$ countably decomposably.

If $e, f \in P(M)$ properly infinite. Then,

$z(p) \leq z(q) \Rightarrow p \preceq q$.

Proposition. Let $M \subset B(H)$ be an abelian vN algebra that admits cylic, separating vecor $\Rightarrow M = M'$.

Theorem 90. Suppose $M \subset B(H)$ vN algebra.

Let $\xi \in H$ with $[M\xi] = [M'\xi] = 1$.

If M is finite, then M' is finite. (Note: Type is preserved under commutant, subtype is not).

Note that any vN algebra can be written as a direct sum of Type I + Type II + Type III.

Proof. Let $q \in P(M')$ a maximal central projection. Then, $M'q$ is finite.

If $q = 1$, we are done. If not, consider the complement and note it is properly infinite.

Assume WLOG, M is finite while M' is properly infinite.

If M is abelian.

[Side Note: $M(1-q) \subset B((1-q)H) \Rightarrow [M(1-q)\xi]? = (1-q)H$. Then,

$$\begin{aligned}
[M(1-q)\xi]? &= (1-q)H \\
[(1-q)M\xi] &= (1-q)\underbrace{(\overline{M\xi})}_{=H} \\
&= (1-q)H
\end{aligned}$$

]

The M also has a cyclic separating vector $\Rightarrow M = M'$ (Abelian algebra cannot be properly infinite. Hence, this is a contradiction).

(Any algebra commute with its central support, then it is Abelian)

Since M not abelian $\Rightarrow \exists p \in P(M)$ such that $p < z(p) = 1$.

Let $r = [Mp\xi] \in M'$. Since M' (properly infinite) has a separating vector.

Exercie: M' is countably decomposable.

$M \subset B(H), \xi \in H, q =$ sup of all projection, $q\xi = 0 \Rightarrow \sup = 0 \Rightarrow 1-q$ is countable.

Consider $r \leq z(r)$ (Central support same). By Prop 2, we have $r \sim z(r)$.

Note, $H =$ closure of $M\xi$.

$$\begin{aligned}
r \sim z(r) &= [M'rH] \\
&= [M'Mp\xi] \\
&- [MpM'\xi] \\
&= [MpH] \\
&= z(p) \\
&= 1
\end{aligned}$$

Now, we have shown that $r = [Mp\xi] \sim 1 = [M\xi]$ (cyclic vector by assumption).

By Prop 1 $\Rightarrow [M'p\xi] \sim [M'\xi] = 1 \Rightarrow p = [pM'xi]$.

Contradiction $\Rightarrow M$ is finite. $\square$

Theorem 91. $M \subset B(H)$ vN algbra. Then,

M is type I $\iff$ M' is type I (respectively for Type II, III).

We only need to show $\Rightarrow$ implication.

Proof. Assume by way of contradiction that M is type I while M' is not type I.

$\exists q \in Z(M)$ such that Mq has no abelian subprojection.

WLOG assume M is type I while M' has no abelian subprojections.

Pick $p \in M$ an abelian projection $\Rightarrow pMp = z(M)p$ is abelian. Now we define two projections:

(i) $e = [pMp\xi] \in M'p$, (ii) $q - [pMp'\xi] \in pMp$ (e is contained in q.

Now look at $qe\xi \in qeH$ (this is a cyclic separating projection for $qMqe$). [Recall, earlier we had to partition H with cyclic separating vectors in each space. Similar here]

$qMqe = (qMqe)' = eM'eq$. (If algebra is certain type that corner, pMp is also same type).

Hence, contradiction.

For Type II, same argument as before. The only difference is that we will replace Abelian projections by finite projections.

For Type III, negation.

□

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9 February 2022

Theorem 92. Let $M \subset B(H)$ vN algebra.

Then M is finite $\iff \exists$ a faithful normal tracial central state.

Definition 19.7. A map $\Phi : M \to Z(M)$ is a called a faithful normal tracial central state if the following hold:

1. $\Phi(x^*x) \geq 0 \quad \forall\, x \in M$ (positive operator)

2. $\Phi(zx) = z\Phi(x)\ \forall z \in Z(M),\ x \in M$

2b. $\Phi(1) \Rightarrow \Phi|_{Z(M)} = \mathrm{Id}$ (unital)

3. Φ is normal (Φ of the sum is summable. A form of continuity. Convergence preserves the W-convergence)

4. $\Phi(x^*x) = 0 \iff x = 0$ (faithful)

5. $\Phi(xy) = \Phi(yx)\ \ \forall x, y \in M$.

(If we drop 5, this is **conditional expectation**).

Proof. $\Leftarrow$

$\exists p\ P(M)$ such that $p \sim_v 1 \Rightarrow v^*v = p, vv^* = 1$.

Consider $1 - p \geq 0$. Since this is a positive operator, we have

$$\begin{aligned} \Phi(1-p) &= \Phi(vv^* - v^*v) \\ &= \Phi(vv^*)\Phi(v^*v) \\ &= 0 \end{aligned}$$

$$\begin{aligned} 0 &= \Phi(1-p) \\ &= \Phi((1-p)^*(1-p)) \\ &= 1 - p = 0 \\ \Rightarrow p &= 1 \end{aligned}$$

□

$\Rightarrow$ (Construction proof - Dyadic numbers)

Lemma 93. Suppose M is a finite vN algebra, $0 \neq p \in P(M)$ (nonzero projection).

Then any family of projections, $\forall\ F = \{p_i \in P(M) | p :\sim p, p_i p_j = 0\ \forall i \neq j\}$ finite.

Proof. Suppose $|F| = \infty,\ \exists F_0 \not\subset F,\ |F_0| = |F|$ (characteristic of infinite sets)

Consider $\phi : F \to F_0$ bijective.

$\sum_{i \in F} p_i \sim \sum_{i \in F_0} p_i$.

This is impossible because one of them is a strict projection of the other $\Rightarrow M$ is not finite. □

Let us build the dyadic numbers first.

Let $p_0 = 0, p_1 = 1$.

Lemma 94. M is II_1.

$\exists\ p_{1/2} \in P(M)$ such that $p_{1/2} = 1 - p_{1/2}$.

Proof. Consider the family $\{p_i, q_i\}$ which are equivalent $p_i \sim q_i$. They are mutually orthogonal.

(By Zorn's Lemma) Pick such a maximal family.

Now consider the sum, $p = \sum p_i \sim q = \sum q_i$. (Essentially, p is perpendicular to q). Then, we look at $1-(p+q)$.

Note that $(1-(p+q))M(1-(p+q))$ is not Abelian. Then,

$\exists 0 \neq p_0, q_0 \in P(1-(p+q))M(1-(p+q))$ such that

$p_0 \sim q_0$ and $p_0 q_0 = 0$.

$\{p_i\}_i \cup p_0$ and $\{q_i\} \cup q_0 \Rightarrow 1-(p+q) = 0 \Rightarrow p+q = 1$.

$\Rightarrow \exists r \in (1-(p+q))M(1-(p+q))$ such that $r < z(r)$.

Take $z(r) - r$ and r. Take the center of this. Will be something nonzero. Then, by comparison need to find something that lives in both.

That is exactly our p_0, q_0. In other words,

$0 \neq z(r) = r, r \neq 0$. Take the central support.

$$\begin{aligned} 0 \neq 0Z(z(r)-r) &\leq Z(r) \\ \Rightarrow Z(z(r)-r) \cdot Z(r) &= Z(z(r)-r) \neq 0 \\ \Rightarrow p_\xi &\leq z(r) - = r \\ 0 &\leq r \end{aligned}$$

Conclusion: $p_0 = 0, p_1 = 1, p_{1/2}$. ∎

So we now have:

$p_{1/2} \sim_v p_{1/2}$

$p_{1/2} = v^* v \Rightarrow 1 - p_{1/2} = vv^*$. Then, $p_1 - p_{1/2} = vv^*$.

Then, $p_{1/2} M p_{1/2}$. Basically, $p_{1/2} = p'(1-q')$.

Iterating it, we get $p_{1/2} = p_{1/4} + (p_{1/2} - p_{1/4} \Rightarrow p_{1/4} \sim p_{1/2} - p_{1/4}$.

Then, $p_{3/4} = p_{1/2} + v p_{1/4} v^*$. When factor, trace = number (1/2, 1/4, 3/4, etc.) Hence, we have $p_r \in P(M)$ where $r \in Q$ dyadic number.

$r \leq t \Rightarrow p_r \leq p_t$.

$r - t = r' - t' \Rightarrow p_r - p_t \sim p'_r - p'_t$.

---

11 February 2022 Recall,

Theorem 95. $M \subset B(H)$ Type II finite vN algebra. Then, $\exists$ family of projections $\{p_r\}_{r \in \text{dyadic}} \subset P(M) : \{\frac{k}{2^n} | k, n \in \mathbb{N}\}$.

$p_0 = 0, p_1 = 1$.

$0 \leq p_r \leq p_t \leq 1, r \leq t$.

$\forall r, t, r', t'$ dyadic numbers. Then, $r - t = r' - t'$.

Lemma 96. $M \subset B(H)$ vN algebra. $\forall 0 \neq q \in P(M)$, $\exists z \in Z(M)$ (for every nonzero projection, there exists a central projection) and a dyadic number, r such that $p_r \cdot z \neq 0$ and

$$p_r \cdot z \preceq qz. \tag{4}$$

Proof. Consider $z_0 = z(q) \in Z(M)$. Working into $M \cdot z_0$.

We can assume WLOG that $z(q) = 1$.

$z(p_r) \leq z(q) = 1$. Now we use the comparison theorem.

If (4) does not hold, $\Rightarrow q \preceq p_r \ \forall r$.

Then,

$$\begin{aligned} q &\preceq p\frac{1}{2^{k+1}} \\ &= p\frac{1}{2^{k+1}} - p_0 \\ &\sim p\frac{1}{2^k} - p\frac{1}{2^{k+1}} \ \forall k \qquad \text{(they are projections)} \end{aligned}$$

$$\Rightarrow (p\frac{1}{2^k} - p\frac{1}{2^{k+1}}) \cdot (p\frac{1}{2^{k+1}} - p\frac{1}{2^{k+2}}) = 0 \ \ \forall k$$

(Next one lives underneath and are mutually orthogonal).

But this contradicts Lemma 1 (cannot keep finding more). $\square$

Definition 19.8 (Monic Projections)**.** $p \in p(M)$ is called 'monic" if $\exists$ finitely many projections, $\{p_1, p_2, ..., p_n\}$ mutually orthogonal with $p_i \sim p$ and $\sum_{i=1}^n p_i \in Z(M)$ (all span something in the center).

Observation $\{p_r\}_{r \in \text{dyadic}}$ are monic projections. We can do this because

$$1 = \underbrace{1}_{p_1} - p_{3/4} + p_{3/4} - p_{1/2} + p_{1/2} - p_{1/4} + p_{1/8}$$

Can do something similar for any k.

Lemma 97. If $M \subset B(H)$ finite vN algebra then any projection is a sum of mutually orthogonal monic projections.

Proof. By a maximality argument it suffices to argue that $\forall p \in P(M) \ \exists p_0 \leq p$ that is.

But this follows from Lemma 1 (Case II$_1$.

Case I$_n$ follows for any abelian projection. $\square$

Lemma 98. $\forall \phi : M \to Z(M)$ central valued state. These are (norm) bounded (with norms, $||\phi|| = 1$).

$\forall x \geq 0 \Rightarrow \phi(x) \geq 0$.

$y = y^* = y_+ - y_-, \ \ y_+, y_- \geq 0$.

$y = y^* \Rightarrow \phi(y) = \phi(y_+ 0 \phi_-) = \phi(y_+) - \phi(y_-) \in Z(M)$.

$$
\begin{aligned}
\phi(y^*) &= \phi((y_1 + iy_2)^*) \\
&= (\phi(y_1) + i\phi(y_2))^* \\
&= \phi(y)^*
\end{aligned}
$$

For $y \in M$,

$$
\begin{aligned}
y + y^* &\leq ||y + y^*|| \cdot 1 \\
\phi(||y + y^*||1 \pm (y + y^*)) &\geq 0 \\
||y + y^*|| \cdot 1 \cdot 2\phi(y + y^*)|
\end{aligned}
$$

$$
\begin{aligned}
||\phi(y)|| &= ||\phi(\frac{y + y^*}{2})|| \\
&\leq ||\frac{y + y}{2}|| \\
&\leq ||y||
\end{aligned}
$$

Also note that $||y^*||^2 = ||y^*y|| = ||yy^*|| = ||y||^2$.

Lemma 99. TFAE

(i) $\phi(xy) = \phi(yx)\ \forall x, y \in M$

(ii) $\phi(x^*x) = \phi(xx^*)\ \forall x \in M$

(ii) $\phi(p) = \phi(q) \forall p \sim q, pq \in P(M)$.

14 Feb 2022 Recall

Theorem 100. $\phi : M \to Z(M)$ center-valued state. Then TFAE:

(1) $\phi(xy) = \phi(yx)\ \forall x, y \in M$

(2) $\phi(xx^*) = \phi(x^*x)\ \forall x \in M$

(3) $\phi(p) = \phi(q)\ \forall p \sim q, p, q \in P(M)$.

$1 \Rightarrow 2 \Rightarrow 3$

Proof. If $u \in U(M)$, $\quad uqu^* \sim q$

By 3, $\phi(uqu^*) = \phi(q) \quad \forall q \in P(M),\ \forall u \in U(M)$.

Then, take linear combination of projections: $\phi(u \sum_{i\lambda_i q_i u^*)=\phi(\sum \lambda_i q_i)}$. Then, by Spectral theorem we have

$\phi(uxu^*) = \phi(x)\ \forall x \geq 0 \Rightarrow \phi(uxu^*) = \phi(x)\ \forall x \in M$.

Now do a change of variables, $x \mapsto xu$. Then, we have

$$\begin{aligned} \phi(u(xu)u^*) &= \phi(xu) \\ \phi(ux) &= \phi(xu)\ \forall x \in M,\ \forall u \in U(M) \\ \Rightarrow \phi(yx) &= \phi(xy)\ \forall x, y \in M \end{aligned}$$

The last line follows since every thing of M is a linear combination of elements of $U(M)$. $\square$

Now we can start building the trace.

Theorem 101. If $M \subset B(H)\ vN$ has a normal central state.

Note that M commutes with every element in the set. The, $M \subset Z(M)'$ (commutant). Note that $Z(M)$ is abelian (of Type I).

We learned that Type is preserved under commutant $\Rightarrow Z(M)'$ is type I as well. Since it is Type I, we can find an abelian projection.

Pick such a $p \in P(Z(M)')$ abelian with $z(p) = 1$. Then,

Note: $pAp = Z(A)p$. (In general center of $Z(M)$ is same as center of $Z(M)'$).

$$\begin{aligned} pMp &\subset pZ(M)'p \qquad \text{(Comment: Not an algebra, just a subspace)} \\ &= Z(Z(M)')p \\ &= Z(Z(M))p \\ &= Z(M)p \\ \therefore pMp &\subset Z(M)p \end{aligned}$$

Note we have a map from

$$\begin{aligned} Z(M) &\mapsto Z(M)p \\ \theta(x) &= xp \end{aligned}$$

This is a surjective homomorphism. Since $z(p) = 1$ (full support), this is precisely an injective map.

Check $\phi(x) = 0 \iff xp = 0 \Rightarrow x \cdot z(p) = 0 \Rightarrow x \cdot 1 = 0 \Rightarrow x = 0$ (no non-trivial kernel, so injective).

Another way to check:

$$\begin{aligned} xp &= 0 \\ \Rightarrow Mxp &= 0 \qquad M \in Z(M)' \\ XMp &= 0 \\ XMp(\xi) &= 0 \qquad (X \text{ operator })\ \forall \xi \in H \\ XZ(p)\xi &= 0 \qquad Zp = [Z(M)'pH] \\ X \cdot Z(p) &= 0 \end{aligned}$$

Also note another map:

$$\begin{aligned} pMp &\mapsto Z(M) \\ \phi(x) &= \phi^{-1}(pxp) \end{aligned}$$

Now this diagram commutes.

Lemma 102. (Property of Corners) Let $\phi : M \to Z(M)$ is a normal central valued state. Then $\forall \epsilon > 0,\ \exists p \in P(M)$ such that $\phi|_{pMp}$ faithful and $\phi(xx^*) \leq (1+\epsilon)\phi(x^*x)\ \forall x \in pMp'$.

Proof. Let $\{q_i\}$ be a maximal family of mutually orthogonal projections such that $\phi(q_i) = 0$.
$\phi(\sum q_i) = \sum \phi(q_i) = 0$.

$$0 \neq q_0 = 1 - \sum_i q_i \Rightarrow \phi \text{ is faithful in } q_0 M q_0$$

Suppose $x > 0 \in q_0 M q_0$ such that $\phi(x) = 0$. Then,
$\exists \chi_{(\epsilon,\infty)}(x) \neq 0 \Rightarrow$
$\phi(x \cdot \chi_{(\epsilon,\infty)}(x)) \geq \epsilon \phi(\xi_{(\epsilon,\infty)}(x)) = 0$.
Let $\{e_i, f_i\}_{i \in J}$ maximal family of projections.
Note that $\{e_i\}, \{f_i\}$ are mutually orthogonal, $e_i \sim f_i$ and $\phi(e_i) > \phi(f_i)\ \forall i$.
Let us consider $e = q_0 - \sum_i e_i, q_0 - \sum_i f_i = f$. We know that $e \sim f$.
Now look at $0 \neq \phi(f) > \phi(e) \geq 0 \Rightarrow e \neq 0$.
Let $\mu \geq 0$ be the smallest number such that $\phi(\tilde{e}_i) \leq \mu\phi(\tilde{f})\ \forall \tilde{e} \leq e, \tilde{f} \leq f, \tilde{e} \sim \tilde{f}$.
First observe that $\mu \neq 0$.

Proof. $\mu \neq 0, \phi(e) \neq 0$.
Fix $\epsilon > 0$. Then, $\exists$ two projections, $0 \neq \tilde{e} \leq e, 0 \neq \tilde{f} \leq f$ such that

$$\begin{aligned} \phi(\tilde{e}) &\geq \frac{\mu}{1+\epsilon}\phi(\tilde{f}) \\ (1+\epsilon)\phi(\tilde{e}) &> \mu\phi(\tilde{f}) \end{aligned}$$

Consider $\{\hat{e}_i, \hat{f}_i\}$ maximal family of $\hat{e}_i \leq \tilde{e}, \hat{f}_i \leq \tilde{f}$ such that
$\hat{e}_i \sim \hat{f}_i$ such that $(1+\epsilon)\phi(\hat{e}_i) \leq \phi(\hat{f}_i)$ Then,
$p = \tilde{e} - \sum \hat{e}_i, q = \tilde{f} - \sum \hat{f}_i$.
$\forall p_1, p_2 \leq p$ with $p_1 \sim p_2$. Then,
$\exists r \leq q$ such that $r \sim p_1$.
$\phi(p_1) \leq \mu\phi(p_2) \leq (1+\epsilon)\phi(p_2)$.
$\phi(p_2) \leq (1+\epsilon)\phi(p_2)$.
$\phi(up_2u^*) \leq (1+\epsilon)\phi(p_2)\ \forall p_2 \in pMp$.
$\phi(uyu^*) \leq (1+\epsilon)\phi(y), \quad y \geq 0$.
$\phi(ux^*xu) \leq (1+\epsilon)\phi(x^*x) \forall x \in pMp$
Something about polar decomposition ∎

∎

---

16 Feb 2022 Recall

Lemma 103. $\exists \phi : M \to Z(M)$ normal, central-valued state.

Lemma 104. Let $M \subset B(H)$ vN algebra, $\exists \phi : M \to Z(M)$ normal central-valued state such that $\forall\ \epsilon > 0\ \exists 0 \neq p \in P(M)$ such that

$\phi|_{pMp}$ is faithful and

$$\phi(xx^*) \leq (1+\epsilon)\phi(x^*x)\ \forall x \in pMp$$

Lemma 105. Let $M \subset B(H)$ finite vN algebra. Then, $\forall\ \epsilon > 0\ \exists \phi : M \to Z(M)$ a normal, central-valued state such that $\phi(xx^*) \leq (1+\epsilon)\phi(x^*x)\ \forall x \in M$.

Observation: It suffices to show the statement for $Mz, z \in Z(M)$.

Proof. From lemma 1 (today), $\exists \phi : M \to Z(M)$ normal central valued trace.

From Lemma 2 (today), $\exists p \in P(Mz)$ such that $\forall x \in pMzp$.

$\phi_1(xx^*) \leq (1+\epsilon)p_0(x^*x)$ (Note: Every projection is a sum of monic, orthogonal projections - due to the existence of the dyadic scale).

Passing to a subprojection WLOG p is monic. Then $\exists$ finitely many projections $p_1, p_2, .., p_n \in P(Mz)$ such that $p_i \sim p$.

$$\sum_{i=1}^{n} p_i = z_0 \in Z(Mz)$$

Now we know to make use of $p_i \sim_{v_i} p$ where $v_i^* v_i = p_i$ and $v_i v_i^* = p_i$.

$\Psi Mz_0 \to Z(Mz_0)$ where

$\Psi(x) = \sum_{i=1}^{n} \phi_0(v_i x v_i^*)$. Is this in $pMzp$?

[Can write partial isometry as $v_i =_i v_i^* v_i$ and $v_i^* = v_i^* v_i v_i^*$. Hence it is in $pMzp$.]

We know that $x \in Mz_0$. Then,

$$\begin{aligned}
0 &\leq \Psi(xx^*) \\
&= \Psi(xz_0x^*) \\
&= \Psi(x(\sum_{i=1}^{n} p_i)x^*) \\
&= \sum_{i=1}^{n} \Psi(xp_ix^*) \\
&= \sum_{i=1}^{n} \Psi(xv_i^*v_ix^*) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \phi_0(v_jxv_i^*v_ix^*v_j^*) \\
&= \sum_{i,j}(1+\epsilon)\phi_0(v_ix^*v_j^*v_jxv_j^*) \\
&= \sum_{i}(1+\epsilon)\phi_0(v_ix^*\underbrace{(\sum_{j} v_j^*v_j)}_{z_0}v_i^*) \\
&= (1+\epsilon)\sum_{i}\phi_0(v_ix^*xv_i^*) \\
&= (1+\epsilon)\Psi(x^*x)
\end{aligned}$$

□

Theorem 106. $M \subset B(H)$ is finite $\iff \exists\, \phi : M \to Z(M)$ normal central-valued trace.

Proof. ($\Leftarrow$) Done.

($\Rightarrow$) Now.

Let $1 + \dfrac{1}{n} = a_n \downarrow \perp$ Then, from Lemma 3 (today), $\exists \tau_n : M \to Z(M)$ normal central valued state.

$$\tau_n(xx^*) \leq a_n \tau_n(x^*x) \;\forall x \in M.$$

Claim: $0 \leq m \leq n$. Consider the map from $M \to Z(M)$ defined by $\Psi_{m,n} = a_m^2 \tau_m - \tau_n$ (still normal. Also this is a positive map).

$$\begin{aligned} \Psi_{m,n}(\perp) &= a + m^2 \cdot 1 \\ &= 1 \\ &= a_m^2 - 1 \\ \Rightarrow ||a_m^2 z_m - z_1|| &= a_m^2 - 1 \end{aligned}$$

□

Now check the above state that it is **positive**.

Proof. Show $\Psi_{m,n}(x) = 0 \;\forall x \geq 0$. Enough to just check for finite projections.

(Recall: $x = \int \lambda$)

$$\begin{aligned} \Psi_{m,n}(p) &\geq 0 \quad \forall p \in M \text{ (This is enough by spectral theorem)} \\ a_m^2 \tau_m(p) - \tau_n(p) &\geq 0 \end{aligned}$$

A map $\Psi(\sum e_i) = \sum \Psi(e_i)$ by normality) and because $\forall p$ is a sum of monic projections.

It suffices to check that $a_m \tau_m(p) - \tau_n(p) \geq 0 \;\forall p$ monic projections. Then,

$\exists p_1, p_2, ..., p_n \in P(M)$ such that

$$\sum_{i=1}^{k} p_i = z$$

for $p_i \sim p$.

$$
\begin{aligned}
\tau_n(p) &\leq a_n \tau_n(p_i) \\
\tau_n(p_i) &\leq a_n b_m(p) \\
k\tau_n(p) &= \sum_{i=1}^{k} \tau_n(p) \\
&\leq \sum_{i=1}^{k} a_n \tau_n(p_i) \\
&= a_n \tau_n(\sum p_i) \\
&= a_n \tau_n(z) - a_n z \\
&= a_n \tau_m(z) \\
&= a + n\tau_m(\sum_{i=1}^{k} p_i) \\
&= \sum_{i=1}^{k} a_n \tau_m(p_i) \\
&\leq a_n \sum_{i}^{k} a_m \tau_m(p) \\
&= k a_n a_m \tau_m(p) \\
&\leq k a_m^2 \tau_m(p) \\
\Rightarrow z_n(p) &\leq a_m^2 z_m(p) \\
\therefore a_m^2 \tau_m - \tau_n &\geq 0 \\
\Rightarrow ||a_m^2 \tau_m - \tau_n|| &= a_m^2 - 1 > 0
\end{aligned}
$$

Therefore, $\exists \tau_m(x) \to \tau(x)$. such that $||\tau_m - \tau||| \to 0$.
Then $\tau(xx^*) \leftarrow \tau_n(xx^*) \leq a_n \tau(x^*x) \to \tau(x^*x)\ \forall x \in M$ and
$\tau(xx^*) \leq \tau(x^*x)$
$\Rightarrow \tau(xx^*) = \tau(x^*x)$.
$\Rightarrow \tau(xy) = \tau(yx)\ \forall x, y \in M$.
$\tau|_{Z(M)} =$ identity.
$\phi \in M_x$.
$||\phi \circ \tau - \phi \circ \tau_m|| \leq ||\phi||||\tau - \tau_m||$
$\Rightarrow M_x$ is closed. $\square$

---

18 February 2022 (B. Yeadon) proved existence of center- valued trace on finite vN Ryll-Narz. fixed point.

Consider $x \mapsto \{\phi(uu^*)|u \in U(M).\}$

Theorem 107. Suppose M is a finite vN algebra.

$\Psi : M \to Z(M)$ faithful, normal central valued state. Then, $\forall e, f \in P(M), e \preceq f \iff \Psi(e) \leq \Psi(f)$ (trace of e is less than or equal to trace of f).

Note we know about trace that $\Psi(zx) - z(\Psi x)\ \forall z \in Z(M)$ (modular with respect to the center).

Proof. $(\Rightarrow)$
$e \preceq f \Rightarrow\ \exists v \in M$ such that $e = v^*v, vv^* \leq f$. Then,
$\Psi(e) = \Psi(v^*v) = \Psi(vv^*) \leq \Psi(f)$ □

Proof. $(\Leftarrow)$ Let $e, f \in P(M)$ such that $\Psi(e) \leq \Psi(f)$.
By comparison, $\exists z \in Z(M)$ such that $ez \leq fz$ and $(1-z)f \preceq (1-z)e$.

$$
\begin{aligned}
(1-z)f &\preceq (1-z)e \\
(1-z)f &= v^*v,\ vv^* \leq (1-z)e \\
\Rightarrow \Psi(vv^*) &\leq \Psi((1-z)e) \\
&= (1-z)\Psi(e) \text{ (since modular with respect to the center)} \\
&\leq (1-z)\Psi(f) \\
&= \Psi((1-z)f) \\
&= \Psi(v^*v) \\
&= \Psi(vv^*) \\
\Rightarrow \Psi((1-z)e - vv^*) &= 0 \\
\Rightarrow (1-z)e &= vv^* \\
\therefore (1-z)f &\sim (1-z)e \\
fz &> ze \\
\Rightarrow f &> e
\end{aligned}
$$

□

Theorem 108 (Dixmier averaging proposition - Jacques (1949)). Let M be a von Neumann algebra.
$\forall x \in M, \overline{K(x)}^{\|\ \|} \cap Z(M) \neq \phi$ where the **convex hull** is defined as

$$K(x) = c_0\{uxu^*|U(M)\} = \{\sum_{i=1}^{n} \alpha_i u_i x u_i^*, \sum \alpha_i = 1$$

(same x, different u's).

- Note, it does not have any topological properties so we close it.

Corollary 108.1. Suppose M is a finite vN algebra, then there exists only one central-valued trace.

Proof. Suppose there exists $\Psi_1, \Psi_2 : M \to Z(M)$ central-valued trace. Then, by (Di 49),
$\Psi_1(\sum_{i=1}^n \alpha_1 u_i x u_i^*) = \sum_{i=1}^n \Psi_1(u_i x u_i^*) = (\sum_{i=1}^n \alpha_i)\Psi(x) = \Psi_1(x)$.
$\Psi_1(\overline{K(x)}) = \Psi_1(x)$.
Now consider $\Psi_1(Z(M)) = \mathrm{Id}_{Z(M)}$. Then,
$\overline{K(x)}^{\|\ \|} \cap Z(M) = \{\Psi_1(x)\} = \{\Psi_2(x)\}$.
$\therefore \Psi_1(x) = \{\Psi_2(x)\}$.
(By the Theorem, there is an element in the center that is approximated by the convex hull. Because z is in the center $\Phi_z = z$. So the right hand side is the trace so they must be the same). □

Lemma 109. Let M vN algebra, $\forall\ x = x^* \in M$.
$\exists u \in U(M), y \in Z(M)$ such that

$$||\frac{1}{2}(x + uxu^*) - y|| \leq \frac{3}{4}||x||$$

(Since self-adjoint, may not be positive, but we select the positive part).

Proof. Consider $p = \chi_{[0,\infty)}(x) \in P(M)$. Let $q = 1 - p$. Now, by comparison, $\exists\ z \in Z(M)$ such that for $q_1, q_2, p_1, p_2 \in P(M)$, we have
$zq \sim_v p_1 \leq p_1 + p_2 = zp$ and $(1-z)p \sim_w q_1 \leq q_1 + q_2 = (1-z)q$ Then,

$$\begin{aligned}
u = v + v^* + w + w^* + p_2 + q_2 \in U(M) \\
u^*p_1u = zq, \quad u^*zqu = p_1, \quad u^*p_2u = p_2 \\
u^*q_1u = (1-z), \quad u^*(1-z)pu = q_1, \quad u^*q_2u = q_2
\end{aligned}$$

We prove one of them. Rest should follow:

$$\begin{aligned}
u^*p_1u &= (v + v^* + u + u^* + p_2 + q_2)p_1(v + v^* + u + u^* + p_2 + q_2) \\
&= (vp_1 + v^*p_1 + up_1 + u^*p_1)(v + v^* + u + u^* + p_2 + q_2) \\
&= (vp_1 + v^*p_1 + up_1 + u^*p_1)(v + v^*) \quad \text{(Rest are 0)} \\
&= (vp_1 + v^*p_1)(v + v^*) \\
&= \underbrace{vp_1v^*}_{zq}
\end{aligned}$$

Note that $vp_1w = vp_1ww^*w$ and $ww^* = q_1 \leq (1-z)q_1$ and $w^*w = (1-z)p$. so things cancel.

□

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23 Feb 2022

Lemma 110. Let $M \subset B(H)$ vN algebra. Let $x \in M^k, x = x^*$. Then, $\exists u \in U(M), y \in Z(M)$ such that

$$\frac{1}{2}(x + uxu^*) - y|| \leq \frac{3}{4}||x||$$

Proof. $x = x^*$ and consider the characteristic function (only positive part) - $p = \chi_{[0,\infty)}(x) \in M$. Assume $||x|| = 1$.

Let $q = 1 - p$. Then, use the Comparison Theorem i.e. $\exists z \in P(Z(M))$ and $p_1, p_2, q_1, q_2 \in P(M)$ such that

$zq \sim_v p_1 \leq p_1 + p_2 = zp$.

$(1-z)p \sim_w q_1 \leq q_1 + q_2 = (1-z)q$.

By the partial isometries defined above, we have

$v^*v = zz, w^* = p_1, w^*w = (1-z)p, ww^* = q_1$.

(By last time, we have)

$u = v + v^* + w + w^* + p_2 + q_2 \in U(M)$.

(i) $u^*p_1u = zq$, $u^*z_qu = p_1$, $u^*p_2u = p_2$. And

(ii) $u^*q_1u = (1-z)p$, $u^*(1-z)pu = q_1$, $u^*q_2u = q_2$. Then,

$$\begin{aligned}
x &\leq \chi_{[0,\infty)}(x) = p \\
zxz^* &\leq zpz^* \\
-qz \leq xz &\leq pz = p_1 + p_2 \qquad \text{(can always conjugate)} \\
-p_1 = -u^*qzu \leq u^*xzu &\leq u^*(p_1 + p_2)u = zq + p_2 \\
\Rightarrow -p_1 \leq u^*xzu &\leq zq + p_2 \\
-qz \leq xz &\leq p_1 + p_2 \qquad \text{(add them up)} \\
\frac{-z}{2} = \frac{-(p_1 + p_2)}{2} &\leq \frac{u^*xzu + xz}{2} \leq \frac{zq + p_1 + zp_2}{z} \\
&\leq zq + p_1 + p_2 \\
&= z
\end{aligned}$$

We also have

$$\begin{aligned}
z &= z1 \\
&= z(p+q) \\
&= zp + zq \\
&= p_1 + p_2 + zq
\end{aligned}$$

Therefore, we have

$\dfrac{-z}{2} \leq \dfrac{u^*xzu + xz}{2} \leq z$

$\dfrac{-3z}{4} \leq \dfrac{1}{2}(u^*xzu + xz) + \dfrac{z}{4} \leq \dfrac{3}{4}z\dfrac{-3}{4}(1-z) \quad \leq \dfrac{1}{2}((1-z)x + u^*(1-z)xu^* + \dfrac{1}{4}(1-z) \leq \dfrac{3}{4}(1-z)u^*$

Similarly for the other one. When you add things up, we get:

$$\begin{aligned}
\frac{-3}{4} &\leq \frac{1}{2}(x + u^*xu) - \frac{1}{4}(2z+1) \leq \frac{3}{4} \\
||\frac{1}{2}(x + u^*xu) &- \underbrace{\frac{1}{4}(2z-1)}_{y}|| \leq \frac{3}{4}
\end{aligned}$$

□

Recall the theorem:

Theorem 111 (Dixmier '49). Suppose $M \subset B(H)$ vN algebra.

Let $x \in M$. Consider $\overline{K(x)}^{|| \ ||} \cap Z(M) \neq \emptyset$, where $K(x)$ is the convex hull i.e. $K(x) =$ co $\{uxu^* | u \in U(M)\}$.

Proof. Suppose $x = a_0 + ib_0$, where a_0, b_0 self-adjoint ($a_0 = a_0^*, b_0 = b_0^*$).
Consider map:

$$\alpha(x) = \sum_{i=1}^{n} \alpha_i u_i x u_i^*, \qquad \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}_+, \alpha_1 + \alpha_2 + ... + \alpha_n = 1, u_1, u_2, ..., u_n \subset U(M)$$

Apply the lemma to $x = a_0$. Then, $\exists y_1 \in Z(M)$ such that

$$||\alpha_1(a_0) - y_1|| \leq \frac{3}{4}$$

Now we want to iterate the process. Apply lemma to $x = \alpha(a_0) - y_1$. $\exists y)2 \in Z(M)^k$

$$\begin{aligned} ||\alpha_2(\alpha_1(a_0) - y_1)) - y_2|| \leq \frac{3}{4} ||\alpha_1(a_0) - y_1|| \leq (\frac{3}{4})^2 \\ ||\alpha_2 \circ \alpha_1(\alpha_0) - (y_1 + y_2)|| \leq (\frac{3}{4})^2 \end{aligned}$$

Continue iterating this to get the following. For $y_k \in Z(M)$, we get

$$||\alpha_k \circ \alpha_{k-1} \circ .... \circ (\alpha_0) - (y_1 + y_2 + ... + y_k)|| \leq (\frac{3}{4})^k \tag{5}$$

Hence, we have

$$|| \underbrace{a_k}_{K(a_0)} - \underbrace{\tilde{y}_k}_{\in Z(M)} || \leq (\frac{3}{4})^k$$

We are done for the real part. Now need to consider the imaginary part. For α averaging operator $\forall \ \epsilon > 0$, $\exists y \in Z(M)$ such that

$$||\alpha(a_0) - y|| \leq \epsilon$$

Similarly, $\exists \ \beta$ an averaging operator. Let $z \in Z(M)$ such that

$$\begin{aligned} ||\beta(\alpha(b_0)) - z|| &< \epsilon \\ ||\beta(\alpha(a_0)) - y|| &= ||\beta(\alpha(a_0) - y|| \\ &\leq ||\alpha(a_0) - y|| \qquad \text{(since Lipschitz)} \\ &< z \end{aligned}$$

Now apply directly to our element $x = a_0 + ib_0$.

$$||\beta(\alpha(a_0 + ib_0) - y + iz|| \leq ||\beta(\alpha(a_0)) - y|| + ||\beta(\alpha(b_0) - z|| \leq zq$$

We have proved that $\forall \ \epsilon > 0$, $\exists x_k \in K(x), z_k \in Z(M)$ such that $||x_k - z_k|| \leq \frac{1}{2^k}$.

$\Rightarrow ||x_{k+1} - x_k|| \leq \frac{1}{2^k}$.

$x_k \xrightarrow{|| \ ||} x$ $\square$

20 Fundamental Group

$M \mathrm{II}_1 \rightsquigarrow \mathcal{F}(M)$

$0 \leq t \leq 1$.

$\exists p \in P(M)$ such that $\sigma(p) = t$.

Look at pMp '$='$ M^t (isomorphism class of pMp).

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25 February 2022

Suppose $p, q \in P(M), \sigma(p) = \sigma(q) = t \Rightarrow p \sim q$.

$p \sim_v q, 1 - p \sim_w 1 - q$.

Let $u = v + w \in U(M)$. Then, we have

$upu^* = q, u(1-p)u^* = (1-q)$.

Consider the map: ad $(u) : pMp \to qMq$ by the map ad $(u)(x) = uxu^*$.

$$\begin{aligned} upMpu^* &= up1M1u^* \\ &- upu^*uMu^*upu^* \\ &= upu^*Mupu^* \\ &= qMq \end{aligned}$$

If $1 \leq t < \infty \Rightarrow \ \exists n \in \mathbb{N}, 0 < \dfrac{t}{n} \leq 1$.

(Then tensorize)

$M^t = p(M \otimes M_n(\mathbb{C}))p \quad M_n(M)$

$(M \otimes M_n(\mathbb{C}), \underbrace{\sigma \otimes \frac{Tr}{n}}_{\tilde{\tau}})$

$p \in M \otimes M_n(\mathbb{C})$

---

28 Feb 2022 Recall Fundamental group.

$M \mathrm{II}_1$ factor. Then, $F(M) \subset (\mathbb{R}_+, \cdot)$

$F(M) = \{t \in (0,\infty) | M^t \cong M\}$

$0 < t \leq 1$. Pick$p \in P(M)$ such that $\tau(p) = t$.

Consider $M \xrightarrow{\text{central trace}} Z(M) \to \mathbb{C}$ and $M \to \mathbb{C}$

Consider pMp II_1 factor.

$M^t =$ equivalence class of pMp.

$p, q \in P(M), \tau(p) = \tau(q) = t \Rightarrow upu^* = q$.

Then, $pMp \cong_{\mathrm{ad}(u)} qMq$.

For $1 \leq t < \infty$, pick $n \in \mathbb{N}$ such that $\dfrac{t}{n} \leq 1$.

$M_n(\mathbb{C}) \otimes M$, II_1 factor. Then, take tensor

$(M_1(\mathbb{C} \otimes M, \mathrm{tr} \times \tau)$ where tr:$M_n(\mathbb{C} \to \mathbb{C}$ where the map is defined by $\mathrm{tr} = \dfrac{1}{4}$ Tr.

Consider $\tilde{\tau(p)} = \dfrac{t}{n}$.

Now take $M^t =$ the equivalence class $p(M_n(\mathbb{C} \otimes M)p$.

Proof idea:

$M_n(\mathbb{C}) \otimes \subset M_{n_r}(\mathbb{C}) \otimes M$

$M_n(\mathbb{C} \otimes M \subset M_{n_r}(\mathbb{C})$

Now we take $(M^t)^s \cong M^{ts}$ for $0 < s, t \leq 1$. Then,

$0 < \leq 1 \rightsquigarrow$ defined by $z(p) = t$.

(pMp, τ_p) where $\tau_p(x) = \dfrac{z(pxp)}{z(p)}$ II_1 factor

$\exists q \in P(pMp)$ such that $\tau_p(q) = s \Rightarrow \dfrac{\tau pqp}{\tau p} = s$

Then, $\dfrac{\tau(q)}{\tau(p)} = s \Rightarrow \tau(q) = ts$.

$q(pMp)q = qMq \cong M^{ts}$

Now suppose s and t are in the Fundamental Group, i.e. $s, t \in F(M)$. By definition, we have

$M^{ts} \cong (M^t)^s \cong (M)^t \cong M$

21 Open Problems and History

Definition 21.1 (Hyperfinite)**.** $A_0 \subset A_1 \subset A_2 \subset .... \subset A_n \subset \overline{\cup A_n}^{\mathrm{SOT}} = M$. (if closed and Type I).

Any two Type I towers with same structures are isomorphic (unique up to isomorphism).

Consequence: Fundamental Group of Type I is whole $\mathbb{R}_+$.

Consequence II: If M finite dimensional, then so is compression pMp.

(M and vN ' (43)) $R =$ hyperfinite factors. Then, $F(R) = \mathbb{R}_+$

(Kadison' 1967)

$PSL_n(\mathbb{Z}), n \geq$. Then, $F(L(G)) =$ countable (cannot be $\mathbb{R}_+$.)

(Dan-Virgil, Vorculescu, Florin Radulescu) ('91)

$F(L(\mathbb{F}_\infty)) = \mathbb{R}_+$

$F(L(\mathbb{F}_n) \subset \{1, \mathbb{R}_+)$

Field Medals Problem (related to FG/ free probability question):

$L(\mathbb{F}_n \ncong L(\mathbb{F}_m), \quad n \neq m$. (Connes' 80)

$L(\mathbb{F}_n)^p = L(\mathbb{F}_{p-1})$

(Popa 2001)

$F(L(\mathbb{Z}^2 \times SL_2(\mathbb{Z})) = \{1\}$

Popa's Deformation/Rigidity Theory

$\forall F$ (countable) $\subset (\mathbb{R}_+, \cdot)$

(Free Product) $F(L(*(\mathbb{Z}^2 \times SL_2(\mathbb{Z})) = F$

22 Radon-Nikodym

Theorem 112. $N \subset (M, \tau)$ finite vN algebras.

Then, $\exists E_N : M \to N$ a conditional expectations with the following properities:

1) Normal

2) $E_n(n_1 \times n_2) = n_1 E_N(X|n_2)$ $\forall n_1, n_2 \in N, X \in M$ (bimodular)

2b) $E_N(1) = 1$ (unital)

3) $\tau \circ E_N = \tau$ (τ- invariant)

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21 March 2022 (Post Spring Break)

Theorem 113. M semi-finite factor has a unique normal "semifinite trace" i.e.

Tr: $M_+ \to \overline{\mathbb{R}} = [0, \infty]$ with $\text{Tr}(x) \geq 0$. and $M_+ - \{x \in M | x \geq 0\}$.

(discussion on weights, ideals, etc.)

Tr (sup x) = sup Tr (x) (normally)

In addition, Trace is **faithful**. (In conclusion, semifinite trace has factors similar to finite).

$p \in M$ finite $\iff$ $\text{Tr}(p) < \infty$.

$p \preceq q \iff \text{Tr}(p) \leq \text{Tr}(q)$.

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23 Sorin Popa's Intertwining Techniques

Result from '05 (Invent. Math.)

Motivation (In group theory), $A, B \leq G$. Does $\exists\, g \in G$ such that $gAg^{-1} = B$ (conjugacy class). Loosen conjugacy to $gA \subset Bg$ and extend idea to algebras instead of just groups.

(Example of Intertwining) Similarly, intertwining is $x_p A_p \subset Bx$ (automorphisms).

Now, we formalize this notion.

Theorem 114. Let (M, σ) be a finite vN algebra, $f \in P(M)$.

$A \subset fMf$ (compression of M by f), $B \subset M$ vN subalgebras.

Then, the following are equivalent.

1. $\exists\, 0 \neq a \in A' \cap (< M, B >)_+, \text{Tr}(a) < \infty$.
2. $\exists\, 0 \neq e \in A' \cap < M, B >$ such that Tr $(e) < \infty$. (projection)
3. $\exists\, 0 \neq q \in P(A), 0 \neq p \in P(B), av \in M$ partial isometry and $*-$homomorphism such that

$\Psi : qAq \to pBp$ such that

$\Psi(x)v = vx\ \forall x \in qAq$ and $vqAq \subset pBpv$.

(For 1, we need to take the basic construction of M by B) $B \subset M \subset < M, B > = JB'J = < M, e_R >$ (note $A \subset M$ as well.

In general: $\eta_\Phi = \{x \in Q, \Phi(x^*x) < \infty\}$ where $\Phi : Q_+ \to [0, \infty]$. (left ideal - check by Cauchy Schwarz inequality)

Consider $m_\Phi = \{\sum_{i=1}^n y_i^* x_i | x_i, y_i \in n_\Phi, x \in N\}, x$ subalgebra (spans n_Φ).

Tr: $< M, B > \to \mathbb{R}$ where $\{\overline{\sum_{i=1}^n x_i e_B y_i} | n \in \mathbb{N}, x_i, y_i \in N\} \subset < M, B >$

Then,

$$\begin{aligned} \text{Tr}(\sum_{i=1}^n x_i e_B y_i) &= \sum_{i=1}^n \text{Tr}(x_i e_B y_i) \\ &= \sum_{i=1}^n \sigma(x_i y_i) \end{aligned}$$

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23 March 2022

Theorem 115 (S. Popa '03). Let (M,τ) finite vN algebra. Let $f \in P(M)$. (projection).

Let $A \subset fMf, B \subset M$ vN subalgebras (inclusions are unital). TFAE:

1. $\exists\ 0 \neq a \in A' \cap f < MB > f$ that has finite trace i.e. $\mathrm{Tr}(a) < \infty$ where $a \geq 0$ (and where $B \subset M \subset <M,B>= \{M,e_B\}'' \subset B(L^2(H)$. Also $A \subset M \subset <M,B>$).

2. $\exists\ 0 \neq e \in A' \cap f < M,B > f,\ \mathrm{Tr}\ (e) < \infty$ i.e. $e(H) = K \subset L(M) \Rightarrow AKB = K$. (range needs A-B bimodule).

3. $\exists\ 0 \neq q \in P(A)$ and $\xi \in qL^2(M)$ such that $qAq\xi \subset \overline{\xi B}$.

4. $\exists\ 0 \neq q \in P(A), p \in P(B), v \in M$ partial isometry and $\Psi : qAq \to pBp$ $*$-homomorphism - unital, injective (NOT surjective) such that $\Psi(x)v = vx \forall x \in qAq$.

i.e. $A \preceq_M B$ (corner of A intertwines with B in M).

5. The following doesn't hold:

$\forall a_1, a_2, ..., a_n \in M,\ \forall \epsilon > 0,\ \exists u \in U(A)$ such that $||E_B(a_i u a_j^*)||_2 < \epsilon\ \forall i,j = \overline{1,n}$.

(for separable algebras, this is equivalent to saying $\exists (u_n)_n \subset U(A)$ such that $||E_B(xu_ny)||_2 \to 0$ as $n \to \infty\ \forall x,y \in M$. (If not separable, have to work with nests instead of sequences but still true).

Proof. $(1 \Rightarrow 2)$

$a \neq 0$. $\exists\ e = \chi(a)_{[\epsilon,\infty)}$ spectral projection of a.

Then, $ae \geq \epsilon e$. Then, we have

$$\begin{aligned} \infty &> \mathrm{Tr}(a) \\ &\geq \mathrm{Tr}(a \cdot e) \\ &\geq \epsilon \mathrm{Tr}(e) \\ \Rightarrow a &\geq e^{1/2} e a^{1/2} \\ &= ae \end{aligned}$$

$\Rightarrow e \in A' \cap f(M,B)f$. □

Proof. (4 and 5). (let us start by negating (5)).

Suppose 5 holds. Then, $\exists a_1, a_2, .., a_n \in M\ \exists \xi_0 > 0$ such that $\forall\ u \in U(A),\ \exists\ i,j$ such that $||E_B(a_i u a_j^*)||_2^2 \geq \xi_0^2$. $\Rightarrow \sum_{i=1,j}^n ||E_B(a_i u a_j^*)||_2^2 \geq \xi_0^2$. (no need to make choices for i,j with the sum).

(Note we will use $\mathrm{Tr}(xe_By) = \tau(xy)$).

$$\begin{aligned} \sum_{i=1,j}^n ||E_B(a_i u a_j^*)||_2^2 \geq \xi_0^2 &= \sum_{i,j=1}^n \tau(E_B(a_i u a_j^*) E_B(a_i u a_j^*)) \\ &= \sum_{i,j=1}^n \mathrm{Tr}(E_B(a_j u^* a_i^*) e_B E_B(a_i u a_j^*)) \\ &= \sum_{i,j=1}^n \mathrm{Tr}(e_B E_B a_j u^* a_i^*) e_B E_B(a_i u a_j^*) e_B) \\ &= \sum_{i,j=1}^n \mathrm{Tr}(e_B a_j u^* a_i^* e_B a_i u a_j^* e_B) \\ &= \sum_{i,j=1}^n \mathrm{Tr}(u^*(a_i^* e_B a_i) u (a_j^* e_B e_j)) \\ &= \mathrm{Tr}(u^*(\sum_{i,j=1}^n \underbrace{a_i^* e_B e_i}_{=y}) u (\sum_{i,j=1}^n e_j^* e_B a_j) \\ &= \mathrm{Tr}(u^* y u y) \geq \xi_0 \end{aligned}$$

□

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25 March 2022 Recall $< M, B > \{M, e_B\}'' \subset B(L^2(M))$. Also recall the theorem:

Theorem 116 (Popa '03). Consider (M, τ) vN algebra, $A \subset fMf$ (corner), $B \subset M$.

Then, TFAE:

(1) $\exists\ 0 \neq a \in (A' \cap f < M, B > f)_+,\ \mathrm{Tr}(a) < \infty$.

(2) $\exists\ 0 \neq e \in P(A' \cap f < M, B > f), \mathrm{Tr}(e) < \infty$.

(3) $\exists\ 0 \neq q_0 \in P(A)$ and $0 \neq \xi \in q_0 L^2(M)$ such that $q_0 A q_0 \xi \subset \overline{\xi B}$.

(4) (Intertwining) $\exists 0 \neq q \in Q, 0 \neq p \in B$, *-homomorphism (unital, injective) with $\Psi : qAq \to pBp$, $\exists v \in m$ such that $xv = v\Psi(x)\ \forall x \in qAq$.

(5) (Mixing Representation) The following (analytic) conditins do not hold:

(i) $\forall a_, a_2, ..., a_n \in M,\ \forall \epsilon > 0,\ \exists u \in U(A)$ (unitary) such that $||E_B(a_i u a_j^*|_2 < \epsilon, 1 \leq i, j \leq n$.

(This is a decay property).

(ii) $\exists (u_n)_n \subset U(A), ||E_B(xu_n(y)||_2 \to 0 \forall x, y \in M$

(This is mixing condition).

Explanation (4) (Idea: Maybe patch the corners)

We need to know what happens to v^*v, vv^*. With the relation above, we will find that

$vv^* \in qAq' \cap qMq$ and $v^*v \in \Psi(qAq)' \cap pMp.$.

In particular for the first one, we have

$$\begin{aligned} xv &= v\Psi(x) \\ \Rightarrow v^*x &= (xv)^* \\ &= (v\Psi(x))^* \\ &= \Psi(x)v^*\ \forall x \in qAq. \end{aligned}$$

$$\begin{aligned} xvv^* &= v\Psi(x)v^* \\ &= vv^*x \\ \Rightarrow xvv^* &= vv^*x \\ \Rightarrow vv^* &\in qAq' \cap qMq \end{aligned}$$

Now let us continue the proofs.

(1) $\Rightarrow$ (2) Done Last time.

Proof. (4) $\Rightarrow$ (1)

Consider $ve_Bv^* \subset Me_BM \subset < M, e_B >$. (essentially in $q < M, e_B > q$).

Fix $x \in qAq$. Then,

$$\begin{aligned} ve_Bv^*x &= ve_B \underbrace{\Psi(x)}_{\in B} v^* \\ &= v\Psi(x)e_Bv^* \\ &= xve_Bv^* \\ \therefore ve_Bv^* &\in qAq' \cap q < M, e_B > q. \end{aligned}$$

(further, Tr $(ve_Bv^*) = \tau(vv^*) < \infty$ (finite trace).

We have $(A' \cap f < M, e_Bf)q$.

We will use the **monoticity** property for the whole algebra.

$\exists v_1, ..., v_n \in M$ partial isometries such that $v_i^*v < q, z = \sum_{i=1}^n v_iv_i^* \in Z(M)$.

Then we build $\sum_{i=1}^{n} v_i v e_B v^* v_i^* \in f < M, e_B > f, \quad \mathrm{Tr}(a) < \infty$.
Fix $x \in A$. Note, since v_i^* is partial isometry, then $v_i^* = \underbrace{v_i^* v_i}_{\leq q} v_i^*$. Now we have

$$
\begin{aligned}
azx &= x \sum_{i=1}^{n} v_i v e_B v^* v_i^* x \sum_{i=1}^{n} v) j v_j^*) \\
&= \sum_{i,j}^{n} v_i v e_B v^* (v_i^* x v_j) v_j^* \\
&= \sum_{i,j}^{n} v_i v_i^* x v_j^* v e_B v^* v_j^* \\
&= zx \sum_{j=1}^{n} v_j^* v e_B v^* v_j \\
&= zxa = xza
\end{aligned}
$$

(Hence, commutes with everybody). □

Proof. (5) ⇒ (1)
$\exists a_1, ..., a_n \in M, q_0 > 0, \quad \forall u \in U(A)$, we have that $||E_B(a_i u a_j^*)||_2 \geq q_0$.
We will work with a weaker condition i.e. $\sum_{i,j=1} ||E_B(a_i u a_j^*])||_2^2 \geq q_0^2 \ \forall u \in U(A)$.. Then,
Tr $(u^* y u y)$ where $y = \sum_{i=1}^{n} a_i e_B a_i^*$.
Tr $(u^* y u y) \geq q_0 \ \forall u \in U(A)$.
Then, we will do an averaging. Consider the convex hull and take its weak closure:
co $\{^W u^* y u | u \in U(A)\} \subset L^2(< M, e_B >)$ **(item (iv) from printed out notes).**
$\exists\ \xi \in K$ s $| \cdot ||_{2\ \mathrm{Tr}} -minimal. In particular, we have$
$\inf_{\xi_0 \in K} ||\xi_0||_{2\ \mathrm{Tr}} = ||\xi||_{2\ \mathrm{Tr}}$. Then,
$||u^* \xi u||_{2\ \mathrm{Tr}}^2 = Tr(\xi^* \xi = ||\xi||_{2\ \mathrm{Tr}}$.
Finally, $u^* \xi u = \xi \ \forall u \in U(A)$.
Then, Tr $\geq \xi_0 > 0$. □

30 March 2022 Let $p \in P(M)$. We are in (M, τ) finite vN algebra. $A \subset pMp, B \subset M$. Then, recall that TFAE:

(i) $\exists\ 0 \neq a \in A \cap p < M, B > p,\ \mathrm{Tr}(a) < \infty$

(ii) $\exists\ 0 \neq e \in P(A' \cap p < M, B > p), \mathrm{Tr}(e) < \infty$

(iii) $\exists\ 0 = p_0 \in P(A), 0 \neq \xi \in L^2(M, \tau)$ such that $p_0 A p_0 \subset \overline{\xi B}$.

(iv) $\exists\ 0 = p_0 \in P(A), q \in P(B), v \in m$ part is the $\Psi : p_0 A p_0 \to qBq\ * -$ such that $\Psi(x)v = vx$.

(v) The following analytic property doesn't hold (weak mixing)

$\forall a_1, ..., a_n \in M\ \forall \epsilon > 0,\ \exists u \in U(A)$ such that $||E_B(a)iua_j^*||_2 < \epsilon,\ \forall 1 \leq i, j \leq n$.

$1 \Rightarrow 2, 4 \Rightarrow 1, 5 \Rightarrow 1$ done

Proof. $(1 \Rightarrow 5)$ (Proof Idea - Actual proof requires Pulldown Lemma)

$f in A' \cap p < M, B >, \quad \mathrm{Tr}(f) < \infty$

$\exists a_1, a_2, ..., a_n \in M$.

$f \equiv \sum_{i=1}^{n} a_e e_B a_i^*$ (can approximate by finite rank. When B is trivial algebra, this is rank 1)

(Can make argument rigorous for the basis if $B \subset L^2(M)$.

Even though ξ_i are in L^2 (not bounded), $\xi_i e_B \xi_i^*$ is a projection.

Let $u \in U(A)$. Take $\mathrm{Tr}(fufu^*) = \mathrm{Tr}(f^2) = \mathrm{Tr}$ [semi-finite trace, on the basic construction]

$\mathrm{Tr}((\sum_{i=1}^{n} a_i e_B a_i) u (\sum e_i e_B e_i^8) u^*) \Rightarrow \sum ||E_B(a_i^* u a_j||_2^2 > \mathrm{Tr}(f) - 2\epsilon\ \forall n \in U(A)$

Hence we found an a and ϵ such that 5 fails (negation done). □

Side Note Usually we have $B \subset M \subset L^2(M, \tau)$ and $L^2(M) = \overline{\sum_{i \in I} \xi_i B}$ (can write this like a Fourier series i.e. $x = \sum \xi_i E_B(\xi_i^*) x)$

$E_B(\xi_i^* \xi_j) = \delta_{i,j} e_i$ (Use bimodule structure and Hilbert analysis)

Proof. $(3 \Rightarrow 4)$

If we look at $\xi^* \xi$ (product of 2 L^2 is in L^1 by C-S) (Map from $M \to B$, but also a map from $L^1(M) \to L^1(B)$).

$||E_B(x)||_1 = \sup\{\tau(uE_B(x)|u \in U(B)\} = \tau(ux) \leq \sup \tau(ux) = ||x||_1 c$ □

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1 April 2022 Recall theorem as above. We will continue the proofs.

($1 \Rightarrow 2, 4 \to 1, 1 \iff 5$ Done)

Proof. ($3 \Rightarrow 4$)

Consider $E_B : L^2M \to L^2B$ where $e_B(x\hat{1}) = E_B(x)\hat{1}$ (expectation as we know it)

It does extend to L^1 space. Consider $E_B : L^1(M,\tau) \to L^1(B,\tau)$ where $||x||_1 = \tau(|x|)$ (Banach space, has all the properties).

Show that $||E_B(x)||_1 \leq ||x||_1 \ \forall x \in M$. (contraction).

We can write $||x||_1 = \sup\{\tau(ux)|u \in U(M)\}$. Then,

$$\begin{aligned} ||E_B(x)||_1 &= \sup\{\tau(uE_B(x))|u \in U(B)\} \\ &- \sup\{\tau(E_B(ux))|u \in U(B)\} \qquad \text{(since bimodular)} \\ &= \sup\{\tau(ux)|u \in U(B) \subset U(M)\} \\ &\leq \{\tau(ux)|u \in U(M)\} \\ &= ||x||_1 \end{aligned}$$

$\therefore$, E_B extends by continuity i.e. $E_B : L^1M \to L^1B$ completely positive map.

We will use this to build our $*$-homomorphism.

Let $\xi \in q_0L^2M \subset L^2M$ (can be seen as **affiliated operator**).

For $x \in L^2, x\xi$ (x acts on L^2 but ξ can also act on x by acting on the right). Hence, can be seenly as a densely defined operator. $\xi^*\xi \in L^1M$(Multiply two vectors in L^2, we get vector in L^1).

(similar idea of polar decomposition for unbounded operators).

$\xi_{(q,\infty)}(T)$ (no notion of support, we build the support)

(Example to have in mind: $L^\infty[0,1] = M$ (or even $L^1[0,1]$). Consider the set of measurable functions Meas[0,1]. In general, Meas[0,1] are affiliated operators. Can always do pointwise approximation.

Even for unbounded functions, can fix a lower and upper bound and truncate the function. 0 outside of this.)

For every affiliated operator T, we have $T = V|T|$ (any of the spectral projections are in the vN algebra)

Consider $E_B(\xi^*\xi) \in L^1B$.

Define $\xi_0 = \xi E_B(\xi^*\xi)^{-1/2}$. Perform functional calculus, take projection $\xi_{(\epsilon,\infty)}(E_B(\xi^*\xi))$. Take inverse and square root, we get the above (compression).

$$\begin{aligned} \xi_0^*\xi_0 &= E_B(\xi^*\xi)^{-1/2}\xi^*\xi E_B(\xi^*\xi)^{-1/2} \qquad \text{(Apply } E_B\text{)} \\ E_B(\xi_0^*\xi_0) &= E_B(E_B(\xi^*\xi)^{-1/2}\xi^*\xi E_B(\xi^*\xi)^{-1/2}) \\ &= E_B(\xi^*\xi)^{-1/2}E_B(\xi^*\xi)E_B(\xi^*\xi)^{-1/2} \\ &= q \in P(B) \end{aligned}$$

Now we are finally ready to define our map.

Define $\Psi : q_0Aq_0 \to L^1(qBq)$ defined by $\Psi(x) = E_B(\xi_0^*x\xi_0)$. (This is a normal, faithful, completely positive map).

In fact, $\Psi(q_0Aq_0) \subset qBq$. [Note: $\Psi(q_0) = E_B(\xi_0^*\xi_0) = p$]

If we show $x\xi_0 = \xi_0\Psi(x)$, then it proves both homorphism and embedding)

$q_0Aq_0 \subset \overline{\xi B}, x\xi_0 = \xi_0 y$

$$\begin{aligned} \Psi(x) = E_B(\xi_0^*x\xi_0) &= E_B(\xi_0^*xi_0y) \\ &= E_B(\xi_0^*\xi_0)y \\ &= y \end{aligned}$$

Let $x_1, x_2 \in q_0 A q_0$. Then,

$$x_1 x_2 \xi_0 = x_1 \xi_0 \Psi(x_2)$$
$$\xi_0 \Psi(x_1 x_2) = \xi_0 \Psi(x_1) \Psi(x_2)$$

$x v_0 \xi_0 = v_0 \Psi(x)$ (can cancel) - Proves the last part.

$\square$

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24 Group - Measure Space vN algebras

4 April 2021 (Murray-von Neumann '36-'43)

Γ-countable discrete group.

$l^2\Gamma = \{\xi : P \to \mathbb{C}, \sum_{g\in P} |\xi(g)|^2 < \infty\}$

$\xi, \eta \in l^2\Gamma$, then

(i) $<\xi,\eta>= \sum_{g\in\Gamma} \xi(g)\overline{\eta(g)}$, (ii) $||\xi|| =<\xi,\xi>^{1/2}= (\sum_{g\in\Gamma} |\xi(g)|^2)^{1/2}$.

This is a **Hilbert space**.

Convolution $\xi, \eta \in l^2\Gamma$, $\xi * \eta : \Gamma \to \mathbb{C}$. Then,

$$\begin{aligned}
\xi * \eta(g) - &= \sum_{h\in\Gamma} \xi(h)\eta(h^{-1}g) \\
&= \sum_{h\in\Gamma} \xi(gh^{-1})\eta(h) \\
|\xi * \eta(g)| &= |\sum_{h\in\Gamma} \xi(h)\eta(h^{-1}g)| \\
&\leq \sum_{h\in\Gamma} |\xi(h)||\eta(h^{-1}g)| \\
&\leq (\sum_{h\in\Gamma} |\xi(h)|^2)^{1/2}(\sum_{h\in\Gamma} |\eta(h^{-1}g)|^2)^{1/2} \\
&= ||\xi||_2 \cdot ||\eta||_2 \\
\therefore ||\xi * \eta||_\infty &\leq ||\xi||_2 \cdot ||\eta||^2
\end{aligned}$$

For $\xi * \eta \in l^\infty\Gamma$.

Ex: $\xi, \eta \in l^1\Gamma \Rightarrow \xi * \eta \in L^1\Gamma$.

Properties:

$$\delta_g(h) = \begin{cases} 1 & g = h \\ 0 & \text{otherwise} \end{cases}$$

for $g \in \Gamma$.

Pick $\xi \in l^2\Gamma$. Then,

$$\begin{aligned}
\xi * \delta_g &= \rho g^{-1}\xi \\
\delta_g * \xi(k) &= \sum_l \delta_g(l)\xi(l^1 k) \\
&= \xi(g^{-1}k) \\
&= \lambda_g \xi(k) \quad \text{by left-regular representation,}
\end{aligned}$$

where

$$\begin{aligned}
&\lambda_g, \rho_g : l^2\Gamma \to l^2\Gamma \\
&\lambda_g\xi(h) = \xi(g^{-1}h) \\
&\rho_g\xi(h) = \xi(hg)
\end{aligned}$$

For $\xi \in l^2\Gamma \rightsquigarrow \overline{\xi} \in l^2\Gamma, \overline{\Gamma}(g) = \overline{\xi(g^{-1})}$

If $\xi, \eta, b \in l^2\Gamma$ and $(\xi * \eta) * b \in l^2\Gamma, \xi * (\eta * b) \in l^2\Gamma$, then $(\xi * \eta) * b = \xi * (\eta * b)$

Let $\xi \in l^2\Gamma$. Then, $D_\xi = \{\eta \in l^2\Gamma : \xi * \eta \in l^2\Gamma\} \supset \{\delta_g | g \in \Gamma\}$ (nonempty set, presrves $l^2\Gamma$).
Note that for $\eta_1, \eta_2 \in D_\xi \Rightarrow \xi * \eta_1, \xi * \eta_2 \in l^2\Gamma \Rightarrow \xi * \eta_1 + \xi * \eta_2 inl^2\Gamma \Rightarrow \xi * (\eta_1 * \eta_2) \in l^2\Gamma$.
Conclusion: $\overline{D_\xi} = l^2\Gamma$.
This allows us to define the convolution operator.

Left Convolution $L_\xi : D_\xi \to l^2\Gamma$ defined by $L_\xi(\eta) = \xi * \eta$ linear operator.
Similarly, let $D'_\xi = \{\xi \in l^2\Gamma : \eta * \xi \in l^2\Gamma\}$.

Right Convolution $R_\xi : D'_\xi \to l^2\Gamma$ defined by $R_\xi(\eta) = \eta * \xi$.
ξ is called a left convolution (resp. right convolution) iff $D_\xi = l^2\Gamma$ (resp. $D'_\xi = l^2\Gamma$)
Example 117. $\forall \xi \in \text{span}\,\{\delta_g | g \in \Gamma\}$ is a left (or right) convolution (There are many).
Theorem 118. $\forall \xi \in l^2\Gamma, L_\xi, R_\xi$ have closed graphs in $l^2\Gamma \oplus l^2\Gamma \Rightarrow (\zeta, L_\xi(\zeta))$.

(Will use Closed Graph Theorem).

Proof. If $(\eta_n, L_\xi(\eta_n)) \to (\eta, \zeta) \Rightarrow \zeta = L_\xi(\eta)$ is a sequence.
(With $||||_2$ norm), $\eta_n \to \eta$ and $||L_\xi(\eta_n) \to \zeta$. (Show $||\xi * \eta_n - \zeta||_2 \to 0$).
Fix $g \in \Gamma$, then

$$\begin{aligned}
|\zeta(g) - L_\xi(\eta)g| &= |\zeta(g) = xi * \eta(g)| \\
&= \lim_{n\to\infty} |\xi * \eta_n(g) - \xi * \eta(g)|| \\
&= \lim_{n\infty} |\xi * (\eta_n - \eta)(q)| \\
&\leq \lim_{n\to\infty} ||\xi||_2 ||\eta_n - \eta||_2 \\
&= 0
\end{aligned}$$

$\xi \in l^2\Gamma$ left convolution $\Rightarrow L_\xi \in B(l^2\Gamma)$ (by Closed Graph Theorem). □

Definition 24.1. $L\Gamma = \{\xi \in l^2\Gamma : \xi \text{ left convolution }\}$.
$R\Gamma = \{\xi \in l^2\Gamma : \xi \text{ right convolution }\}$.

Since every bounded operator has an adjoint and ξis left convolution, so is $\bar{\xi} \Rightarrow (L\xi)^* = L\bar{\xi}$.
Similarly, $(R\xi)^* = R\bar{\xi}$.
Then,

$$\begin{aligned}
L_\xi \circ L_\eta(\zeta) &= L_\xi(\eta * \zeta) \\
&= \xi * (\eta * \zeta) \\
&= (\xi * \eta) * \zeta \\
&= L_{\xi * \eta}
\end{aligned}$$

Similarly,

$$\begin{aligned}
L_\xi \circ R_\eta(\zeta) &= L_{\xi(b * \eta)} \\
&= \xi * (b * \eta) \\
&= (\xi * \zeta) * \zeta \\
&= R_\eta \circ L_\xi
\end{aligned}$$

Hence, $L\Gamma, R\Gamma$ are $*$-subalgebras (we will show that these are, in fact, vN algebras).
$L\Gamma \subset R\Gamma'$.

Definition 24.2. If Γ is a countable discrete, then $L\Gamma, R\Gamma$ are vN algebra, then
(a)
$$L\Gamma = R\Gamma' = \rho(\Gamma)' \Rightarrow L\Gamma = R\Gamma' = \lambda(\Gamma)''$$
(via left regular representation)
(b)
$$R\Gamma = L\Gamma' = \lambda(\Gamma)'$$

Proof. $L\Gamma \subset R\Gamma' \subset \rho(\Gamma)'$
If we can show that $\rho(\Gamma)' \leq L\Gamma$, then we are done.
Let $T \in \rho(\Gamma)'$.
Let $\xi = T\delta_1$ (where T is bounded operator). Show that $T = L_\xi$.
Take $g \in \Gamma$ and consider $L_\xi(\delta_g)$.

$$\begin{aligned} L_\xi(\delta_g) &= \xi * \delta_g \\ &= \rho_{g^{-1}}(\xi) \\ &= \rho_{g^{-1}}(\Gamma(\delta_1) \\ &= \rho_{g^{-1}} \circ T(\delta_1) \\ &= T \circ \rho_{g^{-1}} \\ &= T(\delta_g) \end{aligned}$$

$\therefore, T = L_\xi$ □

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6 April 2022 F countable discrete group.

$$\begin{aligned}
L(\Gamma) &= \{L\xi | \xi \in l^2\Gamma \text{ left convolutions}\} \\
&= \{\lambda_g | g \in \Gamma\}'' \subset B(l^2\Gamma) \\
&= \overline{\mathbb{C}[\Gamma]}^{\text{WOT}} = \overline{\mathbb{C}[\Gamma]}^{\text{SOT}} = \overline{\mathbb{C}[\Gamma]}''
\end{aligned}$$

Note that $\forall x \in L\Gamma$ admits a Fourier expansion.

Theorem 119. $\exists$ normal, faithful, trivial state $\tau : L(\Gamma) \to \mathbb{C}$ (trace). Then, $\tau(x) =< x\delta_1, \delta_1 > \ \forall x \in L(\Gamma)$.

Note that $x = \sum_{g\in\Gamma} x_g u_g$, where x_g is the Fourier expansion (in $\mathbb{C}$).

Proof. Normal is clear since $\tau(x^*x) - \Rightarrow x = 0$.

$$\begin{aligned}
x = L\xi \Rightarrow 0 &= \tau(x^*x) \\
&=< x^*x\delta_1, \delta_1 > \qquad (\text{since } \xi = x\delta_1) \\
&=< x\delta_1, x\delta_1 >= \sum_j |x_j|^2 \\
&=< \xi, \xi > \\
&= ||\xi||^2 \\
&\Rightarrow \xi = 0 \\
&\Rightarrow x = L\xi = 0
\end{aligned}$$

(We will use linearity)

$$\begin{aligned}
\tau(xy) &= \tau(yx) \\
\tau(u_g u_x) &= \tau(u_h u_g) \\
\tau(u_g u_h u) g^{-1}) &= \tau(u_h) \quad \forall g, h \\
\tau(u_g u_h u_g^{-1}) &= \tau(u_{ghg^{-1}}) \\
&=< u_{ghg^{-1}}, \delta_1 > \\
&=< \delta_{ghg^{-1}}, \delta_1 > \\
&= \delta_{ghg^{-1}} \\
&= \tau(u_h)
\end{aligned}$$

Theorem 120 (Murray-von Neumann '43)). $L\Gamma$ is a factor $\iff$ $\Gamma$is an ICC Group $\forall 1 \neq g \in \Gamma$, $|g^\Gamma| = |\{hgh^{-1} | h \in \Gamma\}| = \infty$

□

Proof. Let $g \in \Gamma, |g^\Gamma| < \infty$.
Let $1 \neq x = \sum_{k\in g^\Gamma} u_k \in Z(L\Gamma)$.
Fix $h \in \Gamma$,

$$\begin{aligned}
u_h x u_h^{-1} &= u_h(\sum_{k=g^\Gamma} u_k)u_{h^{-1}}) \\
&= \sum_{k\in g^\Gamma} u_{hkh^{-1}} \\
&= x \quad \ x \in Z(L\Gamma)
\end{aligned}$$

Let $z \in Z(L\Gamma)$ with $u_h x u_{h^{-1}} = x \ \ \forall x \in \Gamma$.

Pick Fourier decomposition $x = \sum_{g \in \Gamma x_g u_g}$. Then,

$$\begin{aligned}
x &= \sum_{g \in \Gamma x_g u_g} \\
&= u_h (\sum_g x_g u_g) u_{h^{-1}} \\
&= \sum_{g \in \Gamma} x_g u_{hgh^{-1}} \\
&= \sum_{g \in \Gamma} x_{hgh} u_g \\
x_g &= x_{h^{-1}gh}
\end{aligned}$$

(Norm 2 convergence, so can permute the group $\to$ Absolute convergence)

Note that $x_g = \sum \tau(x u_{g^{-1}})$

Since Norm 2, $\sum_{g \in \Gamma} |x_g|^2 = ||x||_2^2$ (norm-2 summable. Finite if either finite terms or if infinite, all zeros. Since we are in ICC, except for trivial conjugacy class, orbits are infinite, then the Fourier coeffienets are 0).

i.e. $x_g = 0 \ \forall g \neq 1$. □

$Z(L\Gamma) \subset L(FC\Gamma))$ (still open problem. Equality is not true since left side is Abelian, but right hand side doesn't have to be).

Consider $L(\mathbb{Z} \times \mathbb{F}_2) \cong L(\Gamma))$

25 Examples of ICC Groups

8 April 2022 "You catch a snake by hand of crazy person" "Pure spirit" "Everybody can cook" "Beautiful marriage of mathematics" (C^* algebra and Von Neumann Algebras)

25.1 Group-measure space von Neumann algebra

(Murray-von Neumann '36, '45)

Suppose (X_1, μ_1) and (X_2, μ_2) are two probability spaces.

Then, $(X_1, \mu_1) \cong_\theta (X_2, \mu_2)$ if there is a Borel isomorphism i.e.

$\exists \theta : X_1 \to X_2$ (Borel) measure preserving map such that

$E \subset X_2$ Borel such that $\mu(\theta^{-1}(E_2)) = \mu_2(E)$.

Assume $(X_1, \mu_1) = (X_2, \mu_2) = (X, \mu)$. (special case). Under composition,this becomes a group i.e. $\mathrm{Aut}(X, \mu)$ = group of (class modulo null sets) *pmp* automorphism of X.

$$\Theta \in \mathrm{Aut}(X, mu) \leftrightsquigarrow \mathrm{Aut}(L^\infty(X, \mu)), \tau_1) \ni f(x) \to f \circ \theta = f(\theta(x)).$$

$$\begin{gathered} \tau(f) = \tau(f \circ \theta) \\ \int_X f(x)\ dx = \int_X f \circ \theta(x)\ dx \\ \int_X \chi_E = \int_X \chi_E \circ \theta \end{gathered}$$

(Works since measure preserving. Hence, the induced map).

Action $\Gamma \rightsquigarrow (X, \mu)$ (*pmp* action)

$\mu(\gamma^{-1} E) = \mu(E)\ \forall E X$.

Then, $\Gamma \rightsquigarrow_\sigma (f(X)$, where $f : X \to \mathbb{C}$.

$\sigma_p(f)(x) = f(\gamma^{-1}(x)\ \forall x \in X, \gamma \in \Gamma$. (Respects L^p).

$f \in L^\infty(x) \rightsquigarrow \sigma_{\gamma](f) \in L^\infty(X), ||\sigma_\gamma(f)|| = ||f||}$

$f \in L^2(x) \rightsquigarrow \sigma_\gamma(f)(L^2(X)$

For $f, h \in L^2(X)$

$$\begin{aligned} < \sigma_\gamma(f), \sigma_\gamma(h) > &= \int_X \overline{\sigma_\gamma(h)} \sigma_\gamma(g)\ \ d\mu \\ &= < f, g > \end{aligned}$$

Preserves dot product.

11 April 2022 Γ countable discrete group.

$\Gamma \rightsquigarrow$ action on (X, μ) *pmp* such that $\mu(\gamma E) = \mu(E)\ \forall E \subset X$ measurable, $\forall \gamma \in \Gamma$.

This induces another action.

$\Gamma \rightsquigarrow_\sigma$ (action) on $F(X) = \{f : X \to \sigma | f \text{ function }\}$, where

$\gamma \mapsto \sigma_\gamma(f) = f(\gamma^{-1}(x))\ \forall x \in X, \forall f \in F(x)$.

$f \in L^2(X, \mu) \mapsto \sigma_\gamma(f) \in L^2(X)$, $||f||_2 = ||\sigma_\gamma|f|||_2\ \forall f \forall \gamma$.

$f \in L^\infty(X) \mapsto \sigma_\gamma(f) \in L^\infty(X), ||f||_\infty = ||\sigma_\gamma|f||_\infty\ \forall f \forall \infty$.

$\Gamma \rightsquigarrow_\sigma$ (action) $(L^\infty(X), \int \cdot d\mu = \tau)$

$Gamma \rightsquigarrow (L^2(X), <>), f \in L^\infty(X)$

$$\begin{aligned}
\tau(\sigma_\gamma(f)) &= \int_X \sigma_\gamma(f)(x)\ d\mu(x) \\
&= \int_X f(\gamma^{-1}x)\ d\mu(x) \\
&= \int_X f(x)\ d\mu(x) \\
&= \tau(f)
\end{aligned}$$

(τ-preserving action).

Definition 25.1 (Koopman Representation)**.**

$$< \sigma_g(f), \sigma_g(h) >=< f, g > \qquad f, g \in L^2(X),$$

where $\Gamma \rightsquigarrow$ (action) $U(L^2(X), <, >)$ is a unitary representation.

(will drop μ in our notation from now, but it is there).

Note that $\sigma : \Gamma \to \mathrm{Aut}(L^\infty(X), \int \cdot \mu)$ where $\gamma \mapsto \sigma_\gamma$.

Suppose $f \in L^\infty(X) \rightsquigarrow M_f \in B(L^2(X))$, where $\xi \in L^2(X)$, and it is defined as followed:

$$M_f(\xi)(x) = f(x)\xi(x), ||M_f||_{B(L^2(X))} = ||f||_\infty.$$

Let $\xi \in L^2$.

$$\begin{aligned}
\sigma_g(M_f)\sigma_{g^{-1}}(\xi) &= \sigma_g M_f(\xi \circ g^{-1}) \\
&= \sigma_g(f \cdot \xi \circ g^{-1}) \qquad \text{(Pointwise)} \\
&= \sigma_g(f)\xi \\
&= M_{\sigma_g}(f)
\end{aligned}$$

(Recovers the action. Covariant. Think about cross product or semi-direct product).

Now we build our von Neumann algebra (i.e. cross product algebra).

We now consider a larger Hilbert space, $H = L^2(X) \otimes \ell^2\Gamma$.

$L\infty(X), Gamma \rightsquigarrow$ (action) $B(H)$. (represent)

$L^\infty(X) \ni f \rightsquigarrow f \otimes 1 : L^2(X) \otimes \ell^2\Gamma \to L^2 X \otimes \ell^2\Gamma$. i.e.

$f \otimes 1(\xi \otimes \delta_g) = (f\xi) \otimes \delta_g$.

(Now take element of the group, Γ). $\Gamma \ni g \rightsquigarrow u_g(\xi \otimes \delta_g) = \sigma_g(\xi) \otimes \delta_{gh}$. (on the second component, left regular representation).

$\Rightarrow u_g = \sigma_g \otimes \lambda_g$.

(Do not want to use Mf anymore).

Note that two operators are the same if they agree on some generator (basis or something).

$$\begin{aligned}
u_g f u_g^{-1}(\xi \otimes \delta_h) &= u_g f(\sigma_g(\xi) \otimes \lambda_{g^{-1}}(\delta_h)) \\
&= u_g(f \cdot \sigma_{g^{-1}}(\xi)) \otimes \lambda_{g^{-1}}(\delta_h) \\
&= \sigma_g(f)\xi \otimes \delta_h \\
&= \sigma_g(f)(\xi \otimes \delta_h)
\end{aligned}$$

i.e. $u_g f u_g^{-1} = \sigma_g(f)$ (on this representation).

Implication

$$\begin{aligned}
f u_g \cot k u_h &= f \underbrace{u_g k u_g^{-1}}_{=\sigma_g(k)} u_g u_h \\
&= \sigma_g(k) u_{gh}
\end{aligned}$$

Definition 25.2 (Algebraic (or linear) span)**.** span $= \{\sum_{g \text{ finite}} a_g u_g | a_g \in L^\infty(X), g \in \Gamma\} \subset B(H)$.

Then, this is an algebra, but in particular, it is a **star algebra**.

$$\begin{aligned}
(a u_g)^* &= u_g^* a^* \\
&= u_{g^{-1}} a^* \cdot 1 \\
&= u_{g^{-1}} a^* u_g u_{g^{-1}} \\
&= \sigma_{g^{-1}}(a^*)
\end{aligned}$$

$[L^\infty X]$ $\quad L^\infty(X)$ alg Γ

Therefore, to get the von Neumann algebra, we take the closure as follows

Definition 25.3 (Group Measure Space von Neumann algebra)**.**

$$L^\infty \rtimes_\sigma \Gamma = \overline{L^\infty(X) \rtimes_{\text{alg}} \Gamma}^{\text{SOT}} = \overline{L^\infty(X) \rtimes_{\text{alg}} \Gamma}^{\text{WOT}} = \{\overline{L^\infty(X) \rtimes_{\text{alg}} \Gamma}^{\text{SOT}}\}'' \subset B(H)$$

of $\Gamma(X, \mu)$.

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13 April 2022 $\Gamma \rightsquigarrow$ (acts on) (X,μ) *pmp* standard probability space $([0,1],\lambda)$.

$\rightsquigarrow L^\infty(X)\rtimes_\sigma \Gamma = \overline{L^\infty(X)\rtimes_{\text{alg}}\Gamma}^{\text{SOT}}$

$H = L^2(X)\otimes \ell^2(\Gamma)$.

$$L^\infty(X)\ni f \rightsquigarrow f(\xi\times\delta_g) = (f\xi)\otimes\delta_g \qquad f\in B(H), \forall \xi\in L^2(X)$$
$$\Gamma\ni g\rightsquigarrow u_h(\xi\otimes\delta_g) = \sigma_R(\xi_)\otimes\delta_{h_g} \qquad u_g\in B(H)$$
$$[L^\infty X]\Gamma = L^\infty(X)\rtimes_{\text{alg}}\Gamma = \lim\{au_g | a\in L^\infty X, g\in\Gamma\}$$

Note that $\Gamma\rightsquigarrow_\sigma$ (acts on $L^\infty(X,\mu)$

$\Gamma\rightsquigarrow_\sigma L^2(X,\mu)$

Also note that

$$\begin{aligned} u_h f u_{h^{-1}} &= \sigma_h(f), \qquad f\in L^\infty X \\ \Rightarrow au_g bu_h &= uu_g b u_{g^{-1}} u_g u_h \\ &= au_g(b)u_g h \end{aligned}$$

$L^\infty X\subset L^\infty X\rtimes_\sigma \Gamma\subset L(\Gamma)$ (finite von Neumann algebra)

Let $A = L^\infty(X)$.

Definition 25.4 (Left Convolvers)**.** $A\rtimes_{\text{alg}}\subset \Gamma L(A,\Gamma) = \{\xi = \sum_g \xi_g\delta_g\in H | \exists \infty > c\geq 0$ such that $||\xi * \eta||_2\leq c||\eta||_2\}$ $\forall\eta\in H$.

(In particular, the SOT-closure of the algebra on the left, $A\rtimes_{\text{alg}} = L(A,\Gamma)$).

Similarly,

Definition 25.5 (Right Convolvers)**.** $A\rtimes_{\text{alg}}\subset R(A,\Gamma) = \{\xi = \sum_g \xi_g\delta_g\in H | \exists\infty > c\geq 0$ such that $||\eta * \xi||_2\leq c||\eta||_2$ $\forall\eta\in H\}$

$\xi = (\sum \xi_g\delta_g), \eta = (\sum_g \eta_g\delta_g)$ (cannot multiply pairwise since not scalars)

Therefore, $(\xi*\eta)_g = \sum_h \xi h\sigma_h(hg)$.

$x\in L^\infty(X)\rtimes_\sigma\Gamma\rightsquigarrow$ (Fourier expansion) $x = \sum_{g\in\Gamma} x_g u_g$,

where $x = L\xi = L_{x(1\otimes\delta}$ (representing vector in L^2).

$\tau : L^\infty(X)\rtimes_\sigma\Gamma\to\mathbb{C}$. Hence,

$\tau(x) = \int_X x(t)\ d\mu$.

$x_g = E_{L^\infty(X)}(xu_{g^{-1}})$, where $E(x) = a_e, E_{L\Gamma}(x) = \sum_{g\in\Gamma}\tau(x_g)u_g$ (expectations)

Proof. Fix $h\in\Gamma$. Then,

$$\begin{aligned} E_{L^\infty X}(xu_{h^{-1}}) &= E_{L^\infty X}(\sum_g x_g u_g u_{h^{-1}} \\ &= E_{L^\infty X}(\sum_g x_g u_g h^{-1}) \\ &= \sum_g E_{L^\infty X}(x_g u_{gh^{-1}} \\ &= \sum_g x_g E_{L^\infty X}(u_{gh^{-1}}) \end{aligned}$$

Then,

$$\begin{aligned}
E_{L^\infty X}(u_k) &= \delta_{k,l} \\
\tau \circ E_{L^\infty X} &= \tau \quad a \in L^\infty X \\
\tau(aE_{L^\infty X}(u_k)) &= \tau(E_{L^\infty X}(au_k)) \\
&= \tau(au_k) \\
&= \delta_{k,l}
\end{aligned}$$

If we let $a = E_{L^\infty X}(u_k^*)$. Take norms and we get precisely $\delta_{k,l}$

$\therefore$, $\sum_g x_g E_{L^\infty X}(u_{gh^{-1}}) = \sum_g x_g \delta_{gh^{-1},l} = x_h$

$\square$

Exercise Suppose you have two subgroups, $\Delta, \Sigma \leq \Gamma$ and $L(\Delta) \subset L(\Gamma) \supset L(\Sigma)$.

Show that $L(\Delta) \preceq L(\Sigma) \iff \exists h \in \Gamma[\Delta : h\Sigma h^{-1} \cap \Delta] < \infty$ (index is finite. Can be proved using analytic index and intertwining techniques).

Definition 25.6 (Ergodic)**.** $\Gamma \rightsquigarrow (X, \mu)$ free $\iff \forall g \neq 1 \mu(\{x \in X_0 | yx = x\}) = 0$ or $\mu(gX_0 \Delta X_0) = 0$.

$\Gamma \rightsquigarrow (X, \mu)$ ergodic $\iff \forall X_0 \subset X$ if $gX_0 = X_0$ $\forall g \in \Gamma, \mu(X_0) \in \{0, 1\}$ (for every Borel set)

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Definition 25.7. $\Gamma \rightsquigarrow$ (acts on) (X, μ) is **free** (ess). if $\mu(\{x \in X | x = gx\}) = 0 \; \forall g \neq 1$ (Borel measurable set)

(Keeps thing fixed)

Definition 25.8. $\Gamma \rightsquigarrow$ (acts on) $X, \mu)$ is **ergodic** iff whenever $Y \subset X, \mu(gY \Delta Y) = 0 \; \forall g \Rightarrow \mu(Y) = \{0, 1\}$ (Y is Γ- invariant up to measure 0 sets).

(g can move things within Y but stays within Y).

Lemma 121. The following are equivalent:

1. $\Gamma \rightsquigarrow$ (acts on) (X, μ) is ergodic.
2. If $\Gamma \rightsquigarrow$ (acts on) $L^\infty(X, \mu), L^2(X, \mu)$ then whenever $f \in L^\infty(X)$ such that $\sigma_g(f) = f \; \forall g \in \Gamma \Rightarrow f \in \lambda 1, \lambda \in \mathbb{C}$.
3. If $\Gamma \rightsquigarrow$ (acts on) $M(X)$ then whenever $\phi \in M(X)$ such that $\sigma_g(f) = f \; \forall g \in \Gamma \Rightarrow f = \lambda 1, \lambda \in \mathbb{C}$.

Note that $\mu(gY \delta Y) = 0 \iff \sigma_g(1_Y) = 1_Y$ since $1_Y(g^{-1}\lambda) = 1_Y(x)$. ($1_Y$ is the characteristic function. $gY = Y$ almost everywhere so symmetric difference is 0 ae).

$\Rightarrow 1_Y = 1$.

Proof. $(1 \Rightarrow 3)$

Suppose $f \in M(X) = \{f\text{''} X \to \mathbb{C} \text{ measurable}\}$

$$\begin{aligned} \sigma_g(f) &= f \quad \forall g \in X \\ \sigma_g(f)(x) &= f(x) \qquad \text{a e } x \\ f(g^{-1}x) &= f(x) \qquad \text{a e } x \end{aligned}$$

(Now consider level sets) $E_r = \{x \in X | f(x) > r\}$

E_r is Γ-invariant due to last line of the equation above) $\Rightarrow \mu(E_r) \in \{0, 1\}$.

Take $\alpha = \sup\{r | \mu(E_r) = 0\}$ (As you increase r, the set gets smaller and smaller).

What you can prove, then $r_1 < \alpha < r_2, \mu(E_{r_1}) = 0$ while $\mu(E_{r_2}) = 1$. So, $f \cong \alpha$. □

Lemma 122. $g \in \mathrm{Aut}(X, \mu) \rightsquigarrow \sigma_g \in \mathrm{Aut}(L^\infty(X, \mu))$. Then TFAE:

1. g is free.
2. $\forall A \subset X, \mu(A) > 0 \; \exists B \subset A$ such that $\mu(B) > 0$ such that $gB \cap B = \emptyset$.
3. If $a \in L^\infty(X)$ such that $a\sigma_j(x) = xa \; \forall x \in L^{infty}(X, \mu) \to a = 0$.

Proof. $(1 \Rightarrow 2)$ - HW

$(2 \Rightarrow 3$

$a\sigma_g(x) = xa$

If $a \neq 0, v|a|\sigma_g(x) = v|a|x \; \forall x$.

$f = \dfrac{f}{|f|}|f|$ (partial isometry). But have to be careful since it may have 0. To avoid this, we will multiply by characteristic function, then the support will take care of things). So now we have

$f = \dfrac{f}{|f|}\chi_{\text{supp}}$

$p\sigma_g(x) = px \; \forall x$.

$p = 1_Y, m(Y) > 0$.

$1_Y(y) = x(g^{-1}y) = 1_Y(y)x(y)\forall y \in X$.

$x(g^{-1}y) = \chi(y)\ \forall y \in Y$.

By 2, we can find a B with $g^{-1}B \cap B = \emptyset$ (take y in this B) such that $B(g^{-1}y) = 1_B y$.

Hence $gB = B$. Therefore, contradiction, $a = 0$. □

Theorem 123 (M vN ' 36). 1. $\Gamma \rightsquigarrow$ (acts on) (X, μ) free $\iff$ $L^\infty(X) \rtimes_\sigma \Gamma$ MASA (maximal Abelian *-subalgebra)

$A' \cap (A \rtimes_\sigma \Gamma) = A$.

2. If $\Gamma \rightsquigarrow$ (acts on) (X, μ) is free, then $A \rtimes_\sigma \Gamma$is a factor $\iff$ $\Gamma \rightsquigarrow$ (acts on) (X, μ) is ergodic.

(By Zorn's Lemma, we can always find a maximal Abelian subalgebra. This theorem states this is exactly what is it).

Proof. $(\Rightarrow)$

Suppose $x \in A' \cap (A \rtimes \Gamma)$,

$x = \sigma_g x_g u_g, x_g \in A$.

Then $xa = ax\ \forall a \in A$. Then, we have

$xa = (\sum_g x_g u_g)a = \sum_g x_g u_g a||u_g^{-1}||_g = \sum(x_g\sigma_g(a))u_g$.

Also we have $ax = a(\sum_g x_g u_g)\sum_g(ax_g)u_g$.

$ax_g = x_g\sigma_g(a)\forall g$ and $\forall a \in A$. Since $\Gamma \rightsquigarrow$ (acts on) (X, μ) free, then By above Lemma, $x_g = 0\ \forall g \neq 1$. Then,

$x = \sum x_g u_g = x \in A$. □

Proof. $(\Leftarrow)$

By Lemma 2, $ax = \sigma_g(x)a\ \forall x \in A$.

$\Rightarrow ax = u_g a u_g^{-1} a$.

$\Rightarrow ug^{-1}ax = x(u_{g^{-1}})a\ \forall x \in A$. The, it is in the commutant ie.e $u_{g^{-1}}a \in A' \cap (A \rtimes \Gamma) = A$.

□

$z \in Z(A \rtimes \Gamma) \subset A' \cap (A \rtimes E) = A$.

$A \ni z$. (z is in center).

We have $zu_g = u_g z \Rightarrow z - u_g z u_{g^{-1}} = \sigma_g(z)\ \forall g \Rightarrow z \in \mathbb{C}1$. So if you assume ergodic, then it will be a scalar.

Proof. $\sigma_g(a) = a\ \forall g$

$u_g a u_{g^{-1}} = a \Rightarrow u_g a = a u_g$.

Then $bu_g a = bau_g = a(bu_g) = a \in Z(A \rtimes \Gamma) = 1\mathbb{C}$. □

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26 Amenable Groups

18 April 2022 Suppose Γ is a discrete, countable group (doesn't have to be).

Definition 26.1 (Amenable)**.** Γ is amenable $\iff$ $\exists 0 \neq \phi \in \ell^\infty(\Gamma)^*$ left invariant state (i.e. ϕ positive $\phi \geq 0$, unital $\phi(1) = 0$, linear functional and $\phi(tf) = f)$ $\forall f \in \ell^\infty\Gamma,\ \forall t \in \Gamma)$

$tf(x) = \phi(f^{-1}x), \forall f, x \in \Gamma.$

Example 124. $\exists \mu : 2^\Gamma \to [0,1]$ such that

$\mu(A \sqcup B) = \mu(A) + \mu(B), AcapB = \emptyset$

$\mu(\Gamma) = 1$

$\mu(gA) = \mu(A)$

Example 125. Every finite group is amenable.

Theorem 126. The following groups are amenable.

1) $\forall$ finite groups, $\mu : 2^\Gamma \to [0,1]$ given by $\mu(A) = \dfrac{|A|}{|\Gamma|}$

2) $\forall$ abelian group (consequence of Kakutani FPT - HW)

3) Class of Amenable groups is closed under taking subgroups, extensions, quotients, and inductive limits

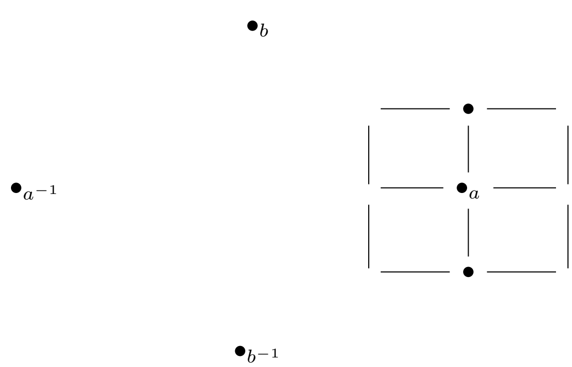
4) (Asymptotic Analysis) All groups of sub-exponential growth i.e. $\Gamma =< s >$ where $(s = s^{-1}), |s| < \infty$ (finitely generated group).

Can define a notion of growth using the Cayley graph. Can define distance between two words i.e. $(s,t) = \ell(t^{-1}s)$ (length). Then count how many elements you have. Now this is is a metric space.

Then, $B(e,g) = \{g \in \Gamma | d(e,g) \leq r\}$ where e is the identity. Finally, $|B(e,r)| \leq$ exponential, then we have sub-exponential growth.

Theorem 127. $\mathbb{F}_2$ is not amenable.

Proof.



Disjoint Union

$\mathbb{F}_2 =< a, b >$

$A^+ = \{w \in \mathbb{F}_2 | w = a...\}$

$A^- = \{w \in \mathbb{F}_2 | w = a^{-1}...\}$

$B^+ = \{w \in \mathbb{F}_2 | w = b...\}$

$B^- = \{w \in \mathbb{F}_2 | w = b^{-1}...\}$

$\mathbb{F}_2 = A^+ \sqcup A^{-1} \sqcup B^+ \sqcup B^{-1} \sqcup \{e\}$

$\mathbb{F}_2 = A^+ \sqcup aA$ and $\mathbb{F}_2 = B^+ \sqcup bB^{-1}$ (Paradoxical Decomposition)

Now assume by contradiction $\exists \phi(\ell^\infty(\mathbb{F}_2)^*)$ left-invariant state. Then,

$$\begin{aligned}
\phi(1) &= \phi(1_{A^+} + 1_{A^-} + 1_{B^+} + 1_{B^-} + 1_e)_{S_1} \\
&= \phi(1_{A^+}) + \phi(1_{A^-}) + \phi(1_{B^+}) + \phi(1_{B^-}) + \phi(1_e)_{S_1}) \\
&= \phi(1_{A^+}) + \phi(a1_{A^-}) + \phi(1_{B^+}) + \phi(b1_{B^{-1}}) + \phi(1_e) \\
&= \phi(1_{A^+} + 1_{aA^-}) + \phi(1_{B^+} + 1_{bB^-}) + \phi(1_e) \\
&\geq \phi(1) + \phi(1) \\
\phi(1) &\geq \phi(1) + \phi(1) \qquad \text{for } 0 \leq \phi(1) \leq 0
\end{aligned}$$

Contradiction □

(Thompson Group - famous group. Amenable or not? (3 each so far)) - so close to border "It is a joke. I am not sure if it is a funny joke." - Conferences/Voting on Amenable Groups.

Theorem 128. TFAE:

1) Γ is amenable

2) $\exists\ \mu \in P(\Gamma) \subset \ell^1\Gamma$ such that $||\mu_i - t\mu_i||_1 \to 0$.

3) The left regular representation $\lambda : \Gamma \to U(\ell^2\Gamma)$ has almost invariant vectors i.e. $\exists \xi \in \ell^2\Gamma, ||\xi||_2 = 1$ such that $||\lambda_g(\xi_i) - \xi_i||_2 \to 0\ \forall g$.

4) $\exists (F_i) \subset \Gamma$ (Folner nets) $\exists F_i \subset \Gamma$ finite such that $\dfrac{F_i \Delta t f_i}{|F_i|} \to 0\ \forall t \in \Gamma$

Proof. $(4 \Rightarrow 2)$

Take $\mu_i = \dfrac{1}{|F_i|} \cdot 1_{F_i}$. Then,

$||\mu_i - t\mu_i||_1 = \sum |\mu(s) - \mu(t^{-1}s) = \sum_g |\dfrac{1}{|F_i|} \cdot 1_{F_i}(t^{-1}s)| = \dfrac{F_i \Delta t f_i}{|F_i|} \to 0$ □

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20 April 2022 Prob $(\Gamma) = \{\mu : \Gamma \to \mathbb{C} | \mu(g) > 0, \sum_{g\in\Gamma} \mu(g0 = 1, ||\mu||_1 = 1\}$

Theorem 129. TFAE

1. Γ is amenable
2. $\exists(\mu_i) \in \text{Prob}(\Gamma) \subset (\ell^1\Gamma)$ such that $||\mu_i - g\mu_i||_1 \to 0\ \forall g \in \Gamma$.
3. The left regular representation $\lambda : \Gamma \to U(\ell^2\Gamma)$ as almost invarint vectors $(1_\Gamma \preceq \lambda_\Gamma)$

$\exists\ (\xi_i)_i \subset \ell^2\Gamma, ||\xi_i||_2 = 1, ||\lambda_g(\xi_i) - \xi_i||_2 \to 0 \quad \forall g \in \Gamma.$

4. $\exists$ Folner net $(F_i)_i \subset \Gamma$,

$\dfrac{F_i \delta g F_i}{|F_i|} \to 0\ \forall g \in \Gamma$

$\forall \epsilon > 0, \forall E \subset \Gamma\ \exists F \subset\subset \Gamma$ such that

$$\frac{|F\delta sF}{|F|} < M\epsilon\ \forall s \subset E$$

Last time, we proved (4) $\Rightarrow$ (2)

Proof. (4) $\Rightarrow$ (3)

Let $\xi_i = \dfrac{1}{|F_i|^{1/2}} 1_{F_i} \in \ell^2\Gamma, ||\xi_i||_2 = 1$

$||\xi\lambda_g\xi_i||_2^2 = ||\dfrac{1}{|F_i|^2} 1_{F_i} - \dfrac{1}{|F_i|^{1/2}g} 1_{F_i}||_2^2$. Then,

$$\begin{aligned}\sum_t \frac{1}{|F_i|} |1_{F_i}(t) - g1_{F_i}(t)|^2 &= \sum_t \frac{1}{|F_i|} |1_{F_i}(t) - 1_{F_i}(g^{-1}t)|^2 \\ &= \frac{1}{|F_i|} |F_i \Delta g F_i| \\ &= \frac{|F_i \Delta g f_i|}{|F_i|} \to 0\end{aligned}$$

□

Proof. (2) $\Rightarrow$ (3) If $\exists \mu_i \in \text{Prob}(\Gamma)$

$$\mu_i - gu_i||_1 \to 0\ \forall g$$

$\xi_i : \Gamma rightarrow \mathbb{C}, \xi_i(t) =_\Delta \mu_i(t)^{1/2}, \quad \xi \in (\ell^2(\Gamma))_1$. Then,

$$\begin{aligned}||\xi_i - \lambda_g \xi_i||_2^2 &= \sum_t |\xi_i(t) - \xi_i(g^{-1}t)|^2 \\ &= \sum_t |\mu_i(t)^{1/2} - \mu_i(g^{-1}t)^{1/2}|^2 \\ &\leq \sum_t |\mu_i(t) - \mu_i(g^{-1}t)| \\ &= ||\mu_i - g\mu_i||_1 \to 0\end{aligned}$$

(We use the fact that $|a-b|^2 \leq |a^2 - b^2|$ since $|a-b||a0b| \leq |a-b||a+b|$)

((3) $\Rightarrow$ (2) Similarly) □

Proof. (2) $\Rightarrow$ (4) (Namioka's trace)

$\forall \epsilon > 0,\ \exists \mu \in$ Prob (Γ) such that $\sum_{g\in E} ||\mu - g\mu||_1 < \epsilon = \epsilon \cdot 1 = \epsilon||\mu||_1$.

Observe that $\forall \alpha, \beta \geq 0$, we have

$$|\alpha - \beta| = \int_0^\infty |1_{(r,\infty)}(\alpha) - 1_{(r,\infty)}(\beta)| \ \ dr$$

$$\Rightarrow \sum_{g \in E} \sum_t |\mu(t) - \mu(g^{-1}t)| < \epsilon \sum_t \mu(t)$$

$$\sum_{g \in E} \sum_{t \in \Gamma} \int_0^\infty |1_{(r,\infty)}\mu(t) - 1_{(r,\infty)} - 1_{(r,\infty)}(\mu(g^{-1}t)| \ dr < \epsilon \sum_t \int_0^\infty 1_{(r,\infty)}\mu(t)$$

$$\int_0^\infty (\sum_{g \in E} \sum_{t \in \Gamma} |1_{(r,\infty)}(\mu(t)) - 1_{(r,\infty))}(\mu(g^{-1}t)) \ dr < \int_0^\infty \epsilon \sum_t 1\mu(t)$$

$$\exists r > 0 \text{ s t } \sum_{g \in E} \sum_{t \in \Gamma} |1_{(r,\infty)}(\mu(t)) - 1_{(r,\infty))}(\mu(g^{-1}t)) \ dr < \epsilon \sum_t 1_{(r,\infty)}(\mu(t))$$

Now consider the set $F = \{t \in \Gamma | \mu(t) > r\}$ (Level Set - Finite, nonempty subset of Γ). Then, we have

$$\sum_g |F \Delta_g F| < \epsilon |F|$$

$$|F \Delta g F| < \epsilon |F|$$

$$\frac{F\delta_g F|}{|F|} < \epsilon \ \forall g \in E$$

□

Proof. (2) $\Rightarrow$ (1) Trivial since $||\mu_i - g\mu_i||_1 \to 0 \ \ \ell^1\Gamma \subset (\ell^\infty\Gamma)^*$

Then,

$\mu \rightsquigarrow \phi_m \in (\ell^2(\Gamma)^*$. Then,

$$\phi_M(F) = \int_\Gamma f \ dm$$

$$= \sum_{g \in \Gamma} f(g)\mu(g)$$

□

Proof. (1) $\Rightarrow$ (2) (Day's Trick)

Suppose $\phi : \ell^\infty(\Gamma) \to \mathbb{C}$ state left invariant i.e. $g\phi = \phi \ \forall g \in \Gamma$.

Claim $\exists \mu_i \in \text{Prob}(\Gamma)$ (net) such that in weak-$*$ topology, μ_i converges to ϕ (pointwise) as elements of $(\ell^\infty\Gamma)^*$.

Separation argument (via Hahn-Banach)

Suppose that this does not hold (by way of contradiction). Then, Prob $(\Gamma) \subset (\ell^\infty\Gamma)^*$ (convex subspace)

Note that after the closure these two are disjoint subspaces of $(\ell^\infty\Gamma)^*$. Then, by Hahn-Banach separation theorem, $\exists f \in \ell^\infty(\Gamma)$

$\exists r > 0, s < t \in \mathbb{R}$ such that Re $(\nu(f)) \le s < t \le$ Re $\phi(f)$ $\forall \nu \in$ Prob (Γ).

Now consider $f \in \ell^\infty(\Gamma)$ with $1/2(f + \overline{f})$ (real-valued) since Re $(\phi(f)) = \phi(\frac{f + \overline{f}}{2})$.

Then $||f||_\infty + \nu(f) \le ||f||_\infty + s < ||f||_\infty + t \le ||f||_\infty + \phi(f) \ \forall \nu$

Then,

$\nu(f + ||f||_\infty) \le s_0 < t_0 \le \phi(f + ||f||_\infty)$. Take supremum to get

$\sup_{\nu \in \text{Prob}}(\nu(f)) \le s - 0 < t \le \phi(f) \forall \nu \in \text{Prob}(\Gamma)$

Then, supremum is norm-infinity. So they are equal.

□

22 April 2022

1. Γ is amenable $\iff \exists \phi \in (\ell^\infty)^*$ state left-invariant i.e. $s\phi = \phi$.
2. $\exists \mu_i \in \text{Prob}(\Gamma) \in \ell^2\Gamma$ i.e. $\forall s \in \Gamma,\ \forall \epsilon > 0, \forall t \in \Gamma, ||\mu_i - s\mu_i|| < \epsilon$ for all $s < t$.

   (Day's Trick)

Proof. (1) $\Rightarrow$ (2)

$\phi : \ell^2\phi \to \mathbb{C}$ positive, unital function such that $s\phi = \phi\ \forall s$

Claim: $\exists \mu_i \in \text{Prob}(\Gamma)$ such that $\mu_i \xrightarrow{\text{weak}*} \phi$ in $(\ell^\infty\Gamma)^*$.

$\overline{\text{Prob}(\Gamma)}^{\text{Weak}*}, \phi \subset (\ell^\infty\Gamma)^*$

Assume by contradiction these are disjoint sets. By Hahn-Banach separation theorem, $\exists\ \phi \in \ell^\infty(\Gamma)$ such that for $s < t$,

$$\text{Re } \nu(f) \leq s < t \leq \phi(f) \forall \text{Re } \phi(f) \quad \forall \nu \in \overline{\text{Prob } \Gamma}^{\text{Weak}}$$

$$f \to \frac{f + \overline{f}}{2}$$

$$\nu(f) \leq < t < \phi(f)\ \forall \nu \in \text{Prob } (\Gamma)$$

$$f \geq 0$$

$$||f||_\infty = \sup_{\nu \in \text{Prob } \Gamma} \nu(f) \leq s < t < \phi(f) \leq ||f||_\infty$$

That is a contradiction. (Why is the last line equal? $g \in \Gamma$ and consider $\delta_g(f) = \sum_{t \in \Gamma} f(t)\delta_g(t) = f(g)$

Hence, the claim is true.

Fix $s \in \Gamma, f \in \ell^\infty\Gamma$ Now consider

$$\begin{aligned} s\mu_i - \mu_i(\phi) &= s\mu_i(f) - s\phi(f) + s\phi(f) - \phi(f) + \phi(f) - \mu_i(f) \\ &= s(\mu_i - \phi)(f) + (\phi + \mu_i)(f) \to 0 \end{aligned}$$

Hence, $s\mu_i - \mu_i \xrightarrow{\text{Weak}*} 0$.

$$\overline{\oplus_{s \in E}\{s\mu - \mu | \mu \in \text{Prob } (\Gamma)\}}^{\text{Weak}} \ni \{0\}$$

K"" $\overline{K}^{\text{Weak}*} \subset X$

The, $|| \oplus s\mu - \mu||_1 < \epsilon$. Then, we have

$\sum s \in E||||s\mu - \mu||_1 < \epsilon$ (if each is less than ϵ, then all of it is less). ∎

Theorem 130. TFAE:

1. Γ is amenable.
2. $\exists \phi : \Gamma \to \mathbb{C}$ positive definite finitely supported.

   $|\phi_i(g) - 1| \to 0$.
3. $C^*(\Gamma) \cong C^*_\lambda(\Gamma)$
4. $C^*\Gamma$ admits a $1-$ dimensional representation.
5. (Kesten's criteria) $\forall E \subset \Gamma$

   $||\frac{1}{|E|}\sum_{s \in E} \lambda_s||_\infty = 1$ such that $\lambda_s : \ell^2\Gamma \to \ell^2\Gamma$ defined by $(\lambda_s\xi)(h) = \xi(s^{-1}h) \quad h \in \Gamma$

Definition 26.2 (Positive Definite)**.** $\forall F$ finite $\subset \Gamma$, then,
$[\phi_i(s^{-1}t)] \in M_{|F|}(\mathbb{C}) \;\; s, t \in F$.
$< \phi_i(s^{-1}t), \vec{v}, \vec{v} > \leq 0 \;\; \vec{v} \in \mathbb{C}^{|F|}$.

Proof. (1) $\Rightarrow$ (2)
$\xi \in \ell^2\Gamma$ such that $||\xi||_2 = 1$ such that $||\lambda_s(\xi_i) - \xi_i||_2 \to 0$
Consider $\phi_i : \gamma \to \mathbb{C}$ given by $\phi(g) =< \lambda_g(\xi), \xi >$ such that
$< [\phi(s^{-1}t)\vec{v}, \vec{v} >= || \sum_{i=1}^{n} v_i \lambda_{s_i}(\xi)|| \geq 0$ for $F \subset \Gamma$.
$\phi_i(g) - 1|| = | < \lambda_g(\xi_), \xi_i > - < \xi_i, \xi_i > | = | < \lambda_g(\xi_i) - \xi_i, \xi_i > | \leq ||\lambda_(\xi_i) - \xi_i|| \to 0$ (follows from C-s)
(kind of like baby version of GNS construction) $\square$

25 April 2021 Recall the theorem

Theorem 131. 1. Γ is amenable.

2. $\forall E \subset \Gamma, E = E^{-1}$ such that $||\frac{1}{|E|}\sum_{s\in E}\lambda_s||_\infty = 1$. (Kesten)

Last time, we proved ($\Rightarrow$).

Proof. ($\Leftarrow$)

$LetS = \frac{1}{|E|}\sum_{s\in E} s\lambda_s$. Then, $S^* = \frac{1}{|E|}\sum_{s\in E}\lambda_{s^{-1}} = S$ (self-adjoint operator).

Also, $||S||_\infty = \sup_{||\xi||=1} =< S\xi, \xi >$ (HW)

Note that $||S||_\infty = 1$.

$\forall\ \epsilon > 0, \exists \xi \in (\ell^2\Gamma)_1$ such that$||\xi||_2 = 1$.

Also $1 - \epsilon \leq |< S\xi, \xi >|$

Then, $|\xi|(g) = |\xi(g), g \in \Gamma$.

$$
\begin{aligned}
1 - \epsilon &\leq |< S\xi, \xi >| \\
&= |\sum_{t\in\Gamma}(S\xi)(t) - \overline{\xi(t)} \\
&\leq \sum_{t\in\Gamma}|(S\xi)(t)||\xi(t)| \\
&=< |S\xi|, |\xi| > \\
&\leq \sum_{t\in\Gamma} S\xi|(t)|\xi(t)| \\
&=< S(|\xi|), |\xi|) \\
&= \sum_{s\in E}\lambda_s(|\xi|), |\xi| > \\
\frac{\sum_{s\in E}(1-\epsilon)}{|E|} = 1 - \epsilon &\leq \frac{1}{|E|}\sum_{s\in E} < \lambda|\xi|, |\xi| > \\
1 = \epsilon &\leq < \lambda_s|\xi|, |\xi| >
\end{aligned}
$$

For $\epsilon >$ small, all $< \lambda_s|\xi|, |\xi| >$ are arbitrarily close to 1 $\forall s \in E$.

Also notice that

$$
\begin{aligned}
||\lambda_s|\xi| - |\xi|||_2^2 &= ||\lambda_s(\xi)||_2^2 + |||\xi|||_2^2 - 2 < \lambda_s|\xi|, |\xi| > \\
&= 1 + 1 - 2 < \lambda_s|\xi|, \xi >
\end{aligned}
$$

By the thing we proved above, the right most term is arbitrarily close to 1, so everything goes to 0.

$\therefore$, $|\lambda_s|\xi| - |\xi||_2 < \epsilon$.

(Nets)

$\forall E = E^{-1} \subset\subset \Gamma,\ \forall\ \epsilon > 0,\ \exists|\xi| \in \ell^2\Gamma, |||\xi|||_2 = 1$ such that $||\lambda_s|\xi| - |\xi|||_2 < \epsilon\ \forall s \in E$.

$(E, \epsilon), E \subset\subset \Gamma, \epsilon > 0$.

Then, $(E, \epsilon) \leq (F, \delta) \iff E \subset F$ for $\delta < \epsilon$.

Finally, $\exists|\xi| = \xi_{(E,\epsilon)} \in \ell^2\Gamma\ ||\xi_{(E,\epsilon)}||_2 = 1$.

Hence, conclusion follows. $\square$

27 Von Neumann Algebras

(M, τ) von Neumann algebra, M^{op} - opposite von Neumann algebras.

Definition 27.1. M, N are von Neumann algebras.

A H-Hilbert space is called $M - N$ bimodule if there exists two representations if comes equipped with two representations:

$$\begin{aligned} \pi : M &\to B(H) \\ \rho : N^{\text{opp}} &\to B(H) \qquad \text{such that } \pi \text{ commutes with } \rho \\ x\xi y &= \pi(x)\rho(y)\xi, \end{aligned}$$

where $x \in M, y \in N, \xi \in H$.

Consider the bimodules ${}_M H_N$

$\pi_H : M \otimes_{\text{alg}} N^{\text{op}} \to B({}_M H_N)$ unital *-representation.

Example 132. If (von neumann algebra finite, can form the L^2 space) then, ${}_M L^2(M)_M$ is the **trivial bimodule** given by the action

$x\xi y = xJy^* J\xi.$

Example 133. ${}_M L^2(M) \overline{\otimes} L^2(M)_M$ given by

$x(\xi_1 \otimes \xi_2)_y = (x\xi_1) \otimes (\xi_2 y) \quad x, y \in M.$

$1_\Gamma \to L^2(M).$

$\lambda_\Gamma \to L^2 2(M) \otimes L^2(M).$

Example 134. $\theta \in \text{Aut}(M)\tau \circ \theta = \tau$ given by

$L^2_\theta(M) = L^2(M) \ni \xi$ such that

$x\xi y = x\xi\theta(y).$

Definition 27.2. Two correspondences/bimodules are isomorphic i.e. ${}_M H_N \cong_M K_N$ if

$\exists U : H \to K$ unitary such that

$U(x\xi y) = xU(\xi)y, x, y \in M, \xi \in H.$

Definition 27.3. ${}_M H_N \subset_{\text{weakly}}$ (weakly contained in) ${}_M K_n$ given by

$||\pi_H(t)||_\infty \leq ||\pi_K(t)||_\infty \ \forall t \in M \otimes_{\text{alg}} N^{\text{opP}}$ (pointwise)

27 April 2022 Let $\pi : \Gamma \to U(H_\pi)$ unitary representation.

$M = L(\Gamma) = \{u_g, g \in \Gamma\}$, where Γ is countable, discrete.

Consider the Hilbert space $K_\pi = H_\pi \bar{\otimes} \ell^2(\Gamma)$. Let $\xi \in H_\pi$.

Define left action as follows:

$u_s(\xi \otimes \delta_t) = \pi_s(\xi) \otimes \delta_{st}$.

Similarly the right action is defined as:

$(\xi \otimes \delta_t)u_s = \xi \otimes \delta_{ts}$.

The right action extends to all $x \in M$.

How about left action?

Lemma 135 (Fall absorption lemma). Let

$$\begin{aligned} K_\pi \to K_\pi \quad & \text{unitary} \\ H_\pi \otimes \ell^2\Gamma \to H_\pi \otimes \ell^2\Gamma & \\ U(\xi \otimes \delta_t) = \pi_t(\xi) \otimes \delta_t \quad & t \in \Gamma \end{aligned}$$

unitary, then,

$U(11 \otimes \lambda)U^* = \pi \otimes \lambda$, where λ is the left regular representation.

Proof.

$$\begin{aligned} U(1 \otimes \lambda)U^*(\xi \otimes \delta_s) &= \pi \otimes \lambda(\xi \otimes \delta_s) \quad \forall s \in H_\pi, s \in \Gamma \\ U(1 \otimes \lambda)_t &= (\pi_{s^{-1}}(\xi) \otimes \delta_s) \\ U(\pi_{s^{-1}}(\xi) \otimes \delta_{ts}) & \\ &= \pi_{ts}(\pi_{s^{-1}}(\xi) \times \delta_{ts} \\ &= \pi_t \otimes \delta_{ts} \end{aligned}$$

Hence, we have a ${}_M K_{\pi\ M}$ bimodule. □

Theorem 136. 1. K_π is $M - M$ bimodule.

2. ${}_M K_{1_{\Gamma M}} =_M L^2(M)_M$

3. ${}_M K_{\lambda_{\Gamma M}} \equiv_M L^2(M) \otimes L^2(M)_M$

4. $\pi_1 \subset_{\text{weak}} \pi_2 \to K_{\pi_1} \subset_{\text{weak}} K_{\pi_2}$

Proof. Proof left as an exercise/HW to the reader. □

27.1 Preliminaries

$M \subset N$ v N algebra.

Definition 27.4. $\phi \in N^*$ is M-central if $\phi(xTx) = \phi(Tx)\ \forall T \in N, x \in M$.

$x \in M, \bar{x} = (x^{\text{op}})^* \in M^{\text{op}}$.

Then, consider $M \otimes_{\text{alg}} M^{\text{op}} \in B(L^2M \otimes L^2M)$.

$|| \cdot ||_{\min}$ operational norm on $M \otimes_{\text{alg}} M^{\text{op}}$ induced by $B(L^2M \bar{\otimes} L^2(M))$

This is the **minimal tensor norm**.

Definition 27.5 (p-Scheten class)**.** For every $p \geq 1$,

$$S_p(H) = \{T \in B(H)l, Tr(|T|^p < \infty\}$$

$S_p(H)$ is a Banach space and $||T||_p = Tr(|T|^p)^{1/p}$.

When $p = 1$, this is trace class. When $p = 2$, this is Hilbert-Schmidt class.

(M, τ) von Neumann algebra.

$$U : HS(L^2 M) \to L^2(M)\bar{\otimes} L^2(M)$$
$$\xi \otimes \eta \mapsto \xi \otimes \tau_\eta \quad M - \text{bimodule isomorphism}$$

Lemma 137. Suppose A is a C^* algebra and $u \in U(A)$ unitary of A and $w \in A^*$ state. Then,

$$\max\{||\omega - \omega(u)||, ||\omega - \omega(\cdot u^*)||\} \leq \sqrt{2|1 - \omega(u)}$$

(Note, direct proof is difficult, but can be eased through GNS construction - can turn functional state into vector state - via representation).

Proof. Let (π_w, H_w, ξ_w) be a GNS triple for the state ω on A, where $\omega(a) =< \pi + w(a)\xi_w, \xi_w >$ and $\pi_w : A \to B$. Then,

$$\begin{aligned}
||\omega - \omega(u\cdot)|| &= \sup_{||x|| \leq 1} ||\omega(x) - \omega(ux)| \\
&= \sup ||x|| \leq 1| < \pi_\omega(x)\xi_\omega - \pi_\omega(ux)\xi_\omega, \xi_\omega > \\
&\leq ||\pi_\omega(x)\xi_\omega, \xi_\omega 0 \pi_\omega(u^*)\xi_\omega > \\
&\leq_{||x|| \leq 1} ||\pi_\omega(x)\xi_\omega||, ||\xi_\omega - \pi_\omega(u^*)\xi_\omega|| \\
&\leq (||\xi_\omega - \pi_\omega(u^*)\xi_\omega||^2)^{1/2}(||\xi_u||^2 + |\pi(u^*)\xi_u - 2Re < \xi_\omega, \pi_\omega(u^*\xi_\omega)^{1/2} \qquad \text{C-S)} \\
&= (2 - 2Re\omega(u))^{1/2} \\
&\leq 2|1 - \omega(u)|^{1/2}
\end{aligned}$$

∎

29 April 2022

Theorem 138 (Powers-Stormer Inequality). Let H be a Hilbert space.
Suppose $S, T \in S_2(H), S, T \geq 0$.

$$\begin{aligned}||S-T||_2^2 &\leq ||S^2-T^2||_1 \\ &\leq ||S-T||_2||S+T||_2\end{aligned}$$

Proof. **Fact 1.** $A, B \in B(H)$ of finite rank, $A, B \leq 0$.
$AB = V|AB|$ (polar decomposition) $\rightsquigarrow V^*AB = V^*V|AB| = |AB|$

$$\begin{aligned}||AB||_1 &= Tr(|AB|) \\ &= Tr(V^*AB)| \\ &\leq ||V^*A||_2||B||2 \\ &\leq ||A||_2||B||_2\end{aligned}$$

Fact 2. $A, B \in B(H), A, B \geq 0$ and at lest one of finite rank, then $Tr(AB) \geq 0$.
$B = \sum_{i=1}^k \lambda_i < \cdot, \xi_i > \xi_i$ (old notation: $\sum_{i=1}^k \lambda_i \xi_i \bar{\otimes} \bar{\xi_i}$

$$\begin{aligned}B &= \sum_{i=1}^k \lambda_i < \cdot, \xi_i > \xi_i \\ AB &= A(\sum \lambda_i < \cdot, \xi_i > \xi_i > \\ &= \sum \lambda_i \xi_i \otimes A\xi_i \\ \Rightarrow Tr(AB) &= \sum \lambda_i < \xi_i, A\xi_i > \\ &\geq 0\end{aligned}$$

(Note that $Tr(\xi \bar{\otimes} \bar{\eta}) =< \xi, \eta >$.

Now WLOG, $S, T \geq 0$, have finite rank. Consider $S^2 - T^2$.

$$\begin{aligned}S^2 - T^2 &= \frac{1}{2}((S+T)(S-T) + (S-T)(S+T)) \\ ||S^2-T^2||_1 &= \frac{1}{2}Tr(|(S+T)(S-T) + (S-T)(S+T)|) \\ &= \frac{1}{2}Tr(V^*((S+T)(S-T) + (S-T)(S+T)) \\ &= \frac{1}{2}Tr(V^*((S+T)(S-T))) + Tr(V^*((S-T)(S+T))) \\ &\leq \frac{1}{2}||(S+T)||_2||S-T||_2 + ||S-T||_2||S+T||_2 \\ &\leq ||S-T||_2||S+T||_2\end{aligned}$$

(Follows from fact 1).
If $S-T$ is self-adjoint. Then consider the spectral projection:
$p = 1_{[0,\infty)}(S-T)$, $p^\perp = 1-p$. The,
$(S-T)p \geq 0$ and $(T-S)p^\perp \geq 0$.

Note that $p + p^{\perp} = 1$.

$$\begin{aligned}
||S-T||_2 &= Tr((S-T)(S-T)(p+p^{\perp})) \\
&= Tr((S-T)(S-T)p + (T-S)(T-S)p^{\perp}) \\
&\leq Tr((S+T)(S-T)p + (T+S)(T-S)p^{\perp}) \qquad \text{(use positivity from above)} \\
&= Tr((S^2-T^2)p + (T^2-S^2)p^{\perp}) \\
&= ||S^2-T^2||_1
\end{aligned}$$

(Follows from fact 2). □

28 Amenable Algebra

Definition 28.1. (M, τ) vN algebra is **amenable** iff $\exists \phi$ state on $B(L^2M)$ is $M-$central ($\phi(xu) = \phi(ux)$ $\forall u \in M, x \in B(L^2M)$.

Theorem 139 (Connes '76). (M, τ) vN algebra (with separable predual)

TFAE:

1. There exists a conditional expectation $E : B(L^2M) \to M$ (where $M \subset B(L^2M)$. (injectivity).

2. There exists a state ϕ on $B(L^2M)$ that is M-central where $\phi_M = \tau$.

3. There exists a net $\xi_n \in L^2M \otimes L^2M, |\xi_n||_2 = 1$ such that $\lim_{n\to\infty} ||x\xi_n - \xi_n x||_2 = 0$ $\forall x \in M$ and $\lim < x\xi_n, \xi_n >= \tau(x)$. (Invariant factors for Folner sets)

4. ${}_M L^2(M)_M \subset_{\text{weak}} {}_M L^2M \bar{\otimes} L^2M_M$.

5. $\forall a_1, a_2, ..., a_k, b_1, .., b_k \in M$,

$$\tau(\sum_{i=1}^{k} a_i b_i) \leq ||\sum_{i=1}^{k} a_i \otimes b_i^{\text{op}}||_{\min}$$

(minimum tensor norm)

6. M is hyperfinite ($\exists Q_n \subset Q_{n+1} \subset ... \subset M$, $*-$ finite, $\dim_{\mathbb{C}}(Q_n) < \infty$ and $\overline{\cup_n Q_n}^{\text{WOT}} = M$.

If $M = L(\Gamma)$ group vN algebra where Γ is countable discrete.

7. Γ is amenable (i.e. $L(\Gamma)$ is amenable iff Γ amenable. i.e. Group amenable iff von Neumann algebra amenable)

Proof. (1) $\Rightarrow$ (2)

$\phi = \tau \circ \bar{t}$ since $\phi(uxu^*) = \tau(E(uxu^*)) = \tau(uE(x)u^*) = \tau \circ E(x) = \phi(x)$.

$\tau \circ E(m))$. □

---

29 April 2022 (Make up class online)

2 May 2022 Recall,

1. $\exists B(L^2(M)) \to M$ conditional expectation.

6. $\exists Q_n \subset M, Q_n \subset Q_{n+1}\ \forall n, \dim_{\mathbb{C}}(Q_n) < \infty$ (some direct sum). These are like block matrices. So consider
$\overline{\cup_n Q_n}^{\text{SOT}} = M$.

(1) $\Rightarrow$ (6) (Very Hard. Skip).

Proof. (1) $\Leftarrow$ (6)

Consider $U(Q_n)$ finite compact group. Now take $(U(Q_n), \mu_n)$, which represents the Haar measure.

Let $T \in B(L^2M)$. Then,

$$\begin{aligned}
\Phi_n(T) &= \int_{U_n} uTu^*\ d\mu_n(u) \in B(L^2(M)) \qquad u \mapsto vu \\
\Phi(T) &= v(\int_{U_n} uTu^* d\mu_n(u))v^* \\
\Phi(T) &= v\Phi(T)v^*\ \forall v \in U_n v \\
\Rightarrow \Phi(T)v = v\Phi(T).
\end{aligned}$$

Therefore, $\Phi_n(T)$ commutes with v.

This is true for all $n \in \mathbb{N}$. Take $(\mathbb{N}, \omega)$

ω ultra-filter on $\mathbb{N}$ and $\Phi(T) = \lim_{n\to\omega} \Phi_n(T)$ is the ultra-limit. Then,

$$\begin{tikzcd}
\Phi B(L^2M) \arrow[r] \arrow[dr, dashed, "T\mapsto J\Phi(T)J"] & M' \cap B(L^2M) \arrow[d, "\text{ad } J"] \\
& M
\end{tikzcd}$$

□

Proposition. Γ is amenable $\iff$ $L(P)$ is amenable.

Corollary 139.1. Γ is ICC amenable $\Rightarrow L(P) \cong R$ (the hyperfinite)

$\mathbb{Z}S\mathbb{Z} = \oplus\mathbb{Z}^{(\mathbb{Z})} \rtimes \mathbb{Z}$

$\mathbb{Z}_2 S\mathbb{Z} = \otimes_{\mathbb{Z}}\mathbb{Z}_2 \rtimes \mathbb{Z}$

$\cup_n \tilde{S_n} = S_\infty$ (torsion free - no infinite terms). Tower of subgroups procedure.

Proof. ($\Leftarrow$)

M is amenable, then $\exists \phi : B(L^2M) \to \mathbb{C}$ a state, M-central $\phi|_M = \tau$.

In esssence, $B(L^2M)$ is like

$$\begin{aligned}
\ell^\infty(\Gamma) &\subset B(\ell^2\Gamma) \\
f &\rightsquigarrow M_f : \ell^2\Gamma \to \ell^2\Gamma,
\end{aligned}$$

where the mapping is given by $(M_f\xi)H_1 = f(h)\xi h$ and $||M_f||_\infty = ||f||_\infty$ and $M = \phi|_{\ell^\infty(\Gamma)}$ □

(Turning things into bimodule via lifting)

Proof. ($\Rightarrow$)

Γ- amenable $\exists(\xi_n) \subset \ell^2\Gamma, ||\xi_n||_2 = 1, ||\lambda_1(\xi_n) - \xi_n||_2 \to .$

$M = L(\Gamma)$.

$H(\lambda) =_M \ell^2\Gamma \otimes \ell^2\Gamma_M$ given by $u_g(\xi \otimes \delta_h) = \lambda_g(\xi) \otimes \delta_{gh}$ (left).

Similarly, $(\xi \otimes \delta_h)u_g = \xi \otimes \delta_{hg}$ (right action) to define the bimodule structure.

Let $b_n = x\xi_n \otimes \hat{1} \in H_\lambda$. Then

$$\begin{aligned} ||u_g b_n - b_n u_g||^2 &= ||u_(\xi_n \otimes 1) - (\xi_n \otimes 1)u_g||^2 \\ &= \lambda_g(\xi_n) \otimes \delta_g - \xi_n \otimes \delta_g||^2 \\ &= ||(\lambda_g(\xi_n) - \xi_n) \otimes \delta_g||^2 \\ &= ||\lambda_g(\xi_n) - \xi_n||_2 \to 0 \ \forall n \end{aligned}$$

Therefore, $(\xi_n)$is M-central (i.e. almost invariant, then almost central). $\square$

Can check that $< x, \xi_n, \xi_n >= \tau(x) \ \forall x \in M$ (tracial).

Definition 28.2. $\Phi(T) = \lim_{n\to\omega} < Tb_n, b_n >; \phi(u_sT) = \phi(Tu_s)$.

$$\begin{aligned} \phi(u_sT) &= \lim_n < u_sTb_n, b_n > \\ &= \lim_n < Tb_n, u_sb_n > \\ &= \lim_n < Tb_n, b_nu_s > \\ &= \lim_n < Tb_nu_s, b_n > \\ &= \lim_n < Tu_sb_n, b_n > \\ &= \phi(Tu_s) \end{aligned}$$

28.1 Take Home Exam 2

1. Show that the hyperfinite factor is unique (You can find this in the chapter 11 of the attached book).

2. Show that the free group factors $L(F_n)$ associated with the free group with $n \geq 2$ generators does not have property Gamma of Murray and von Neumann. Use this to deduce that $L(F_n)$ is not $*$-isomorphic to $L(F_n \times S_\infty)$, where S_∞ is the group of finite permutations of the natural numbers. (You can find the relevant definitions and the result done in chapter 15 of the book attached.

29 Appendix - Selected Exercises

[font=]**Show that $(B(H), ||\cdot||_\infty)$ is a complete space.**

1.

Definitions 1. **A sequence $\{x_n\}$ in a metric space (X,d) is Cauchy if $\forall\ \epsilon > 0,\ \exists N \in \mathbb{N}$ such that for** $m, n > N, d(x_m, x_n) = ||x_m - x_n|| < \epsilon$.

1b. Note that every Cauchy sequence in a metric space is bounded.

2. A metric space (X,d) is complete if every Cauchy sequence in X converges in X.

Solution *Proof.* First recall that $B(H) = \{T : H \to H|$ linear, bounded $\}$, where H is a Hilbert space. Note that $||T||_\infty = \sup_{||\xi|| \leq 1} ||T(\xi)||$.

Pick a Cauchy sequence T_n in T. Since it is Cauchy, by definition we have $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ such that for $m, n > N, ||T_n - T_m|| = \sup_{||\xi|| \leq 1||} ||T_n(\xi) - T_m(\xi)|| < \epsilon$.

We need to show there is some $T \in B(H)$ such that $\lim_{n\to\infty} ||T_n - T|| = 0$.

$$
\begin{aligned}
||T_n - T_m|| &= \sup_{||\xi|| \leq 1||} ||T_n(\xi) - T_m(\xi)|| \\
&< \epsilon \\
\Rightarrow ||T_n - T|| &= ||T_n - T_m + T_m - T|| \\
&\leq ||T_n - T_m|| + ||T - T_m|| \\
&= \underbrace{\sup_{||\xi|| \leq 1||} ||T_n(\xi) - T_m(\xi)||}_{\text{(Converges since Cauchy)}} + \sup_{||\xi|| \leq 1||} ||T(\xi) - T_m(\xi)||
\end{aligned}
$$

Since T is linear and bounded, by taking limits on both sides, we see that $T_n \to T \in B(H)$ as $m \to \infty$. $\square$

2. Fix $I \in M(A) \to A/I$ (This is a field since I is maximal).

Suppose $(A, ||\cdot||)$ is a Banach Algebra. $\bar{I} = I \leq A$ is a closed ideal $\Rightarrow (A/I, ||\cdot_I)$ is a Banach algebra, where

$$||\alpha + I||_I = \inf_{x \in \lambda} ||\alpha + x||_A \leq ||a||.$$

Solution We need to show that A is a normed space, is sub-multiplicative, and is complete.

(i) Show $||c \cdot x|| = |c| \cdot ||x|| \; \forall x \in A, c \in \mathbb{C}$.

Proof. Let $\alpha + I \in A/I$ and $c \in \mathbb{C}$. Then, we have

$$\begin{aligned}
||c \cdot (\alpha + I)|| &= |c\alpha + I|| \\
&= \inf_{x \in I} ||ca + cx||_A \\
&= \inf_{x \in I} ||c(a + x)||_A \\
&= |c| \cdot \inf_{x \in I} ||\alpha + x||_A \quad \text{(since } A \text{ is a Banach Algebra)} \\
&= |c| \cdot ||\alpha + I||
\end{aligned}$$

□

(ii) Show $||x + y|| \leq ||x|| + ||y||$.

Proof. Let $\alpha + I, \beta + I \in A/I$. Then,

$$\begin{aligned}
||\alpha + I + \beta + I|| &= ||(\alpha + \beta) + I|| \\
&= \inf_{x \in I} ||(\alpha + \beta) + x||_A \\
&= \inf_{x \in I} ||\alpha + x||_A + \inf_{x \in I} ||\beta + x||_A \text{ (since } A \text{ is a Banach Algebra)} \\
&\leq ||\alpha + I|| + ||\beta + I||
\end{aligned}$$

□

(iii) Show $||x|| = 0 \iff x = 0_A$.

Proof. Let $\alpha + I \in A/I$. Then, we have

$$\begin{aligned}
&||\alpha + I|| = 0 \\
\Rightarrow &\inf_{x \in I} ||\alpha + x||_A \leq ||\alpha|| = 0 \\
&\iff \alpha = 0_A \quad \text{(since } A \text{ is a Banach Algebra)} \\
&\iff \alpha + I = 0_{A/I}
\end{aligned}$$

□

(iv) Show sub-multiplicative i.e. $||xy|| \leq ||x|| \cdot ||y||$.

Proof. Let $\alpha + I, \beta + I \in A/I$. Then,

$$
\begin{aligned}
||(\alpha+I)(\beta+I)|| &= ||\alpha\beta+I|| \\
&= \inf_{x\in I} ||\alpha\beta+x||_A \\
&\leq ||\alpha\beta|| \\
&\leq ||\alpha||\cdot||\beta|| \text{ (since } A \text{ is a Banach Algebra)} \\
&= \inf_{x\in I} ||\alpha+x||_A \cdot \inf_{x\in I} ||\beta+x||_A \\
&\leq ||\alpha+I||\cdot||\beta+I||
\end{aligned}
$$

□

(v) Show that $(A, ||\cdot||)$ is complete, i.e. all $||\cdot||$-Cauchy sequences converge.

Proof. Suppose $(\alpha+I)_n$ is a Cauchy sequence in A/I i.e. $\forall\ \epsilon>0$, $\exists N\in\mathbb{N}$ such that for all $m,n>N, ||(\alpha+I)_n-(\alpha+I)_m||<\epsilon$. We need to show that this sequence converges *in* A/I. Note by definition,

$||(\alpha+I)_n-(\alpha+I)_m|| = \inf_{x\in I} ||(\alpha+x)_n-(\alpha+x)_m||_A$.

Since A is a Banach algebra, every Cauchy sequence of A converges in A. On the other, we assume that $||(\alpha+I)_n-(\alpha+I)_m||<\epsilon$ and by property (ii), the norm of the sums is bounded by their sum of the norms. Combining these two facts and taking the infimum on the right hand side, we have that $(A/I, ||\cdot||)$ is a complete space. □

3. Show that $\{f \in C(K) | f(K) = 0$ continuous$\}$ is a maximal ideal of $C(K) = \{f : K \to \mathbb{C}$ continuous$\}$, where K is a Hausdorff compact space.

Know 1. There exists a natural homomorphism:

$$\begin{aligned} K &\to \sigma(C(K)) \\ k &\mapsto \phi_k \end{aligned}$$

defined by $\phi_k(f) = f(k)\ \forall f \in C(K)$. This map is a homeomorphism.

2. A two-sided ideal I of a ring R is both a left ideal and a right ideal i.e. it is a subring, $rI \subset I$, and $Ir \subset I$ for all $r \in I$. (absorbs products).

Solution *Proof.* We first recall that M is a maximal ideal iff the quotient ring $C(K)/M$ is a field.

By the First Isomorphism Theorem for rings, $C(K)/\ker(\phi_k) \cong \mathbb{C}$.

In particular, the kernel of ϕ is the set of all continuous functions such that $\phi_k(f) = f(k) = 0\} = M$.

Since $\mathbb{C}$ is a field, $C(K)/\ker(\phi_k) = C(K)/M$ is a field $\Rightarrow M$ is a maximal ideal. □

Alternatively (Assume that this is not a maximal ideal, i.e. $\exists\ J$ ideal such that $I \subset J \subset C(K)$, where $I = \{f \in C(K) | f(K) = 0$ continuous $\}$.

Suppose $f_1 \in J$ such that $f_1 \notin I$. Hence, we have $1 = \dfrac{f_1(k) - f_1 + f_1}{f_1(k)}$ for some k. Since I is an ideal the first part of the sum goes to 0. Hence, we have $1 = \dfrac{f_1}{f_1(k)} \Rightarrow 1 \in J$ and so $J = C(K)$, the entire space. Therefore, $I = M$ is a maximal ideal).

4. (24 September 2021)

Show the following: (a) A **abelian, then** $x \leq y \Rightarrow x^2 \leq y^2$.

Solution (a)

$$\begin{aligned} x &\leq y \\ xx &\leq yx \\ \Rightarrow x^2 &\leq yx \\ &= xy \qquad \textbf{(since } A \textbf{ abelian)} \\ &\leq yy \qquad \textbf{(since } x \leq y) \\ &= y^2 \end{aligned}$$

$\therefore x \leq y \Rightarrow x^2 \leq y^2$.

(b)

5. (25 October 2021) Show that the finite rank operator $FR(H) \subset L^1(B(H))$ and moreover $FR(H) = L^1(B(H))$. (dense)

Hint $\mathbf{Tr}(\xi \otimes \bar{\eta}) =< \xi, \eta >$ **(finite linear combination). Show densy.**

Apply definition. Let $(b_i)_{i\in I}$ be an ONB. Then,

$$
\begin{aligned}
\mathbf{Tr}(\xi \otimes \bar{\eta}) &= \sum_{i=1}^{\infty} \xi \otimes \bar{\eta} < b_i, b_i > \\
&= \sum_{i\in I} << b_i, \eta > \xi, b_i > \\
&= \sum_{i\in I} < b_i, \eta >< \xi, b_i > \qquad \textbf{Prove this is the same as dot product} \\
&= \sum_{i\in I} < \xi_i, < \xi_i, \eta >, \xi_i > \\
&=< \xi_i, \sum_i \overline{< b_i, \eta >} b_i > \\
&=< \xi_i, \sum < \eta, b_i > b_i > \\
&=< \xi, \eta >, \qquad i \in I
\end{aligned}
$$

---

6. (27 October 2021) $L^2(B(H))$ **is a complete space with** $||\cdot||_2$. **(Any Hilbert space = Dual by Riesz representation theorem).**

7. (29 October 2021) (Lemma) $HS(H, K)$ form a closed subspace $L^2(B(H \oplus K))$