Algebraic Topology Notes

Nitesh Mathur

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Contents

1 Foreword

This was a group project conducted by the students for MATH 6400: Algebraic Topology in Fall 2021 under the kind guidance of Professor Ben Cooper.

The following color coding denotes the author of the notes on that particular day.

- Elise Askelsen
- Juan Felipe Ariza Mejia
- Kevin Del Real Ramos
- Quanqi Hu
- Steven Un
- Nitesh Mathur (or regular ink)

Thanks to Ryan (Justin) Bianconi for thoughtful discussions in class all semester.

2 Introduction

Monday, 23 August 2021

Definition 2.1 (Define a Circle S^1)**.**

$$S^1 = \{(x, y) \in \mathbb{R} : X^2 + y^2 = 1\} \subset \mathbb{R}^2.$$

S^1 has the subspace topology. Now, consider S^1 with base point $p := (1, 0) \in S^1$. For X a topological space, $q \in X$ a base point, there is a topological space:

Definition 2.2 ($\Omega_q(X)$)**.**

$$\Omega_q X := \{f : (S^1, p) \to (X, q) : f \text{ continuous}, f(p) = q\}.$$

. Compact-open topology subbasis is

$$\mathcal{S} := \{\cup(K, W) : K \subset S^1 \text{ compact}, W \subset X\text{open}\}.$$

Here,

$$\cup(K, W) = \{f \in \Omega_p X : f(K) \subset W\}.$$

- We call $\Omega_q(x)$ the **loop space** of X at q.
- A point $p \in \Omega_q X$ is a loop based at q.
- A path $H : \gamma \to \gamma'$ between two points is a **homotopy** of loops.

Recall **Fundamental Groups**. If $\Pi_0(X, x_0)$ is a set of path component (with some special one containing x_0), then,

$$\Pi_0(\Omega_q x, \bar{q}) = \Pi_q(X, q)$$

, where $\bar{q}$ = constant loop at q.

*Note, this can be used as an alternative definition of the Fundamental Group.

We can iterate the procedure as follows:

$$\Pi_{k+1}(X, q) := \Pi_k(\Omega_q X, \bar{q})$$

Example 1.

$$\begin{aligned} \Pi_2(X, q) &= \Pi_1(\Omega_q X, \bar{q}) \\ &= \Pi_0(\Omega \bar{q} \Omega q X, \bar{\bar{q}}) \end{aligned}$$

"loops of loops"

An element of $\Pi_2(X)$ is a homotopy class of paths in the space of double loops $\Omega^2(x) \iff$ it is a homotopy class of maps $S^2 \to X$. Here are some key points:

- $\Pi_3(x) = \{S^3 \to X\}$/homotopy (is a finitely generated Abelian group).
- $\Pi_k(X)$ is a group for $k \geq 1$.
- $\Pi_k(X)$ is an abelian group for $k > 1$.

However, it is "imossible to compute" Homotopy groups of X. In other words, homotopy groups are purely geometric invariants (no algebra), but **they aren't computable.**

3 Cohomology Overview

Alternatively **Cohomology** is a dual construction, which is much more computable.

Theorem 2 (Dold-Kohn). (Co)homology is linearization of homotopy groups.

Either way, it is a *dramatic simplicatiion*, but still powerful. Because it is computable, it contains higher dimensional information for $k > 1$.

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25 August 2021

Last Time

$$\Pi_n(X, p) = \{S^n \xrightarrow{f} X : f(0, 0, ..., 1) = p\}/f \sim g$$

if $\exists\ H : I \times S^n \to X$continuous such that$H(0, \theta) = f(\theta), H(1, \theta) = g(\theta)$

Example 3 ($X = S^2$). $\equiv \bar{p}$ constant loop at p by Upper Hemisphere Homotopy.

Recall that Π_1 is the fundamental group (non-abelian, can be computed).

With Seifert-Van-Kampen theorem, one can figure out the $\Pi_1(x)$.

If $X = U \cup V$, then $\Pi_1(X) = \Pi_1(U) *_{\Pi_1(U \cap V)} \Pi_2(V)$

For Π_n(abelian, high dimensional), we do not have such formula.

Example 4 ($\Pi_3(S^2) \cong \mathbb{Z}$). Consider the map $S^3 \xrightarrow{f} S^2$. This corresponds to $f = 1$.

Then, we have $S^3 = S^1 \times D^2 \cup S^1 \times D^2$ (each point on either solid torus lies on S^1 which intersects a $S^2 \hookrightarrow S^3$ with f (pt) $= q$. [So, $f^{-1}(g) = S^1$]). This is known as a **Hopf-Fibration**. Here is a cool video to help us visualize this.

https://www.youtube.com/watch?v=AKotMPGFJYk

3.1 de Rham Cohomology

Definition 3.1 (de Rham Cohomology)**.** This is the dual to homotopy groups.

$$X \text{ space (small manifold)} \longleftrightarrow C^\infty(X)\text{ring of smooth functions on } X.$$

$$\text{pt} \mapsto \text{ maximal ideal coming from evaluation map i.e.} 0 \mapsto m_p \to C^\infty(X) \to \mathbb{R} \to 0.$$

In particular, we have the mapping pt $\mapsto \ker(\text{ ev pt}) = m_{\text{pt}}$ (This is a short exact sequence).

Recall,

$$\Pi_0(X) = \text{ path components of } X$$

. Then,

$$|\Pi_0(x)| = \dim_{\mathbb{R}}\{f : \frac{\partial f}{\partial x^i}(p) = 0\}$$

These are the "locally constant functions."

Recall $\Omega^*(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \to ....$

In local coordinates, $df = [\frac{\partial f}{\partial x^i} dx^i]_{i=1}^n$, where $f \in \Omega_0(X) = C^\infty(X)$.

So f is locally constant $\iff$ $df = 0$. We can now rewrite

$$|\Pi_0(X)| = \dim_{\mathbb{R}}(\ker d : \Omega^0(X) \to \Omega^1(X))$$

Q How do we get higher dimensional thing?

A $\Pi(X, p) = \Pi_0(\Omega_p X, \bar{p})$, where $\bar{p}$ is the constant loop.

The dual things should be functions from $\Omega_p(X) \to \mathbb{R}$, which are locally constant!

3.2 Differential Forms and Stokes

If $\theta \in \Omega^1(X), \theta = f_1 dx^1 + f_2 dx^2 + ... + f_n dx^n$ in coordinates.
$\omega \in \Omega_p X$ a curve, then

$$\int_\gamma \theta = \int_0^{2\pi} \gamma^* \theta \; d\theta \in \mathbb{R}.$$

So a 1-form θ defines a function on curves:

$$\gamma \mapsto \int_\gamma \theta \in \mathbb{R},$$

where $\gamma \in \Omega_p X$.
Locally constant $\Rightarrow \gamma^1$ is near γ, then

$$\int_\gamma \theta = \int_{\gamma^1} \theta.$$

By Stokes' Theorem, $\int_\gamma - \int_{\gamma^1} \theta = \int_D d\theta$, where γ, γ_1 "cobound a region D."
Hence, $d\theta = 0$ tells us that a function is locally constant.
If $\theta = df$, then the $d\theta = 0$condition is trivially satisfied since

$$\begin{aligned} d\theta &= d(df) \\ &= d^2 f \\ &= 0 \end{aligned}$$

$$\begin{aligned} |\Pi_0(\Omega X)| &= \dim_{\mathbb{R}} \{\theta \in \Omega^1(X) = d\theta = 0\} \\ &= \dim_{\mathbb{R}} \{\theta : \Omega^1(X) : \theta = df\} \end{aligned}$$

$|\Pi_0(\Omega X)| \longleftrightarrow \Pi_1(X)$

3.3 de Rham Cohomology

$$H^1(X) = (\frac{\ker d : \Omega^1(X) \to \Omega^2(X)}{\mathrm{im} d : \Omega^0(X) \to \Omega^1(X)})$$

Inductively, we have

$$H^n(X) = (\frac{\ker d : \Omega^n(X) \to \Omega^{n+1}(X)}{\mathrm{im} d : \Omega^{n-1}(X) \to \Omega^n(X)})$$

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27 August 2021 We now discuss the properties associated with $H^n(X)$.

(0) $\dim H^n(X) < \infty$

(1) Poincare Duality (If X compact), then $H^n(X) = \dim H^{m-n}(X)$ (consequence of Stokes')

(2) Observation If X is diffeomorphic to Y, then $\Omega(X) \cong \Omega(Y)$isomorphic.

Exercise Moreover, $H^n(X) \cong H^n(Y)$ for all $n \in \mathbb{Z}_{\geq 0}$
Much stronger statement, if $X \simeq Y$ (homotopic), then $H^n(X) \cong H^n(Y)$ isomorphic for $n \in \mathbb{Z}_{\geq 0}$.

3.4 Homotopy

Pictures (Refer to the Schematic picture and concrete picture in class).

Example 5 (Concrete Picture). If X is a disk $D^2 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$, then the cone $C = \{(x,y,z) : x^2+y^2 \leq 1-z^2, 0 \leq z \leq 1\}$.

This represents the homotopy $D^2 \simeq \text{pt } = (0,0,1)$.

Definition 3.2. $f, g : X \to Y$ continuous

$(f \simeq g) f$ is homotopic to g if there is a continuous map

$$H : X \times I \to Y$$
$$H(x,0) = f, H(x,1) = g$$

Definition 3.3. $X \simeq Y$ if $\exists$ maps

$$f : X \to Y \text{ and } g : Y \to X \text{ such that}$$
$$gf = 1_X : X \to X$$
$$fb = 1_Y : Y \to Y$$

3.5 Mayer-Vietoris Sequence

There is a **long exact sequence** (at least three nonzero terms, often infinite exact sequence) relating the cohomology of X, U, V, and $U \cap V$ when $X = U \cup V$, where $U, V \subset$ X (open).

Suppose we are dealing with vector spaces. We now define the notion of an exact sequence (can do similarly with Abelian groups).

Definition 3.4 (Exact Sequence)**.** $(\star) W^i \xrightarrow{j^i} W^{i+1} \xrightarrow{j^{i+1}} W^{i+2} \to$ ..., where W is a vector space.

Here $f^i : W^i \to W^{i+1}$ linear maps (so we can talk about images and kernels).

$(\star)$ is exact when $(\star\star)$ $\text{im}(f^i) = \ker(f^{i+1})$ in W^{i+1} for all i.

Example 6 (1). $0 \xrightarrow{0} A \xrightarrow{f} B$

$\text{im } 0 = 0 = \ker(f) \iff f$ is injective $\iff$ exactness.

Example 7 (2). $A \xrightarrow{g} \xrightarrow{B} \xrightarrow{0} 0$

$\text{im } (g) = \ker(0) = B \iff g$ is survjective $\iff$ exactness.

Example 8 (3). $0 \to A \xrightarrow{h} B \to 0$ exactness $\iff h$ is an isomorphism.

Definition 3.5 (Short Exact Sequence)**.** $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ By 1st Isomorphism Theorem, we can rewrite this as follows: $0 \to \ker g \to B \to C (=$ co kernel$) \to 0$

3.6 Overview

In the first part, we will look at cohomology groups.

If $X = U \cup V$, then there are exact sequence of cohomology groups.

$\leftarrow H^{i+1}(U \cap V) \leftarrow H^i(U) \oplus H^i(V) \leftarrow H^i(U \cap V) \leftarrow H^{i-1}(X) \leftarrow$ ... At each place the exactness condition holds.

Easy to Compute

$$H^n(S^m) = \begin{cases} \mathbb{R}, & n = m, 0 \\ 0, & n \neq m, 0 \end{cases}$$

For genus g,

$$H^n(\sum_g) = \begin{cases} \mathbb{R}, & n=0 \\ \mathbb{R}^{2g}, & n=1 \\ \mathbb{R}, & n=2 \\ 0, & n>2 \end{cases}$$

We will deal with **spectral sequences in the second part**.

Let $X = \cup_{\lambda\in\Lambda} U_\lambda$, where $\{U_{\lambda,\lambda\in A}$ is an open cover of X.

There is an analogue of the long exact sequence for open cover, which is the spectral sequence.

When $\{U_{\lambda,\lambda\in A}$ is good, $\forall \alpha_0, \alpha_1, ..\alpha_n \in \Lambda$,

$$U_{\alpha_0} \cap U_{\alpha_1} \cap ... \cap U_{alpha_n} \cong \begin{cases} D^{\dim X} \\ \emptyset \end{cases}$$

Triangulations are nice.

Know

$$H^n(D^{\dim X}) = \begin{cases} \mathbb{R}, & n=0 \\ 0, & n\neq 0 \end{cases}$$

$H^n(\emptyset) \neq 0 \forall n \Rightarrow H^n(X)$ can be computed completely combinatorially from a good cover (or triangulation).

So the combinatorial definition resulting from is homotopy invariant. The solutions to the differential equations in $H^n_{\text{de Rham}}$ are completely combinatorially determined!

So, $H^n_{\text{de Rham}} = H^n(X) \otimes_{\mathbb{Z}} \mathbb{R}$ (will also introduce $H_n(X)$ over $\mathbb{Z}$ too).

Later Either (1) try to compute $\Pi_n(X)$ examples or (2) characteristic class of vector bundles (more useful).

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30 August 2021

4 $\mathbb{R}$-Algebra

$\mathbb{R}^n$ affine space with the usual topology on it.

$$||x|| = \sqrt{\sum_{i=1}^n x_i^2},$$

where $x_1, ..., x_n$ coordingates on $\mathbb{R}^n$, $U \subset \mathbb{R}^n$ any open subset.

Definition 4.1 ($C^0(\mathbb{R}^n)$)**.**

$$C^0(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} : f \text{ is continuous}\}$$

It is a subset of $C^\infty(\mathbb{R}^n$

Definition 4.2 ($C^\infty(\mathbb{R}^n)$)**.**

$$C^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} : \frac{\partial^k}{\partial x_{i_1} ... \partial x_{i_k}} \text{exists and is continuous} \forall k \geq 0, 0 \leq i_1, ..., i_k \leq n\}$$

These two are $\mathbb{R}$-algebras i.e. $C^0(\mathbb{R}^n), C^\infty(\mathbb{R}^n)$ are $\mathbb{R}$-vector spaces, are rings, contain the unit, and multiplication is bilinear:

$$\begin{aligned} \mathbb{R} &\to C^\infty(\mathbb{R}^n) \\ x &\mapsto f_x \\ f_x(v) = x &\text{ is } \mathbb{R} \text{ linear} \end{aligned}$$

Exterior Algebra Formal $\mathbb{R}$-algebra.

$\Omega^{\cdot}$ generated by symbols $dx_1, dx_2, ..., dx_n$ subject to the relations:

$$a)\ dx_i^2 = 0 \quad \forall i = 1, ..., n$$
$$b)\ dx_i dx_j = -dx_j dx_i \quad \forall i, j = 1, ..., n$$

Proposition. $\exists$ a basis $\mathcal{B}$ for $\Omega^{\cdot}$ given by:

$$dx_I := dx_{i_1} dx_{i_2} \cdot ...dx_{i_k}$$

for all $1 \leq i_1 < i_2 < ... < i_k \leq n$.

Corollary 8.1. $\dim \Omega^{\cdot} = 2^n$

We can grade this algebra: $|dx_i| = 1$ on generators. This extends to all of $\Omega^{\cdot}$ because relations are **homogeneous** (all monomial term in polynomial have same degree.

We can write $\Omega^{\cdot}$ as follows:

$$\Omega^{\cdot} = \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \oplus +... \oplus \Omega^n.$$

Note that $1 \in \Omega^0, dx_i \in \Omega^1, dx_i dx_j \in \Omega^2, ..., dx_i....dx_n \in \Omega^n$.

Corollary 8.2. $\dim \Omega^k = \binom{n}{k}$

Binomial Theorem $\Rightarrow \sum_{k=0}^n \binom{n}{k} = 2^n$.

4.1 de Rham Complex

de Rham complex is a combination of 2nd $\mathbb{R}$-algebra $C^\infty(\mathbb{R}^n)$ and 3rd $\mathbb{R}$-algebra $\Omega^{\cdot}$.

Definition 4.3. $\Omega^{\cdot}(\mathbb{R}^n) := C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \Omega^{\cdot}$

Similarly, $U \subset \mathbb{R}^n$ open. $\Omega^{\cdot}(U) := C^\infty(U) \otimes_{\mathbb{R}} \Omega^{\cdot}$

An element $\omega \in \Omega^{\cdot}(U)$, $\omega = [f_I \cdot dx_I]$, where $I = (i_1 < i_2 < ... < i_k)$ for various k.

Can write as Direct Sum

$$\Omega^{\cdot}(U) = \Omega^0(U) \oplus \Omega^1(U) \oplus ... \oplus \Omega^n(U).$$

Note that $f \in C^\infty(U)$ is in $\Omega^0(U)$, $\sum_{i=1}^n f_i dx_i$ is in $\Omega^1(U)$, and $f_{i=1,...,n} dx_1 dx_2 ... dx_n$.

$\omega \in \Omega^2(U), \omega = \sum_{i<j} f_{ij} dx_i dx_j$.

4.2 Exterior Derivative

There is an operator:

$$d : \Omega^i(U) \to \Omega^{i+1}(U).$$

1. IF $f \in Omega^0(U)$, then

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

2. If $\omega = \sum f_I dx_I$, then

$$d\omega = \sum df_I dx_I.$$

There is a **wedge product** on $\Omega^{\cdot}(U)$, which comes from the products on $C^\infty(U)$ and Ω.

If $\Omega = \sum f_I dx_I$ and $\tau = \sum g_J dx_J$. Then,

$$\begin{aligned}\omega \wedge \tau &= (\sum I f_I dx_I) \cdot (\sum_J g_J dx_J) \\ &= \sum_{I,J} f_I dx_I \cdot g_J dx_J \\ &= \sum (f_I g_J) \cdot dx_I dx_J\end{aligned}$$

Note that the following relation holds:

$$\omega \wedge \tau = (-1)^{|\omega||\tau|} \tau \wedge \omega$$

for ω, τ homogeneous.

To prove Let $\omega = f dx_I, \tau = g dx_J$. Then

$$\begin{aligned}fg dx_I dx_J &= fg dx_{i_1} \ldots. dx_{i_k} dx_{j_1} \ldots. dx_{j\ell}, \text{ where } k = |w|, \ell = |\tau| \\ &= (-1)^{k \cdot \ell} g f dx_{j_1} \ldots. dx_{j\ell} \ldots. dx_{i_k} \\ &= g f dx_I dx_J\end{aligned}$$

The exterior derivative is compatible with the wedge product.
$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^{|\tau|} \omega \wedge d\tau$.
Assume $\omega = f dx_I, \tau = g dx_J$. Then,

$$\begin{aligned}d(\omega \wedge \tau) &= d(fg dx_I \wedge dx_J) \text{ (by definition)} \\ &= d(fg) dx_I \wedge dx_J \\ &= (d(f)g + f d(g)) dx_I \wedge dx_J\end{aligned}$$

Proposition. $d^2 = 0$

Proof.

$$\begin{aligned}\text{(WLOG) } \omega &= f dx_I \\ d^2\omega &= d(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i) dx_I \\ &= \sum_{i=1}^n d(\frac{\partial f}{\partial x_i} dx_i dx_I \\ &= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_j} dx_j dx_j dx_I \\ &= [\sum_{i<j} f_{ij} dx_j dx_i + \underbrace{\sum_{i=j} f_{ij} dx_i dx_j}_{0} + \sum_{j<i} f_{ij} dx_j dx_i] \cdot dx_I \\ \Rightarrow f_{ij} &= f_{ji} \; \forall f \in C^\infty(U) \\ &= 0\end{aligned}$$

The last line follows from property (b) $dx_j dx_i = -dx_i dx_j$. $\square$

4.3 de Rham Cohomology Groups

Definition 4.4. de Rham cohomology groups of $U \subset \mathbb{R}^n$open. Then,

$$H^m(U) :- \frac{\ker d : \Omega^m(U) \to \Omega^{m+1}(U)}{\text{im } d : \Omega^{m-1}(U) \to \Omega^m(U)}$$

for $m = 0, ..., n$.

Twist Story $C_c^\infty(U) \subset C^\infty(U) = \{f : \overline{f^{-1}(0)}$ is compact.

Definition 4.5.

$$H_c^m(U) :- \frac{\ker d : \Omega_c^m(U) \to \Omega_c^{m+1}(U)}{\text{im } d : \Omega_c^{m-1}(U) \to \Omega_c^m(U)}$$

for $m = 0, ..., n$.

1 September 2021

Last Time **de Rham cohomology**

$$H^n(U) = \frac{\ker d : \Omega^n - D \to \Omega^{n-1}(D)}{\text{im } d : \Omega^{n-1}(U) \to \Omega^n(U)}$$

$= \mathbb{R} < \omega \in \Omega^n(U) : d\omega = 0 > /\omega \sim \tau \iff \omega - \tau = d\theta$

Example 9. When $U = \mathbb{R}$, let us compute $H^n(\mathbb{R})$ and $H_c^n(\mathbb{R})$ for all n.

$$H^n(\mathbb{R}) : \Omega^*(\mathbb{R}) = 0 \to \Omega^0(\mathbb{R}) \xrightarrow{d} \Omega^1(\mathbb{R}) \to 0$$
$$C^\infty(\mathbb{R}) C^\infty(\mathbb{R})\ dx$$
$$f \mapsto \frac{\partial f}{\partial x} \cdot dx$$

$n = 0$ $\Omega^{-1}(\mathbb{R}) = 0 \Rightarrow \text{im} d : \Omega^{-1}(\mathbb{R}) \to \Omega^0(\mathbb{R})$is 0.

(kernel) $d(f) = \frac{\partial f}{\partial x} \cdot dx = 0 \iff \frac{\partial f}{\partial x} = 0 \Rightarrow f = c$ for some constant $c \in \mathbb{R}$.

$\frac{\ker d}{0} \cong \frac{\mathbb{R}}{0} \cong \mathbb{R}$.

$\therefore H^0(\mathbb{R}) = \mathbb{R}$.

n = 1 $d : \Omega^1(\mathbb{R}) \to \Omega^2(\mathbb{R}) = 0$. So $\ker d = \Omega^1(\mathbb{R})$.

Claim im $d = \Omega^1(\mathbb{R}) \Rightarrow$ im $d \subset \Omega^1(\mathbb{R})$ by definition.

Suppose $g\ dx \in \Omega^1(\mathbb{R})$.

FTC $f(y) := \int_0^y g\ dx \iff \frac{\partial f}{\partial y} = g\ dx$ (by FTC).

$f \in \Omega^0(\mathbb{R}),\ df = gdx = \{g\ dx : g \in C^\infty(\mathbb{R})\}$

Since im $d = \ker(d), H^1(\mathbb{R}) = 0$.

Exact im () = ker (). How far is it being from being exact? Insight for (co) homology.

$$H^n(\mathbb{R}) = \begin{cases} \mathbb{R}, & n = 0 \\ 0, n \neq 0 \end{cases}$$

Poincare Lemma Same answer for $\mathbb{R}^n$.

Example 10.

$H_c^n(\mathbb{R})$ $0 \to \Omega_c^0(\mathbb{R}) \xrightarrow{d}\to \Omega_C^1(\mathbb{R}) \to 0$

$n = 0$ im $(d) = 0$.

$\ker(d) = \{f \in C_c^\infty(\mathbb{R}) | \frac{\partial f}{\partial x} = 0\} = 0$. (has to be constant and compact).

If $f = C$, then supp $(f) = \overline{\mathbb{R} - f^{-1}(0)} = \mathbb{R}$ unless $C = 0$.

$n = 1$ $d(f\ dx) = 0$ for all $f\ dx \in \Omega_C^1(\mathbb{R})$.

$\ker d = \Omega_C^1(\mathbb{R})$.

Want to compute $\frac{\Omega_C^1(\mathbb{R}}{\text{im} d}$

There is a linear map $\int : \Omega_C^1(\mathbb{R}) \to \mathbb{R}$.

$f\ dx \mapsto \int_{\mathbb{R}} f\ dx$ and onto $(\exists f\ dx$ such that $\int f\ dx = X$.

So, $\frac{D}{X}\ f\ dx \mapsto D \in \mathbb{R}$.

Claim $\ker(\int) = \text{im } d$.

If $g\ dx \in$im d then

$g\ dx = \dfrac{\partial f}{\partial x}\ dx \iff \int_{\mathbb{R}} g\ dx = \int_{\mathbb{R}} \dfrac{\partial f}{\partial x}\ dx$

Then, $\dfrac{\partial f}{\partial x}$ is compactly supported because f is compactly supported.

$\int_{\mathbb{R}} \dfrac{\partial f}{\partial x}\ dx = \lim_{N\to\infty} \int_{-N}^{N} \dfrac{\partial}{\partial x}\ dx = \lim_{N\to\infty}[f(N) - f(-N)]$ where f is compactly supported.

Since supp (f) bounded $\subset \mathbb{R}$, then supp $(f) \subset [-M, M]$.

$f(M), f(-M) = 0$ and $f(N), f(-N) = 0$ for all $N > M$.

Hence, $\lim_{N\to\infty} f(N) - f(-N) = 0 \Rightarrow$ im $d \subset \ker(\int)$

Claim $\ker(\int) \subset$ im d if $\int_{\mathbb{R}} g\ dx = 0$.

Set $h(y) = \int_{-\infty}^{y} g\ dx \Rightarrow dh = gdX$

Claim $\int g\ dx = 0 \Rightarrow$ supp (h) is compact.

Exercise (Turn this into a proof).

Rank-Nullity $\Omega^1_C(\mathbb{R})/\ker(\int) \to \mathbb{R} = H^2_C(\mathbb{R}) = \mathbb{C}$ (by 1st Isomorphism Theorem, this is an isomorphism).

Corollary 10.1.

$$H^n_C(\mathbb{R}) = \begin{cases} \mathbb{R}, & n = 1 \\ 0, & n \neq 1 \end{cases}$$

The de Rham Complex is an example of chain complex with differential $d's$.

$... \to C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \to ...$

Question Cohomology: How fair is a chain complex from being a long exact sequence with $d^2 = 0$? (since im $d^{i+1} \leq \ker d^i\ \forall i \in \mathbb{Z}$) C^i vector space.

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5 Category Theory

Nitesh 3 September 2021

Definition 5.1 (Category)**.** A **category** $\mathcal{C}$ is a collection of objects Ob $(\mathcal{C})$ and for each $X, Y \in$ Ob $(\mathcal{C}), \text{Hom}(X, Y), f : X \to Y$.

$f : X \to Y, g : Y \to Z$, then $g \circ f : X \to Z$. Then, the following properties are satisfied:

1.

$$\text{Hom } (Y, Z) * \text{ Hom } (X, Y) \to \text{ Hom } (X, Z) \tag{1}$$
$$gf \mapsto g \circ f \tag{2}$$

2. $\forall X \in$ Ob (X), $\exists 1_X \in$ Hom (X, X) such that
   (a) $(f \circ g) \circ h = f \circ (g \circ h)$
   (b) $\forall f : X \to Y$, we have that

$$f \circ 1_X = f, 1_Y \circ f = f.$$

Now we will look some examples of categories.

Example 11. If R ring, $\mathcal{C}_R$ is the category and Ob $(\mathcal{C}_R) = \{\star\}$ and Hom $(\star, \star) = R$.

Example 12. For all **rings** (unital),

Hom $(R, R') = \{f : R \to R' | f(ab) = f(a)f(b), f(a + b) = f(a) + f(b), f(1_R) = 1_{R'}$.

Example 13. If R is a ring, $R-$ mod is a category consisting of objects that are R_M **modules** M.

Then, $\exists R \to \text{End}(M)$ homomorphism, i.e. Hom (M, M') is a R-module homomorphism.

Example 14. **Vector Spaces**

Vect_F is a category for F a field, Ob $(\text{Vect})_F$ are finite dimensional vector spaces and Hom $(V, W) = F$-linear maps.

Example 15. **Abelian Groups**

Ch(Ab) chain complex of abelian groups denoted as Ob(Ch(Ab)). The corresponding homomorphism is

Hom $(C, D) = \{f^i : C^i \to D^i$ so that the following diagram commutes.

$$\begin{tikzcd}
\bullet \arrow[r] & C^i \arrow[r, "dc_i"] \arrow[d, "f^i"'] & C^{i+1} \arrow[r] \arrow[d, dashed, "f^{i+1}"] & \bullet \\
\bullet \arrow[r] & D^i \arrow[r] & D^{i+1} \arrow[r] & \bullet
\end{tikzcd}$$

Note that $f^{i+1} d^i_c = d_{D^i} f^i \ \forall i$.

Example 16. **Topological Spaces**

Ob (Top) are topological spaces, Hom $(X, Y) = \{f : X \to Y, f$ continuous $\}$.

Example 17. **Manifolds**

Ob (Man) are smooth manifolds, Hom (X, Y) is a smooth map.

5.1 Functor

Definition 5.2 (Functor)**.** If $\mathcal{C}$ and $\mathcal{D}$ are categories, $F : \mathcal{C} \to \mathcal{D}$ is a Functor if:

(i) $F :$ Ob $(\mathcal{C}) \to$ Ob(D) i.e. $c \in$ Ob (C) maps to $f(c) \in$ Ob (D).

(ii) $f : X \to Y$ in $\mathcal{C}$, $\exists F(f) : F(X) \to F(Y)$, i.e. $\text{Hom}_{\mathcal{C}}(X, Y) = \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ such that

(iii) $F(1_X) = 1_{F(x)}$

Covariant (iv) $F(f \circ g) = F(f) \circ F(g)$

Contravariant (v) $F(f \circ g) = F(g) \circ F(f)$

Algebraic Topology The study of functors from $\mathcal{C} =$ Top, Man (category containing topological objects) to $\mathcal{C} = R-$ mod , Ab, Ch (Ab) (algebraic categories).

Example 18 (Main Example)*.* Consider the Fundamental Group, $\Pi_1 : \text{Top}_* \to \text{Group}$

Example 19*.* $\Omega^* :$ Man $\to$ Ch (Ab) maps

$f : M \to N \mapsto f^* : \Omega^*(N) \to \Omega^*(M)))$ (Pullback)

Example 20*.* $H^n :$ Ch (Ab) $\to$ Ab for each $n \in \mathbb{Z}$, there is a Functor such that:

Ob $(C, d) = ... \to C^i \xrightarrow[d_i]{C^{i+1}} C^{i+1}$.

$H^n(C, d) = \dfrac{\ker d^n : C^n \to C^{n+1}}{\text{im} d^{n-1} c^{n-1} \to C^n}$.

If $f(C, d_C), (D, d_D)$ is a map, then there is a map:

$H^*(f) := f_* : H^n(c, d) \to H^n(D, d_D)$.

Note that $[c] \in \dfrac{\ker d_c}{\text{im} d_c}$ (equivalence class).

Pick $c \in [c]$. Then, define $f_*([c]) = [f(c)]$.

If $e \in [c]$, then $c - e = d_c x$. Next, we have

$f(c - e) = f(c) - f(e) = f d_c x = d_D f(x)$. Finally, we check that

$$\begin{tikzcd}
\alpha \arrow[r, "\rho"] \arrow[dr, "\pi\circ\rho"'] & A \arrow[d, "\pi"] \\
& \bullet
\end{tikzcd}$$

$$\begin{aligned}
[f(c)] &= [f(e) + d_D f(x)] \\
&= [f(e)] + [d_D f(x)] \\
&= [f(e)] + 0 \\
&= [f(e)]
\end{aligned}$$

Hence, this is well-defined. Finally we check that:

$H^n(1_c : C \to C)$.

$(1_c)_*([c]) = [1_c(c)] = [c]$.

So, $H^n(1_c) = 1_{H^n(c)}$ if $f : (C, d)$ and $g : (D, d) \to (E, d)$, then

$H^n(g) \circ H^n(f)([c]) = H^n(g)(f[c]) = [gf(c)] = H^n(g \circ f)$

6 Exact Sequences

8 September 2021

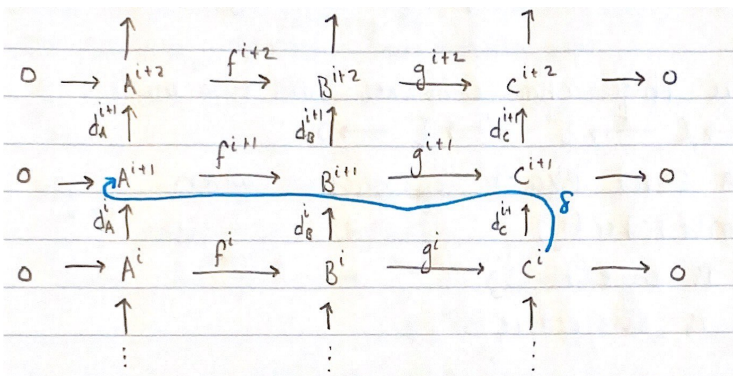
Elise

Theorem 21 (The Snake Lemma). Every short exact sequence of chain complexes

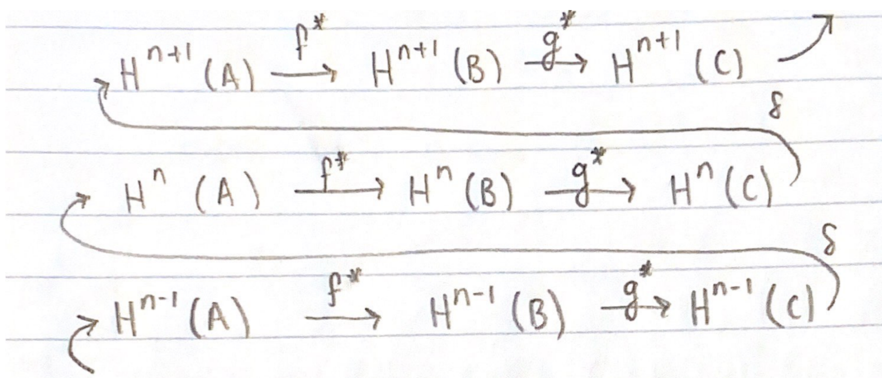
$$0 \to A \to B \to C \to 0$$

gives a long exact sequence of cohomology groups.

Proof. Note f and g are chain maps so $df = fd$ meaning $d_B^{i+1}f^{i+1} = f^{i+2}d_A^{i+1}$ and similarly $dg = gd$. Moreover $d^2 = 0$ or more specifically $d_A^i d_A^{i+1} = 0, d_B^i d_B^{i+1} = 0, d_C^i d_C^{i+1} = 0$ for all $i \in \mathbb{Z}$. Moreover, by the exactness of the sequence, we know for all $i \in \mathbb{Z}$, f^i is injective, g^i is surjective, and $Im(f^i) = Ker(g^i)$.



Define $H^n(A) = \frac{Ker(d_A^n : A^n \to A^{n+1})}{Im(d_A^{n-1} : A^{n-1} \to A^n)}$ as well as $H^n(B)$ and $H^n(C)$ similarly. Then $f^* : H^n(A) \to H^n(B)$ where for $[a] \in H^n(A)$, $f^*([a]) = [f^n(a)] \in H^n(B)$ and $g^* : H^n(B) \to H^n(C)$ where for $[b] \in H^n(B)$, $g^*([b]) = [g^n(b)] \in H^n(C)$.



Now, we define a map, $\delta := (\delta^n) : H^n(C) \to H^{n+1}(A)$ using the following motivation. (The map δ is typically called **the connecting map**.

Pick $[c] \in H^n(C)$ and let $c \in C^n$ be such that $d_C^{n+1}(c) = 0$. Now, since g^n is onto, there exists some $b \in B^n$ such that $g^n(b) = c$. Then $d_B^n(b) \in B^{n+1}$. However, the diagram commutes so $g^{n+1}(d_B^n(b)) = 0 = d_C^n g^n(b) = d_C(c)$. But this shows that $d_B^n(b) \in Ker(g^{n+1}) = Im(f^{n+1})$ by the exactness. Thus, there exists a unique $a \in A^{n+1}$ (since f is injective) such that $f^{n+1}(a) = d_B^n(b)$.

Claim: $d_A(a) = 0$.

Proof. First note that $d_B^{n+1}(d_B^n(b)) = 0$. Then using the definition of $d_B^n(b)$, we find $d_A(f^{n+1}(a)) = d_B^{n+1}(d_B^n(b)) = 0$. On the other hand, $0 = f^{n+1}d_A(a)) = d_B^n(f^n(a))$. Then we have $d_A^n(a) \in Ker(f^{n+1}) = \{0\}$ since f^{n+1} is injective. Therefore $d_A(a) = 0$. □

But now this gives that $a \in Ker(d_A^n)$. Hence $[a] \in \frac{Ker(d^{n+1})}{Im(d^n)} = H^{n+1}(A)$.

With this motivation, we set $\delta([c]) = [a]$.

Claim: δ is well defined.

Subclaim 1: Our choice of $c \in [c]$ is well defined.

Proof. Note it suffices to show $\delta(d(c')) = 0$ where $c = c + d(c')$ for $c' \in C^{n-1}$. Choose $b \in B^{n-1}$ such that $g^{n-1}(b') = c^{'}$ and notice that $d(g(b')) = d(c') = g(d(b'))$. Now, assume $b = d_{B(b')}$. Then $d(d_{B(b')}) = 0$ so $d_{B(b')} \in Ker(g)$. Therefore, we define $a := f^{-1}(d(d_{B(b')}) = 0$ giving $a = 0$. Hence $\delta(d(c')) = 0$. □

Subclaim 2: g^n is onto which implies there exists some $b \in B^n$ such that $g^n(b) = c$.

Proof. Suppose $g(b+b') = c$ for $b' \in Ker(g^n)$. Now, because of the exactness, $Ker(g^n) = Im(f^n)$ so $b' \in Im(f^n)$. Then we have that $d(b + b') \in Ker(g^{n+1})$ since $g^{n+1}(d(b + b')) = 0 = d_C(c) = d_C(g(b + b'))$. Therefore, there exists a unique $a' \in A$ with $f(a') = b'$. Then $d(b + f(a')) \in Ker(g^{n+1})$ so $g^{n+1}(d(b + f(a')) = 0 = d_C(g^n(b + f(a'))$.

Following this argument through gives $\delta(a + da') = b + b'$ giving $[a + d(a')] = [a] \in H^{n+1}(A)$. □

□

10 September 2021

Elise For every open set of $\mathbb{R}^n$, $U \subseteq R^n$, the deRham complex is a commutative differential graded algebra denoted Ω^*.

If $f : U \to V$ is a smooth map for $g \in C^\infty(V)$, there exists $f^* \in C^\infty(U)$ defined by $f^*(g) := g \circ f$. This is a contravariant functor in the sense that $(g \circ f) : U \to V \to W$ and $(f^* \circ g^*) : \Omega^0(U) \to \Omega^0(V) \to \Omega^0(W)$. One other property this function possesses is that $f^* \circ g^* = (g \circ f)^*$ and for $1_U : U \to U$, $1_U^* = 1_{C^\infty(U)}$.

Note if $\omega \in \Omega^k(V)$ for $k > 0$, then $\omega = \sum_I g_I dy_I$.

Moreover $f^*w = \sum_I f^*(g_I)df_I = \sum_I (g_I \circ f)d(y_{i_1} \circ f) \cdots d(y_{i_n} \circ f) = \sum_I (g_I \circ f)df_{i_1} \cdots df_{i_n}$ where $I = (i_1, ..., i_n)$. This extension is a contravariant functor.

Proposition. $f^*(\omega \wedge \tau) = f^*(\omega) \wedge f^*(\tau)$

Proposition. $df^* = f^*d$

This shows that the assignment does indeed define a contracariant functor from the category with objects being open subsets of $\mathbb{R}^n$, $\mathcal{O}_p(Man) = \{M : M \text{is a smooth manifold}\}$, to the differential graded algebra where $Hom(U, V) = \{f : U \to V : f \text{is a smooth map}\}$.

Now, suppose $(x_1, ..., x_n)$ and $(u_1, ..., u_n)$ are two coordinate systems for $U \subseteq \mathbb{R}^n$ where U is an open subset, then there exists a diffeomorphism $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ where $f^*(x_i) = x_i \circ f = u_i$ for all $1 \leq i \leq n$.

Now, if $g : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map, then $dg = \sum_{i=1}^n \frac{\partial g}{\partial u^i} du^i = \sum_{i=1}^n \frac{\partial g}{\partial u^i} \frac{\partial u^i}{\partial x^j} dx^j = \sum_{i=1}^n \frac{\partial g}{\partial x^j} dx^j$.

Similarly, $(d\omega)(u_1, ..., u_n) = (d\omega)(x_1, ..., x_n)$ for all $\omega in \Omega^k(U)$.

Proposition. If $f : U \to V$ is a diffeomorphism, then $f^* : \Omega^*(U) \to \Omega^*(U)$ is an isomorphism.

If M is a smooth manifold, let $\{U_\alpha, \phi_\alpha : U_\alpha \to M\}_{\alpha \in \lambda}$ be a smooth atlas with $M = \bigcup_{\alpha \in \lambda} U_\alpha$. Then $\Omega^*(M) = \{\omega_\alpha : \alpha \in \lambda, \omega_\alpha \in \Omega^*(U_\alpha)\}$ for ω_α to be in this set, we must have that for all $\alpha, \beta \in \lambda$ with $U_\alpha \cap U_\beta \neq \varnothing$, then $\omega_\alpha|_{U_\alpha} \cap U_\beta = \omega_\beta|_{U_\alpha \cap U_\beta}$. If this is the case, $\omega_\alpha \in \Omega^*|(U_\alpha) \cong \Omega^*(phi_\alpha(U_\alpha))$ where $\phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$.

This gives the following diagram for $\alpha, \beta \in \lambda$;

$$\begin{tikzcd}
\Omega^*(U_\beta) \arrow[dr, "i^*"] & \\
\Omega^*(U_\alpha) \arrow[r, "j^*"'] & \Omega^*(U_\alpha \cap U_\beta)
\end{tikzcd}$$

This diagram is induced by the following diagram;

$$\begin{tikzcd}
U_\beta \arrow[dr, "i"] & \\
U_\alpha \arrow[r, "j"'] & U_\alpha \cap U_\beta
\end{tikzcd}$$

Now, if $M = U \cup V$ for open sets U and V, then

$$\begin{tikzcd}
U \cap V \arrow[r, hook] & U \arrow[r, "i"] & M \\
U \cap V \arrow[r, hook] & V \arrow[r, hook, "j"'] & M
\end{tikzcd}$$

Now, applying the functor, we find the following diagram.

$$\begin{tikzcd} U \cap V \arrow[r, hook] \arrow[d] & U \arrow[d] \\ V \arrow[r, hook] & M \end{tikzcd}$$

This gives

$$\begin{tikzcd} \Omega^*(U \cap V) & \Omega^*(U) \arrow[l] \\ \Omega^*(V) \arrow[u] & \Omega^*(M) \arrow[l] \arrow[u] \end{tikzcd}$$

Now from this, we obtain a sequence of chain complexes,

$$\Omega^*(U \cap V) \longleftarrow \Omega^*(U) \oplus \Omega^*(V) \longleftarrow \Omega^*(M)$$

Proposition. The above sequence of chain complexes is a short exact sequence.

13 September 2021

Kevin Let $M = u \cup v$

by M.V. the LES computes $H*(M)$ in terms of $H^*(u)$ and $H^*(v)$ and $H^*(u \cap v)$ with the following diagram

$$\begin{tikzcd}
u \cap v \arrow[r, hook] \arrow[d, hook] & u \arrow[d, hook] \\
v \arrow[r, hook] & M = u \sqcup_{u \cap v} v \arrow[r] & \pi_1(u \sqcup_{u \cap v}) \cong \pi_1(u) *_{\pi_1(u \cap v)} \pi_1(v)
\end{tikzcd}$$

with the diagram above it induces the following diagram

$$\begin{tikzcd}
\Omega^*(u \cap v) & \Omega^*(u) \arrow[l] \\
\Omega^*(v) \arrow[u] \arrow[ur, dash, dotted, "\oplus"] & \Omega^*(M) \arrow[l] \arrow[u]
\end{tikzcd}$$

which then their is a chain complex

$$\Omega^*(M) \longrightarrow \Omega^*(u) \oplus \Omega^*(v) \longrightarrow \Omega^*(v \cap v)$$

we let $O_P = ob(O_p)$ be open sets in M and

$HOm_{O_p}(u,v) = \begin{cases} inclusion u \subset v \\ \emptyset & u \not\subset v \end{cases}$

So we look at the a sheef with the covaraint functor

- (0)
  $\Omega^* = \mathcal{F} : O_P \to AB$
- (1)
  if $\tau, \omega \in \mathcal{F}(u)$ and for all $u_i \subset u$ we have the following:
  $\omega|_{ui} = \tau|_{ui} = (\mathcal{F}(u_i \subset u)(\omega)$ then $\omega = \tau$
- (2)
  if $\{u_i\}_{i \in I}$ and $\omega_i \in \mathcal{F}(u_i)$ such that $\omega_i|u_I \cap u_j = \omega_j|_{u_i \cap v_j}$ for all $i, j \in I$ then their is $\omega \in \mathcal{F}(\bigcup_{i \in I})$ such that $\omega|_{ui} = u_i$

We note that $\mathcal{F} = \Omega^*$ is a sheef and (*) is determined by (0) thus we get

$$\Omega^*(M) \xrightarrow{A} \Omega^*(u) \oplus \Omega^*(v) \xrightarrow{B} \Omega^*(v \cap v)$$

$$\omega \longmapsto (\omega|_u, \omega|_v)$$

$$(\alpha, \beta) \longmapsto \alpha|_{u \cap v} - \beta|_{u \cap v}$$

A injective: if $\omega|_u = 0$ and $\omega|_v = 0$ then by (1) we get that $\omega = 0$

im(A)$\subset$ ker(B): if $\omega \mapsto (\omega|_u, \omega_{|v}) \mapsto \omega|_u - \omega|_v$

$\iff BA(\omega) = 0$ for all $\omega \in \Omega^*(M) = 0$

ker(B) $\subseteq$ Im(A):

If $B(\alpha, \beta) = 0$ then $\alpha|_{u\cap v} - \beta|_{u\cap v} = 0$ iff $\alpha|_{u\cap v} = \beta|_{u\cap v} = 0$

so there is $\omega \in \Omega^*(M)$ so $\omega|_u = \alpha$ and $\omega|_v = \beta$

B is onto:

if $\omega \in Omega^*(u \cap v)$ then we want $(\alpha, \beta) \in \Omega(u) \oplus \Omega^*(v)$ sp $B(\alpha, \beta = \alpha|_{u\cap v} - \beta|_{u\cap v} = \omega$

then there is a partion of unity $\{\rho_u, \rho_v\}$ subordniate to the open cover $\{u, v\}$ of M

$\rho_u, \rho_u : M \to \mathbb{R}_{\geq 0} \in C^\infty(M)$ with $supp\rho_u \in u$ and $supp\rho_v \in v$ and $\rho_u + \rho_v = 1$

thus we get $\alpha := \rho_v\omega \in \Omega^*(u)$ and $\beta := -\rho_u\omega \in \Omega^*(v)$

So $B(\alpha, \beta) = B(\rho_u, \rho_v) = 1\omega|_{u\cap v}$ and we also have $u \mapsto \Omega_c^*(\breve{)}$ there are smooth maps $f : u \to v$ so that f^* doesnt preserve the compact support property $\Omega_c^*(u)$ not a contravariant functor for all smooth maps. BUT

1) if $f : v \to v$ is proper (iff $f^{-1}(c)$ is compact if c is compact) then f^* does preserve the compactly supported functions

2) If $u \to v$ and inclusion then their is a map $\Omega_c^*(u) \to \Omega_c^*(v)$ $f(\omega) = \omega$ so Ω_c^* is a covariant funcotr. form open sets and inclusion to chain complexes ie a SES as follows

$0 \leftarrow \Omega_c^*(M) \leftarrow \Omega_c^*(u) \oplus \omega_c^*(v) \leftarrow \Omega_c^*(u \cap v) \leftarrow 0$

Elise 6.1 Computing $H^*(S^1)$ and $H^*_C(S^1)$

Goal: Compute $H^*(S^1)$ and $H^*_C(S^1)$

In order to use the Mayer-Vietoris long exact sequence to compute the above homology, recall:

$$H^*(\mathbb{R}) = \begin{cases} \mathbb{R} & * = 0 \\ 0 & * \neq 0 \end{cases} \qquad H^*_C(\mathbb{R}) = \begin{cases} \mathbb{R} & * = 1 \\ 0 & * \neq 1 \end{cases}$$

Note $S^1 = U \cup V$ where U and V are open intervals and diffeomorphic to $\mathbb{R}$. Moreover $U \sqcup V$ is a disjoint union of open intervals so $U \sqcup V \cong \mathbb{R} \sqcup \mathbb{R}$.

We note the de Rham complex, Ω^* is diffeomorphism invariant since it is a functor and a diffeomorphism is an isomorphism. Hence we have $\Omega^*(U) \cong \Omega^(\mathbb{R})$, $\Omega^*(V) \cong \Omega^*(\mathbb{R})$, and $\Omega^*(U \cap V) \cong \Omega^*(\mathbb{R} \sqcup \mathbb{R}) \cong \Omega^*(\mathbb{R}) \oplus \Omega^*(\mathbb{R})$. This gives that $H^*(U) \cong H^*(\mathbb{R})$, $H^*(V) \cong H^*(\mathbb{R})$, and $H^*(U \cap V) \cong H^*(\mathbb{R} \sqcup \mathbb{R}) \cong H^*(\mathbb{R}) \oplus H^*(\mathbb{R})$. From this we obtain the following sequence

$$\begin{tikzcd}
 & 0 & & \\
H^1(S^1) \arrow[r, "A"] & H^1(U) \oplus H^1(V) \arrow[r] & H^1(U \cap V) \arrow[ul, "\delta"'] & \\
H^0(S^1) \arrow[r, "C"] \arrow[urr, "\delta"] & H^0(U) \oplus H^0(V) \arrow[r, "B"] & H^0(U \cap V) & \\
 & & & 0 \arrow[ulll, "\delta"]
\end{tikzcd}$$

Now, by exactness, $\ker(\delta) = \operatorname{Im}(A) = H^1(S^1)$ since $H^1(U) \oplus H^1(V) = 0$ so δ is onto $H^1(S^1)$.

Also by exactness (for α and β being the intersections of U and V), $\ker(\delta) = \operatorname{Im}(B)$. But $B : \mathbb{R}^2 \to \mathbb{R}^2$ by $\langle 1_U, 1_V \rangle$ so $B(1_U) = 1_\alpha + 1_\beta$ so $Ker(B) = \langle 1_U + 1_V \rangle \cong \mathbb{R}$ and $Ker(\delta) = Im(B) = \langle 1_\alpha + 1_\beta \rangle \cong \mathbb{R}$.

Therefore, $H^1(S^1) \cong \mathbb{R}$ and $\ker(B) = \operatorname{im}(C) = H^0(S^1)$ since C is injective. Hence $H^0(S^1) \cong \mathbb{R}$.

From this, we can conclude the following;

$$H^n(S^1) = \begin{cases} \mathbb{R} & n = 1 \\ \mathbb{R} & n = 0 \end{cases}$$

Definition 6.1. The n^th *Betti number* is defined as $b_n = dim(H^(M)) \in \mathbb{Z}_{\geq 0}$.

Note: b_n determines $H^n(M)$ up to isomorphism.

As an example, if $M = S^1$, then $b_0 = 1$ and $b_1 = 1$.

This is called the *reductionist.*

The nonreductionist has the following view; $H^n(M)$ is a space of solutions to equations. We can build our solution using a Mayer Vietoris argument where $H^0(S^1)$ is the collection of locally constant functions on S^1.

Now, we know $\delta : H^0(U \cap V)/Im(B) \to H^1(S^1)$ is an isomorphism since $Im(B) = Ker(\delta)$ and $Im(B)$ is the collection of locally constant functions agreeing on α and β, i.e. $Im(B) = Ker(\delta)$ consists of functions $f : U \cap V \to \mathbb{R}$ such that $f|_\alpha = f|_\beta$ so $1_\alpha \notin Im(B)$ so $1_\alpha \in H^0(U \cap V)$ so $\delta(1_\alpha)$ generates $H^1(S^1)$ as an $\mathbb{R}$ vector space.

$$d(\rho_U 1_\alpha + \rho_V 1_\beta) \longrightarrow 0$$

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1(S^1) \arrow[r] & \Omega^1(U) \oplus \Omega^1(V) \arrow[r] & \Omega^1(U \cap V) \arrow[r] & 0 \\
0 \arrow[r] & \Omega^0(S^1) \arrow[r] \arrow[u, "d"] & \Omega^0(U) \oplus \Omega^0(V) \arrow[r] \arrow[u, "d"] & \Omega^0(U \cap V) \arrow[r] \arrow[u, "d"] & 0
\end{tikzcd}$$

$$\rho_U \alpha + \rho_V \beta \longrightarrow 1_\alpha$$

Therefore $d(\rho_U 1_\alpha)|_{U\cap V} = -d(\rho_V 1_\beta)|_{U\cap V}$. This gives $\omega == \begin{cases} d(\rho_U 1_\alpha) & V \\ -d(\rho_V 1_\beta) & U \end{cases}$

17 September 2021

Elise 6.2 Closed, Exact, and Boundary

Recall

$H^*(\mathbb{R}) = \begin{cases} \mathbb{R} & n=0 \\ 0 & n=1 \end{cases}$ $\quad H^n_C(\mathbb{R}) = \begin{cases} \mathbb{R} & n=1 \\ 0 & n=0 \end{cases}$ $\quad H^n_C(S^1) = H^n(S^1) = \begin{cases} \mathbb{R} & n=0,1 \\ 0 & otherwise \end{cases}$

For a general manifold N of dimension n, this switch is cause because $\Omega^k(N) \otimes \Omega^{n-k}(N) \to \Omega^n_C(N) \to \mathbb{R}$ for all $0 \leq k \leq n$. Then for $\omega \in \Omega^k(N)$ and $\tau \in \Omega^{n-k}(N)$ $(\omega, \tau) \to \omega \wedge \tau$. Then, applying the integral, the bilinear form $< \omega, \tau > \in \mathbb{R}$.

Theorem 22. Suppose N is an n-dimensional orientable manifold with boundary. Then for $\omega \in \Omega^{n-1}(N)$,

$$\int_N d\omega = \int_{\partial N} \omega$$

Now, fix N and let $\omega \in \Omega^k(N)$. If M is a submanifold of N and $dim(M) = k$, we can always integegrate ω over any submanifold M of dimension k where

$$\int_M \omega = \int_M i^*\omega \in \mathbb{R}$$

.

For example, if $\omega \in \Omega^1(N)$, then we can integrate ω over M for $dim(M) = 1$.

Note, if $\partial Z = M \sqcup M'$ for a subset Z of N and $\omega \in \Omega^1(N)$ and $d\omega = 0$, then because M and M' have opposite orientations,

$$0 = \int_Z d\omega = \int_{\partial Z} \omega = \int_{M \sqcup M'} \omega = \int_M \omega - \int_{M'} \omega$$

.

The condition that $d\omega = 0$ gives, by Stoke's theorem,

$$\int_M \omega = \int_{M'} \omega$$

when there exists $Z \subset N$ and $\partial Z = M \sqcup M'$. Similarly, if $M' = \varnothing$, then

$$\int_M \omega = \int_\varnothing \omega = 0$$

.

Now, recall that $H^n(N) = Ker(d)/Im(d)$. Now if $\omega - \omega' = d\tau \in \Omega^k(N)$ and $M^k \subset N$, then

$$\int_M \omega - \int_M^{\prime} \omega = \int_M \omega - \omega' = \int_M \tau = \int_{\partial M} \tau.$$

Moreover, if $\partial M = \varnothing$ and $\omega - \omega' = d\tau$, then

$$\int_M \omega = \int_M^{\prime} \omega$$

Definition 6.2.
1. If $d\omega = 0$, then ω is *closed* or a *cycle*.
2. If $\omega = d\tau$, then ω is *exact* or a *boundary*.
3. If M^k is a smooth manifold and $\partial M = \varnothing$, M is a *closed manifold*.
4. If there exists Z such that $\partial Z = M$, them M is a *boundary*

Definition 6.3. We can define the *Cobordism groups of* N by $C^k_C(N) = \mathbb{Z}\{M^k \subseteq N : M^k$is closed, compact,

$$\text{and a k-manifold mapping into } N\}/M \sim M'$$

This definitions tells us there is a canonical map $C^k_C(N) \to H^*_C(N)^*$ defined by $M \mapsto (\omega \mapsto \int_M \omega$ for $[\omega] \in H^*(N)$.

If our manifolds are not compact, we have $C^k(N) \to H^*_C(N)^*$.

Having developed this background, if we again consider

$$\Omega^k(N) \otimes \Omega^{n-k}(N) \to \Omega^n_C(N) \to \mathbb{R}$$

, the pairing $\Omega^k(N) \otimes \Omega^{n-k}(N) \to \mathbb{R}$ descends to a pairing on cohomology $H^k(N) \otimes H^{n-k}_C(N) \to \mathbb{R}$ (by Stoke's theorem).

We can also prove that

1. $dim(H^k(N)) < \infty$ and $dim(H^k_C(N)) < \infty$ for any smooth manifold N
2. The pairing $H^k(N) \otimes H^{n-k}_C(N) \to \mathbb{R}$ is non-degenerate.

Now 1) and 2) from above imply that $H^k(N) \cong H^{n-k}_C(N)^*$. Therefore, $\dim(H^k(N)) = \dim(H^{n-k}_C(N))$. If N is compact, $H^{n-k}_C(N) = H^{n-k}(N)$ so $H^k(N) \cong H^{n-k}(N)$.

20/22 September 2021

Elise 7 Poincare Lemma

Let M be a manifold in the category of smooth manifolds, Man. Let $f : M \to N$ be a smooth map. Then we obtain the sequence,

$$\begin{array}{ccccc} Man & \xrightarrow[\Omega^*]{} & Ch & \xrightarrow[H^*]{} & gVect \\ \\ M & \xrightarrow{\qquad} & f^* & & \end{array}$$

where $f^* : \Omega^*(N) \to \Omega^*(M)$ and $f^* : \oplus_k H^k(N) \to \oplus_k H^k(M)$.

Thus, the **Question** then becomes; Is there a relation/ideal of maps $I \subseteq Ch$ so that $K(Vect) = Ch/I$ which is the homotopy category of vector spaces.

[The obvious relation, $\mathcal{O}$ that we will not use, is the relations where if $f : A \to B$ with $f \in Ch$ and $H^*(f) : H^*(A) \to H^*(B)$ is an isomorphism, then we require $f^{-1} \in Ch/\mathcal{O} = D(Vect)$ which we use to define a new category.

In this case, $D(Vect) = Ch/\mathcal{O}$ is the *derived category of chain complexes* and f is an *inverting quasi-isomorphism.*]

On the other hand, there is a constructive relation on chain maps called *homotopy equivalence.*

Let $f, g : A \to B$ be chain maps which gives the following diagram;

$$\begin{tikzcd} \cdots \arrow[r] & A^i \arrow[r, "d_A^i"] \arrow[d, "{f^i, g^i}"'] & A^{i+1} \arrow[r, "d_A^{i+1}"] \arrow[d, "{f^{i+1}, g^{i+1}}"] \arrow[dl, "k^i"'] & A^{i+2} \arrow[r] \arrow[d, "{f^{i+2}, g^{i+2}}"] \arrow[dl, "k^{i+1}"'] & \cdots \\ \cdots \arrow[r] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r, "d_B^{i+1}"] & B^{i+2} \arrow[r] & \cdots \end{tikzcd}$$

Definition 7.1. Define the relation where $f \simeq g$ if and only if there exists a map $k : A^{*+1} \to B^*$ such that $f - g = (-1)^{*-1}(dk - kd)$.

Proposition. 1. $f \simeq g$ by the above definition is an equivalence relation.

2. if $f \simeq g$, then $f \circ h \simeq g \circ h$ and $h \circ f \simeq h \circ g$.
3. if $f \simeq g$, then $H^*(f) = H^*(g)$.

Proof. 1. To show this is an equivalence relation, first we note that $f - f = 0$ so letting $k : A^{*+1} \to B^*$ be the zero map, i.e. $k \equiv 0$, gives $f - f \equiv 0 \equiv (-1)^{*-1}(dk - kd)$.

Next, suppose $f \simeq g$, then there exists a map $k : A^{*-1} \to B^*$ where $f - g = (-1)^{*-1}(dk - kd)$. Now notice that $g - f = -(f - g)$ so if we let $\tilde{k} : A^{*-1} to B^*$ be defined by $-k$, we find that $g - f = -(f - g) = -(dk - kd) = (kd - dk) = d\tilde{k} - \tilde{k}d$.

Finally, suppose $f \simeq g$ and $g \simeq h$, then there exists a map $k : A^{*-1} \to B^*$ where $f - g = (-1)^{*-1}(dk - kd)$ and there exists a map $\tilde{k} : A^{*-1} \to B^*$ where $f - g = (-1)^{*-1}(d\tilde{k} - \tilde{k}d)$. Now, consider the following using that d is a linear operator $f - h = (f - g) + (g - h) = (-1)^{*-1}(dk - kd) + (-1)^{*-1}(d\tilde{k} - \tilde{k}d) = (-1)^{*-1}(d(k + \tilde{k}) - (k - \tilde{k})d)$. Therefore $k' := k - \tilde{k} : A^{*-1} \to B^*$ satisfies the above conditions so $f \simeq h$.

This shows that the above definition is an equivalence relation.

2. Suppose $f \simeq g$ and let $h : A^{*-1} \to A^{*-1}$ be another map mapping into A^{*-1}. Note that because $f \simeq g$, then there exists a map $k : A^{*-1} \to B^*$ where $f - g = (-1)^{*-1}(dk - kd)$.

Question First we consider $f \circ h$ and $g \circ h$. Now notice that $f \circ h - g \circ h = (f - g) \circ h = (-1)^{*-1}(dk - kd) \circ h = (-1)^{*-1}(d(k \circ h) - k \circ h)d)$. But since $k \circ h : A^{*-1} \to B^*$, then the above relation holds so we can say $f \circ h \simeq g \circ h$.

Question Similarly we consider $h \circ f$ and $h \circ g$. Then $h \circ f - h \circ g = h \circ (f - g) = h \circ [(-1)^{*-1}(dk - kd)] = h \circ [(-1)^{*-1}dk] - h \circ [(-1)^{*-1}kd] = (-1)^{*-1}d(k \circ h) - (-1)^{*-1}(k \circ h)d = (-1)^{*-1}(d(k \circ h) - (k \circ h)d)$. But since $k \circ h : A^{*-1} \to B^*$, then the above relation holds so $h \circ f \simeq h \circ g$.

3. Let $c \in H^*(A)$, then $d(c) = 0$ so $H^*(f)(c) = [f(c)] = [\pm dkc \pm kdc + g(c)] = [dkc + g(c)] = [0 + g(c)] = [g(c)]$ since we are modding out my the exact forms (the image of d since we are in homology).

$\square$

Now, if $g, f : M \to N$ are smooth maps, then they are smoothly homotopic if there exists a homotopy $K : M \times \mathbb{R} \to N$ with $K(M,t) = \begin{cases} f(m) & t > 1 \\ g(m) & t \leq 0 \end{cases}$.

Note M is locally $\mathbb{R}^n$.

Lemma 23 (**Poincaré Lemma:**). $H^*(\mathbb{R}^n \times \mathbb{R}) \cong H^*(\mathbb{R}^n)$.

Proof. TO show this, we first define the following maps; $\pi : \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1} \to R$ where $\pi(x,t) = x$ and $s : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$ by $s(x0 = (x, 0)$. Now, $\pi \circ s = \not\Vdash_{\mathbb{R}^n}$ since s is just a section. However, notice that $s \circ \pi(x) = (x, 0)$ which is not the identity map. Hence, to prove our statement, we need to show that there exists a chain homotpy equivalence

$$\Omega^*(\mathbb{R}^n \times \mathbb{R}) \underset{\pi^*}{\overset{s^*}{\rightleftarrows}} \Omega(\mathbb{R}^n)$$

where $S^* \circ \pi^* = (\pi s)^* = \mathbf{1}_{\Omega^*(\mathbb{R}^n)}$ and $\pi^* \circ s^* = (s\pi)^* \sim \not\Vdash_{\Omega^*(\mathbb{R}^n \times \mathbb{R})}$

Recall $\sim$ if and only if there exists $k : \Omega^{*+1}(\mathbb{R}^n \times \mathbb{R}) \to \Omega * (\mathbb{R}^n \times \mathbb{R})$ so that $1 - \pi^* \circ s^* = \pm dk \pm kd$.

Observe if $f \in C^\infty(\mathbb{R}^n \times \mathbb{R})$, either $f = f(x,t)$ or $f = f(x)$. Similarly, we observe that $\Omega^*(\mathbb{R}^n \times \mathbb{R})$ is defined by where for $\omega \in \Omega^*(\mathbb{R}^n \times \mathbb{R})$, either $\omega = \pi^*(\phi) f(x,t)$ (TYPE I) or $\omega = \pi^*(\phi) f(x,t) dt$ (TYPE II) where $\phi \in \Omega^*(\mathbb{R}^n)$ and $f \in C^\infty(\mathbb{R}^n \times \mathbb{R})$.

Now, define $k = \begin{cases} k\omega = k(\pi^*(\phi) f(x,t) = 0 & \omega \text{ is TYPE I} \\ k\omega = \pi^*(\phi) \left(\int_0^t f(x,t) \ dt \ \right) \text{ and } \omega \text{ is TYPE II} & \end{cases}$.

Note that $1 - s^*\pi^* = \pm dk \pm kd$ since $\pi s = 1$ so $s^*\pi^* = 1$ because $1 - s^8\pi^* = \pm dk \pm 0 \cdot d = 0$.

Next we check $1 - \pi^* s^* = \pm dk \pm kd$ holds for our defined k.

For TYPE I, recall $\omega = \pi^*(\phi) f(x,t)$ and by definition $k\omega = 0$. It follows from this definition that $dk(\omega) = 0$. Moreover,

$$\begin{aligned}
kd\omega &= k(d[\pi^*(\phi) f(x,t)]) \\
&= k(d\pi^*(\phi) f(x,t) + (-1)^{|\phi|} \pi^*(\phi) df(x,t)) \\
&= k(\pi^*(d\phi) f(x,t)) + (-1^{|\phi|} k(\pi^*(\phi) df(x,t)) \\
&= 0 + (-1)^{|\phi|} k(\pi^*(\phi)(\sum_i \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial t} dt)) \\
&= (-1)^{|\phi|} [\sum_i k\pi^*(\phi) \frac{\partial f}{\partial x^i} dx^i] + (-1)^{|\phi|} k\pi^*(\phi) \frac{\partial f}{\partial t} dt \\
&= 0 + (-1)^{|\phi|} \pi^*(\phi) \int_0^t \frac{\partial f}{\partial t} \ , dt \\
&= (-1)^{|\phi|} \pi^*(\phi)[f(x,t) - f(x,0)].
\end{aligned} \tag{3}$$

On the other hand,

$$\begin{aligned}(1-\pi^*s^*)\omega &= \omega - \pi^*s^*\omega \\ &= \pi^*(\phi)f(x,t) - \pi^*\circ s^*(\pi^*(\phi)f(x,t)) \\ &= \pi^*(\phi)f(x,t) - \pi^*s^*\pi^*(\phi)\cdot\pi^*s^*f(x,t) \\ &= \pi^*(\phi)f(x,t) - \pi^*(\phi)\cdot\pi^*s^*f(x,t) \\ &= \pi^*[f(x,t) - \pi^*s^*f(x,t)] \\ &= \pi^*(\phi)[f(x,t) - f(x,0)]\end{aligned} \tag{4}$$

since $\pi^*s^*f(x,t) = f(s\cdot\pi(x,t)) = f(s(x)) = f(x,0)$.

Next, we do the same calculations for TYPE II where $\omega = \pi^*(\phi)f(x,t)dt$. Notice that $\pi^*(d(s^*(t)) = \pi^*(0) = 0$. Therefore,

$$\begin{aligned}(1-\pi^*s^*)\omega &= \omega - \pi^*s^*\omega \\ &= \pi^*(\phi)f(x,t)dt - \pi^*\circ s^*(\pi^*(\phi)f(x,t)dt) \\ &= \pi^*(\phi)f(x,t)dt - \pi^*s^*\pi^*(\phi)\cdot\pi^*s^*f(x,t)\cdot\pi^*s^*dt \\ &= \pi^*(\phi)f(x,t)dt - \pi^*(\phi)\cdot\pi^*s^*f(x,t)\cdot 0 \\ &= \pi^*(\phi)f(x,t)dt\end{aligned} \tag{5}$$

On the other hand

$$\begin{aligned}dk\omega &= k(\pi^*(\phi)\int_0^t f(x,t)\,dt \\ &= d(\pi^*(\phi))\int_0^t f(x,t)\ ,dt + (-1)^{|\omega|-1}\pi^*(\phi)d(\int_0^t f(x,t)\ ,dt) \\ &= d(\pi^*(\phi))\int_0^t f(x,t)\ ,dt + (-1)^{|\omega|-1}\pi^*(\phi)(\sum_i \frac{\partial}{\partial x^i}(\int_0^t f(x,t)\ ,dt)dx^i + \frac{\partial}{\partial t}\int_0^t f(x,t)\ ,dt\end{aligned} \tag{6}$$

Moreover,

$$\begin{aligned}kd\omega &= kd(\pi^*(\phi)\int_0^t f(x,t)\,dt \\ &= k(d(\pi^*(\phi))f(x,t)dt) + k(\pi^*(\phi)df(x,t)dt) + 0 \\ &= k\pi^*(d\phi)f(x,t)dt - k(\pi^*(\phi)\sum_i \frac{\partial f}{\partial x^i}dx^i dt) \\ &= \pi^*(d\phi)\int_0^t f(x,t)\,dt + \sum_i (k(\phi^*(\phi)dx^i)\frac{\partial f}{\partial x^i}dt) + \frac{\partial f}{\partial t}\int_0^t f(x,t)\,dt\end{aligned} \tag{7}$$

But by these calculations, we can see that $(1-\pi^*s^*)\omega = \pi^*(\phi)(f(x,t)dt = (dk-kd)\omega$. This proves our statement so $H^*(\mathbb{R}^n\times\mathbb{R}) \cong H^*(\mathbb{R})$ $\square$

24 September 2021

Elise

Corollary 23.1. $H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & * = 0 \\ 0 & * \neq 0 \end{cases}$

Corollary 23.2. For any smooth manifold M, $H^*(M \times \mathbb{R}) \cong H^*(M)$.

sketch. There exists maps π^* and s^* such that

$$\begin{tikzcd} \Omega^*(\mathbb{R}^n \times \mathbb{R}) \arrow[r, shift left, "s^*"] & \Omega(\mathbb{R}^n) \arrow[l, shift left, "\pi^*"] \end{tikzcd}$$

We cover M with charts $\{U_\alpha, \phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n\}$ where $M = \bigcup_\alpha U_\alpha$. The homotopy operator $k_\alpha : \Omega^*(U_\alpha \times \mathbb{R}) \to \Omega^*(U_\alpha)$ where $k_\alpha = \phi_\alpha^*(k|_{\phi(U_\alpha)}$ where k is the one defined in the Poncaré lemma.

Then, by the sheaf property outlined on September 13th, there exists a unique $k : \Omega^*(M \times \mathbb{R}) \to \Omega^*(M)$ where $k|_{U_\alpha} = k_\alpha$. THe uniqueness implies that k must satisfy the same homotopy relation. □

Corollary 23.3. If $f, g : M \to N$ are smooth maps and $f \simeq g$, then $f^* = g^* : H^*(N) \to H^*(M)$.

Proof. Suppose $f \simeq g$. This gives the following diagram where s_0 is the zero section with $s_0(x) = (x, 0)$ and s_1 is the first sections where $s_1(x) = (x, 1)$.

$$\begin{tikzcd} M \times \mathbb{R} \arrow[rr, "F"'] \arrow[d, shift right, "\pi"'] & & N \\ M \arrow[u, shift right, "s_0 s_1"'] \arrow[urr, "f = F \circ s_0", "g = F \circ s_1"'] & & \end{tikzcd}$$

Then, by the chain rule $f^* = (F \circ s_0)^* = s_0^* \circ F^*$ and $g^* = (F \circ s_1)^* = s_1^* \circ F^*$. But notice that by the Poincaré Lemma, $s_0^= (\pi^*)^{-1} = s_1^*$. Hence $f^* = s_0^* \circ F^* = s_1^* \circ F^* = g^*$.

(Notice that this proof is done at the level of homology. □

Definition 7.2. Let M and N be smooth manifolds. We say M and N have the same **homotopy type** or M **is homotopic to** N written $M \simeq N$ if there exists maps $f : M \to N$ and $g : N \to M$ such that $\mathbb{1}_{\mathbb{M}} \simeq f \circ g$ and $\mathbb{1}_{\mathbb{N}} \simeq g \circ f$.

Corollary 23.4. If $M \simeq N$, then $H^*(M) \cong H^*(N)$ (as an isomorphism of vector spaces.)

Definition 7.3. (A special case of homotopy) Let $A \subseteq M$. We say A is a **deformation retract** of M if $i : A \hookrightarrow M$ and there exists a map $r : M \to A$ such that $r \circ i = \mathbb{1}_A$ and $i \circ r \simeq \infty_M$.

Example 24. (The Mobius Band) Consider the Mobius Band, M, where $M = (-1, 1) \times [0, 1]/[(1, t) \sim (-1, 1 - t)]$. Define $A = \{(\theta, \frac{1}{2}) : -1 \leq \theta \leq 1\}$ is the circle around the middle of the band. Therefore $r : M \to A$ where $r(\theta, t) = (\theta, \frac{1}{2})$. Then clearly $r \circ i(\theta, \frac{1}{2}) = (\theta, \frac{1}{2})$ sor $r \circ i \simeq \mathbb{1}_A$ and $i \circ r(\theta, t) = i(\theta, \frac{1}{2}) = (\theta, \frac{1}{2})$ so we see that $i \circ r \simeq \mathbb{1}_M$ where the homotopy is defined as $F : M \times \mathbb{R} \to M$ where $F((\theta, t), s) = (\theta, st + (1 - s)\frac{1}{2}$. Notice that if $s = 0$, then $F((\theta, t), s) = \mathbb{1}_M$ and if $s = 1$, then $F((\theta, t), s) = i \circ r$. Thus A is a deformation retract of M.

Observation: If A is a deformation retract of M, then $A \simeq M$ and $H^*(A) \cong H^*(M)$.

Corollary 24.1. (To above example) $H^*(M) = \begin{cases} \mathbb{R} & * = 0, 1 \\ 0 & * \neq 0, 1 \end{cases}$

We can also prove using the Mayer Vietoris and induction that $H^(S^n) = \begin{cases} \mathbb{R} & * = 0, n \\ 0 & * \neq 0, n \end{cases}$

27 September 2021

Quanqi Hu

Proposition. The maps

$$H_c^*(M \times \mathbb{R}^1) \rightleftarrows^{\pi_*}_{e_*} H_c^{*-1}(M)$$

are isomorphisms.

Note that here, unlike the previous case, the dimension is shifted by one.

Lemma 25 (**Poincaré Lemma for Compact Supports**).

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } n \\ 0 & \text{otherwise} \end{cases}$$

Consider the projection map $\pi : M \times \mathbb{R} \to \mathbb{R}$. Since π is not proper, the pullback map π^* does not send $\Omega_c^*(M)$ to $\Omega_c^*(M \times \mathbb{R}^1)$. However, we can consider the integration over the fiber $\pi_* : \Omega_c^*(M \times \mathbb{R}^1) \to \Omega_c^{*-1}(M)$. Note that a compactly supported form on $M \times \mathbb{R}^1$ is a linear combination of two types of forms:

$$\begin{aligned} &(\text{I}) \quad \pi^*\phi \cdot f(x,t) \\ &(\text{II}) \quad \pi^*\phi \cdot f(x,t)dt, \end{aligned}$$

where ϕ is a form on the base (not necessarily with compact support), and $f(x,t)$ is a function with compact support. We define π_* by

$$(\text{I}) \quad \pi_*(\pi^*\phi \cdot f(x,t)) = 0$$

$$(\text{II}) \quad \pi_*(\pi^*\phi \cdot f(x,t)dt) = \phi \int_{-\infty}^{\infty} f(x,t)dt.$$

Let $e(t)$ be a bump function. Let $e = e(t)dt$ be a compactly supported 1-form on $\mathbb{R}$ with total integral 1 and define

$$e_* : \Omega_c^*(M \times \mathbb{R}^1) \to \Omega_c^{*+1}(M)$$

by

$$\phi \mapsto \phi \wedge e.$$

By construction, $\pi_* e_* = 1_{\Omega_c^*(\mathbb{R}^n)}$.
Check: $d\pi_* = \pi_* d$ and $de_* = e_* d$.

Proof.

$$\begin{aligned} e_*(\omega) &= \pi^*(\omega) \wedge e \\ &= d\pi^*\omega \wedge e + (-1)^{|\omega|}\pi^*\omega \wedge de \\ &= \pi^*(d\omega) \wedge e, \end{aligned}$$

where the last equality is given by

$$de = d(e(t)dt) = \sum_i \frac{\partial}{\partial x_i} e(t) dx_i dt + \frac{\partial}{\partial t} e(t) dt dt = 0.$$

Type I:

$$\begin{aligned} \pi_* d(\pi^*(\phi) f(x,t)) &= \pi_*(d\pi^*(\phi) f(x,t) + (-1)^{|\phi|}\pi^*(\phi) df) \\ &= \pi_*(0 + (-1)^{|\phi|}(\sum_i \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial t} dt)) \\ &= (-1)^{|\phi|}\phi \int_{-\infty}^{\infty} \frac{\partial f}{\partial t}(x,t)dt \\ &= (-1)^{|\phi|}\phi \lim_{N\to\infty} \int_{-N}^{N} \frac{\partial f}{\partial t}(x,t)dt \\ &= (-1)^{|\phi|}\phi \lim_{N\to\infty} (f(x,N) - f(x,-N)) \\ &= 0. \end{aligned}$$

The last equality is given by the assumption that $f(x,t)$ is compactly supported. Moreover, by the definition of π_*, we have

$$d\pi_*(\pi^*(\phi)f(x,y)) = 0.$$

Type II:

$$\begin{aligned} d\pi_*(\pi^*(\phi)f(x,y)dt) &= d(\phi\int_{-\infty}^{\infty} f(x,t)dt) \\ &= d\phi\int_{-\infty}^{\infty} f(x,t)dt + (-1)^{|\phi|}\phi d\int_{\mathbb{R}} f(x,t)dt. \end{aligned}$$

$$\begin{aligned} \pi_* d(\pi^*(\phi)f(x,y)dt) &= \pi_*\left[[]d\pi^*(\phi)]f(x,t)dt + (-1)^{|\phi|\pi^*(\phi)d(f)dt}\right] \\ &= d\phi\int_{\mathbb{R}} f(x,t)dt + (-1)^{|\phi|}\phi\int_{\mathbb{R}} d(f). \end{aligned}$$

Therefore, $d\pi_* = \pi_* d$. □

Proposition. $1 - e_*\pi_* = (-1)^{q-1}(dK - Kd)$ on $H_c^q(M\times\mathbb{R})$, where the homotopy operator $K : \Omega_c^*(M\times\mathbb{R}) \to \Omega_c^{*-1}(M\times\mathbb{R})$ is defined by

$$\text{(I)}\quad K(\pi^*(\phi)f(x,t)) = 0$$

$$\text{(II)}\quad K(\pi^*(\phi)f(x,t)dt) = \phi\int_{-\infty}^{t} f(x,t)dt - \phi A(t)\int_{-\infty}^{\infty} f(x,t),$$

where $A(t) = \int_{-\infty}^{t} e(t)dt$.

29 September 2021

Quanqi Hu

Definition 7.4. An open cover $U = \{U_\alpha\}$ of M^m is called a *good cover* if all finite intersections $U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}$ are diffeomorphic to $\mathbb{R}^m$. A manifold which has a finite good cover is said to be *of finite type.*

Definition 7.5. A cover $\{V_\beta\}_{\beta\in\Lambda'}$ is a *refinement* of $\{U_\alpha\}_{\alpha\in\Lambda}$ if $\forall\alpha\in\Lambda$, $\exists\beta\in\Lambda'$ such that $V_\beta\subset U_\alpha$.

Fact: Every cover has a good refinement.

Theorem 26. Every manifold has a good cover. If the manifold is compact, then the cover may be chosen to be finite.

Proof. A Riemannian metric on M is a positive definite symmetric bilinear form $\langle\cdot,\cdot\rangle_x : T_xM\times T_xM\to\mathbb{R}$. Every smooth manifold has a Riemannian metric. Consider an atlas $\{U_\alpha,\phi_\alpha\}_{\alpha\in\Lambda}$, then a Reimannian metric on M can be defined as $\langle v,w\rangle := \sum_{\alpha\in\Lambda}\rho_\alpha\phi_\alpha^{-1}\langle v,w\rangle_{\mathbb{R}^m}$, where $\{\rho_\alpha\}_{\alpha\in\Lambda}$ is a partition of unity subordinate to $\{U_\alpha\}_{\alpha\in\Lambda}$.
Every Riemannian metric can be used to produce a good cover. Consider $\exp_x : T_xM\to B(x,\epsilon)$. Take $v\in T_xM$, then there is a unique geodesic curve $\gamma : [0,\delta)\to M$ such that $\gamma(0) = x$, $\gamma'(0) = v$ and $\exp_x(v) = \gamma(1)$. $\exp_x$ is a local diffeomorphism from $B(x,\epsilon')\subset T_xM$ onto $B(x,\epsilon)$. $\operatorname{im}(\exp_x)$ is geodesically convex. This property implies that $B(x,\epsilon)\cong\mathbb{R}^m$ and the property holds for intersections. By taking $U_\alpha = \operatorname{im}(\exp_\alpha)$, we obtain a good cover of M. □

1 October 2021

Nitesh If M has a finite good cover, then the cohomology is finite dimension.

Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a good cover and $|\Lambda| < \infty$

If $|\Lambda| = 1$, then there is only finite nonempty intersection, i.e. $\mathbb{R}^n\cong U_\alpha = M$. (implied by Poincare Lemma)

If $|\Lambda| = 2, U = U_\alpha, V = U_\beta, U_\alpha\cong\mathbb{R}^n, V_B\cong\mathbb{R}^n$.

$\star \; V_\alpha \cap V_\beta \cong \begin{cases} \mathbb{R}^n \\ \emptyset \end{cases}$

Mayer-Vietoris $... \leftarrow H^i(U \cap V) \leftarrow H^i(U) \oplus H^i(V) H^i(U \cup V) \leftarrow H^{i-1}(U \cap V) \leftarrow H^{i-1}(U) \oplus H^{i-1}(V)$

$$\begin{tikzcd}
H^i(U) \oplus H^i(V) & H^i(U \cup V) \arrow[l, "res"'] \arrow[d, tail] & H^i(U \cap V) \arrow[l, tail, "d^*"] \arrow[dl] \\
& im(res) &
\end{tikzcd}$$

Note this is finite dimensional.

im (res) $\cong \dfrac{H^i(U \cup V)}{\text{ker(res}}$ (finite dimensional)

$\Rightarrow \ker(res) = \text{im} d^*$ (this is finite dimensional since $H^i(U \cap V)$ is finite dimensional).

$\therefore$, $H^i(U \cup V)$ is finite dimensional $\Rightarrow \dim(H^i(U \cup V)) = \dim(\text{ im (res)} \oplus \ker(\text{res}) < \infty$.

Proof. Assume this is true for $|\Lambda| = p$

Let $\{U_{\alpha_0}, ..., U_{\alpha_{p-1}}\}$ be a good cover.

Consider $(U_{\alpha_0}, U_{\alpha_1}, ... \cup U_{\alpha_{p-1}}) \cap U_{\alpha_p}$. This is a subspace of M, a good cover with p-open sets.

Distributing this, we get $(U_{\alpha_i} \cap U_{\alpha_p} : i = 0, ..., p)$

Let $\mathbf{U} = U_{\alpha_p}$ have finite dimensional cohomology.

Let $\mathbf{V} = U_{\alpha_0} \cup U_{\alpha_1} \cup ... U_{\alpha_{p-1}}$, which has fixed cover $\{U_{\alpha_0}, U_{\alpha_1}, ..., U_{\alpha_{p-1}}$.

Finally, we have $U \cap V = (V_{\alpha_0} \cup V_{\alpha_1} \cup ... \cup V_{\alpha_{p-1}}) \cap V_{\alpha_p}$ also has a finite dimensional cohomology.

By same argument as $\star$, we have finite dimensionality. Therefore, $H^i(U \cup V)$ is finite dimensional.

Splitting

$$\begin{tikzcd}
0 & \text{im}(r) \arrow[l] & H^i(U \cup V) \arrow[l, "r"'] \arrow[dd, dashed] & \ker(V) \arrow[l, "i"'] & 0 \arrow[l] \\
& \alpha & & \beta & \\
& & \text{im}(r) \oplus \ker(r) \arrow[uul] \arrow[uur] & & \\
& & (\alpha, \beta) \arrow[uul, "\pi_1"] \arrow[uur, "\pi_2"'] & &
\end{tikzcd}$$

$\square$

8 5-Lemma

$$\begin{tikzcd}
(I) & A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, dashed, "c"] & D \arrow[r] \arrow[d, "d"] & E \arrow[d, "e"] & \text{exact sequence} \\
(II) & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' & \text{exact}
\end{tikzcd}$$

All squares commute. If rows (I) and (II) are exact and a, b, d, e are homomorphisms, the squares all commute (up to signs), then c is an isomorphism too.

8.1 How Do We Apply This?

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, dashed, "h"] & C \arrow[d, "g"] & & \text{(ses ch ck)} \\
 & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 &
\end{tikzcd}$$

$$\begin{tikzcd}
H^{i-1}(C) \arrow[r] \arrow[d] & H^i(A) \arrow[r] \arrow[d, "f^*"] & H^i(B) \arrow[r] \arrow[d, dashed, "h^*"] & H^i(C) \arrow[r] \arrow[d, "g^*"] & H^{i+1}(A) \arrow[r] \arrow[d] & ... \\
H^{i-1}(C) & H^i(A') \arrow[r] & H^i(B') \arrow[r] & H^i(C') \arrow[r] & H^i(A) \arrow[r] & ...
\end{tikzcd}$$

f^*, g^* isomorphism $\Rightarrow h^*$i is an isomorphism by 5$-$Lemma.

A bilinear map $< \cdot, - >: V \otimes W \to \mathbb{R}$ is **nondegenerate**.

- If $< V, W >= 0$ for all $w \in W$, then $V = 0$.
- If $< V, W >= 0$ for all $v \in V$ then $W = 0$.

Any bilinear map

$$V \otimes W \to \mathbb{R}\text{induces a map}$$
$$V \xrightarrow{\alpha} W^*$$
$$\alpha(V)(w) =< v, w >$$

If V, W finite dimensional then $<, >$ is nondegenerate $\iff$ α is an isomorphism.

9 Poincare Duality

4 October 2021

6 October 2021

Steven Un

Theorem 27. **The Poincare Duality Theorem.** Let M be a compact oriented smooth manifold of dimension n. Then for each $q \in \{0, 1, 2, \dots n\}$, the bilinear form $\mathcal{B}_q : H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R}$ defined by

$$\mathcal{B}_q([\omega], [\tau]) = \int_M \omega \wedge \tau \tag{8}$$

for all $[\omega] \in H^q(M)$ and all $[\tau] \in H_c^{n-q}(M)$ is nondegenerate; that is, $H^q(M)$ and the dual space $(H_c^{n-q}(M))^*$ of $H_c^{n-q}(M)$ are isomorphic as real vector spaces:

$$H^q(M) \cong (H_c^{n-q}(M))^*. \tag{9}$$

We say that **Poincare duality holds for a smooth manifold M' of dimension m'** if $H^q(M') \cong (H_c^{n-q}(M'))^*$ for each $q \in 0, 1, \dots, m'$. Therefore the conclusion of the theorem is that Poincare duality holds for M.

□

Proof. Let $q \in \{0, 1, 2, \dots, n\}$. We want to establish a vector space isomorphism $H^q(M) \cong (H_c^{n-q}(M))^*$.

Step 1.

Claim. Suppose $M = U \cup V$ for two open subsets U, V of M. Then if Poincare duality holds for U, V, and $U \cap V$ then Poincare duality holds for $M = U \cup V$.

Proof. Assume Poincare duality holds for U, V, and $U \cap V$ (of course, as open subsets of M, all of these sets are smooth manifolds). Let $\{\rho_U, \rho_V\}$ be a partition of unity for M subordinate to the cover $\{U, V\}$ (with $\mathrm{Supp}(\rho_U) \subset U$ and $\mathrm{Supp}(\rho_V) \subset V$. We have the Meyer Vietoris long exact sequence of cohomology groups for the decomposition $M = U \cup V$ and the dual of the corresponding Mayer Vietoris sequence for compact supports. The below diagram shows the portion of the sequence about the cohomology group $H^q(U \cup V)$.

$$\begin{tikzcd}
H^{q-1}(U) \oplus H^{q-1}(V) \arrow[r] \arrow[d] & H^{q-1}(U \cap V) \arrow[r] \arrow[d, "f^*"] & H^q(U \cup V) \arrow[r] \arrow[d, dashed, "h^*"] & H^q(U) \oplus H^q(V \arrow[d, "g^*"] \\
H_c^{n-q+1}(U)^* \oplus H_c^{n-q+1}(V)^* & H_c^{n-q}(U \cup V)^* \arrow[l] & H_c^{n-q}(U \cup V)^* \arrow[l] & H_c^{n-q}(U)^* \oplus H_c^{n-} \arrow[l]
\end{tikzcd}$$

The objective is to deduce the isomorphism $H^q(U \cup V) \cong H_c^{n-q}(U \cup V)^*$ by applying the Short-Five Lemma to this diagram. In turn, to apply the Short-Five Lemma we need isomorphisms between the vector spaces in each column adjacent to the column containing $H^q(U \cup V)$ and $H_c^{n-q}(U \cup V)^*$, two isomorphisms on each side. The commutativity of each square in the above diagram is equivalent to the statement that the bilinear form $\mathcal{B}_q$ is *sign-commutative*. There are three cases of sign-commutativity of $\mathcal{B}_q$ to prove, each corresponding to one of three types of squares in the diagram. (Each type of square corresponds to one of the three combinations of the the objects $H^k(U) \cup H^k(V)$, $H^k(U) \cap H^(V)$, and $H^k(U) \oplus H^k(V)$ taken two a time.) Explicity, sign commutativity of $\mathcal{B}_q$ is given and proved by the following:

Type 1:

$$\int_{U \cap V} \omega \wedge d_* \tau = \pm \int_{U \cup V} (d^* \omega) \wedge \tau \tag{10}$$

for each $[\omega] \in H^q(U \cap V)$, $[\tau] \in H_c^{n-q-1} U \cup V$.

Proof.

$$\int_{U\cup V}(d^*\omega)\wedge\tau = \int_{U\cup V}(d(\rho_U\omega) - d(\rho_V\omega))\wedge\tau$$

$$= \int_{U\cup V} d(\rho_U\omega)\wedge\tau - \int_{U\cup V} d(\rho_V\omega)\wedge\tau \tag{11}$$

$$= \int_{U\cup V}(d(\rho_U)\omega + \rho_U d\omega)\wedge\tau - \int_{U\cup V}(d(\rho_V)\omega + \rho_V d\omega)\wedge\tau \tag{12}$$

$$= \int_{U\cup V}(d\rho_U)\omega\wedge\tau - \int_{U\cup V}(d\rho_V)\omega\wedge\tau \qquad (d\omega = 0) \tag{13}$$

$$= \int_U (d\rho_U)\omega\wedge\tau - \int_V (d\rho_V)\omega\wedge\tau \tag{14}$$

Where the last equality holds because the integrand $(d\rho_U)\omega\wedge\tau$ has support in U and the integrand $(d\rho_V)\omega\wedge\tau$ has support in V. This is because the wedge product factors in each integrand involve ρ_U and ρ_V, which are zero in particular outside of U and V respectively. Then, on the other hand,

$$\int_{U\cap V}\omega\wedge d_*\tau = \int_{U\cap V}\omega\wedge(-d(\rho_U\tau) + d(\rho_V\tau))$$

$$= \int_{U\cap V}\omega\wedge(-(d(\rho_U\tau)) + \int_{U\cap V}\omega\wedge d(\rho_V\tau) = \int_{U\cap V}\omega\wedge(-((d\rho_U)\tau)) + \rho_U d\tau)) + \int_{U\cap V}\omega\wedge((d\rho_V)\tau + \rho_V d\tau).$$

Because $d\tau = 0$, we have

$$\int_{U\cap V}\omega\wedge d_*\tau = \int_{U\cap V}\omega\wedge(-(d\rho_U)\tau) + \int_{U\cap V}\omega\wedge(d\rho_V)\tau$$

$$\int_{U\cap V}\omega\wedge d_*\tau = \int_{U\cap V}(-1)^{|\omega|}(-d\rho_U)\omega\wedge\tau + \int_{U\cap V}(-1)^{|\omega|}(d\rho_V)\omega\wedge\tau))$$

$$= (-1)^{|\omega|}\left(-\int_{U\cap V}(d\rho_U)\omega\wedge\tau + (-1)^{|\omega|}\int_{U\cap V}(d\rho_V)\omega\wedge\tau)\right) \tag{15}$$

$$= (-1)^{|\omega|}\left(-\int_{U\cap V}(d\rho_U)\omega\wedge\tau + \int_{U\cap V}(d\rho_V)\omega\wedge\tau\right) \tag{16}$$

$$= (-1)^{|\omega|}\left(-\int_U (d\rho_U)\omega\wedge\tau + \int_V (d\rho_V)\omega\wedge\tau\right) \tag{17}$$

The last equality, holds because of the appearance of the functions ρ_U and ρ_V in the integrands. From equations (14) and (17) we conclude that

$$\int_{U\cap V}\omega\wedge d_*\tau = \pm\int_{U\cup V}(d^*\omega)\wedge\tau$$

as claimed. We used the anticommutativity of the wedge product with the fact that $d\rho_U$ and $d\ rho_V$ are forms of degree one. □

Type 2:

$$\int_{U\cap V}\delta_q((\omega,\tau))\wedge\theta = \pm\left(\int_U \omega\wedge j_{U*}(\theta) + \int_V \tau\wedge j_{V*}(\theta)\right) \tag{18}$$

for each $([\omega],[\tau]) \in H^q(U)\oplus H^q(V)$, $[\theta]\in H_c^{n-q}(U\cap V)$ where δ_q is the usual difference map in the construction of the Mayer-Vietoris sequence, at the joint q, and $\iota : H_c^{n-q}(U\cap V)\to H_c^{n-q}(U)\oplus H_c^{n-q}(V)$ is signed inclusion, that is, $\iota(\theta) = (-j_{U*}(\theta), j_{V*}(\theta))$, where j_{U*} and j_{V*} extend q-forms on $U\cap V$ by zero to q-forms on U and V, respectively. (*Remark.* Only the components j_{U*} and j_{V*} enter into this sign-commutativity equation, but the important concept is the *map* ι.)

Proof. This case is much easier. Recall the definition of the difference map $\delta_q(\omega, \tau) = \omega|_{U\cap V} - \tau|_{U\cap V}$. Then

$$\int_{U\cap V} \delta_q((\omega,\tau)) \wedge \theta = \int_{U\cap V} (\omega|_{U\cap V} - \tau|_{U\cap V}) \wedge \theta = \int_{U\cap V} \omega|_{U\cap V} \wedge \theta - \int_{U\cap V} \tau|_{U\cap V} \wedge \theta$$

$$= \int_{U\cap V} \omega|_{U\cap V} \wedge j_{U*}(\theta) + \int_{U\cap V} \tau|_{U\cap V} \wedge j_{V*}(\theta) = \int_U \omega \wedge j_{U*}(\theta) + \int_V \tau \wedge j_{V*}(\theta).$$

The proof for this type is complete.

Type 3:

$$\int_{U\cup V} \omega \wedge s((\tau_1,\tau_2)) = \pm\left(\int_U \omega|_U \wedge \tau_1 + \int_V \omega|_V \wedge \tau_2\right) \tag{19}$$

for each $[\omega] \in H^q(U\cup V)$, $([\tau_1],[\tau_2]) \in H_c^{n-q}(U) \oplus H_c^{n-q}(V)$,where $i^*(\omega) = (\omega|_U, \omega|_V)$ is as usual the image of ω under the restriction map in cohomology, and s is the sum map, sending (τ_1,τ_2) to the sum $\tau_1+\tau_2$ after extending each by zero to a form on $U\cup V$.

Proof. Calculate

$$\int_{U\cup V} \omega \wedge s((\tau_1,\tau_2)) = \int_{U\cup V} \omega \wedge (\tau_1+\tau_2) = \int_{U\cup V} \omega \wedge \tau_1 + \int_{U\cup V} \omega \wedge \tau_2.$$

Then

$$\int_{U\cup V} \omega \wedge s((\tau_1,\tau_2)) = \int_{U\cup V} \omega \wedge \tau_1 + \int_{U\cup V} \omega \wedge \tau_2 = \left(\int_U \omega|_U \wedge \tau_1 + \int_V \omega|_V \wedge \tau_2\right).$$

This is because τ_1 is extended by zero to $U\cup V$, so that $\omega\wedge\tau_1$ is zero on V, and we have that the integral of this form on $U\cup V$ is equal to its integral on U. Likewise for the integral of $\omega\wedge\tau_2$ by virtue of extension of τ_2 by zero to $U\cup V$. This completes the proof for Type 3.

Therefore the bilinear form $\mathcal{B}_q : H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R}$ is sign-commutative.

Returning to the above diagram, we have by the hypothesis that Poincare duality holds for U, V, and $U\cap V$ the vector space isomorphisms

$$H^k(U) \cong (H_c^{n-k}(U))^*,$$

$$H^k(V) \cong (H_c^{n-k}(V))^*,$$

and

$$H^k(U\cap V) \cong (H_c^{n-k}(U\cap V))^*,$$

for each $k \in \{0,1,\ldots,n\}$, and in particular for $k=q$ and $k=q-1$ as in the above diagram. The sign-commutativity of the bilinear form $\mathcal{B}_q : H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R}$ implies that each square in the above diagram commutes.

We conclude by the Short-Five Lemma that $H^q(U\cup V) \cong H_c^{n-q}(U\cup V)^*$, so we have Poincare duality for $M = U\cup V$. The proof of the claim is complete.

Step 2. Choose a finite good cover $\mathcal{C} = \{U_1, U_2, \ldots, U_p\}$ of M. We may do so because M is compact. We complete the proof by induction on the cardinality p of $\mathcal{C}$.

If $p=1$, then, because $\mathcal{C}$ is a good cover of M, we have that $M = U_1$ is diffeomorphic to $\mathbb{R}^n$ We use the Poincare Lemmas for the cohomology and compactly supported cohomology of Euclidean spaces. If $q=0$, then

$$H^q(M) \cong H^q(\mathbb{R}^n) = H^0(\mathbb{R}^n) \cong \mathbb{R} \cong H_c^n(\mathbb{R}^n) \cong H_c^n(M) = H_c^{n-q}(M) \cong (H_c^{n-q}(M))^*,$$

so Poincare duality holds if $q=0$. If $q \in \{1,\ldots,n\}$, then by the Poincare Lemmas,

$$H^q(M) \cong H^q(\mathbb{R}^n) = \{0\} = H_c^{n-q}(\mathbb{R}^n) \cong H_c^{n-q}(M) \cong H_c^{n-q}(M)^*.$$

We conclude that Poincare duality holds for M if $p = 1$.

Now suppose that Poincare duality holds for each compact smooth manifold with a finite good cover of cardinality less than or equal to p. Let $\{U_1, U_2, \ldots, U_{p+1}\}$ be a finite good cover of M with cardinality $p+1$. Then $M = (U_1 \cup \cdots \cup U_p) \cup U_{p+1}$, and $\{U_1, \ldots, U_p\}$ is a finite good cover of $U_1 \cup \cdots \cup U_p$, and of $(U_1 \cup \cdots \cup U_p) \cap U_{p+1}$. $\{U_{p+1}\}$ is a finite good cover of U_{p+1}. By the induction hypothesis, we have that Poincare duality holds for $U_1 \cup \cdots \cup U_p$,U_{p+1}, and $(U_1 \cup \cdots \cup U_p) \cap U_{p+1}$.

By Step 1 of the proof, we conclude that Poincare duality holds for $M = (U_1 \cup \cdots \cup U_p) \cup U_{p+1}$. We conclude by mathematical induction that Poincare duality holds for all compact smooth manifolds. $\square$

10 Künneth Theorem

8 October 2021

Juan Felipe We now want to get a formula for the cohomology of a product manifold, for this we will use again a Mayer Vietoris argument and proceed by induction.

Theorem 28. Suppose M is a smooth manifold that admits a finite good cover, and let F be smooth manifold. Consider the product manifold $M \times F$ and denote the projections by $p : M \times F \longrightarrow F$ and $\pi : M \times F \longrightarrow M$. Then the map $\psi : (\Omega^*(M) \bigotimes \Omega^*(F))_n \longrightarrow \Omega_n(M \times F)$ given by $\omega \otimes \tau \mapsto \pi^*\omega \wedge p^*\tau$ induces an isomorphism

$$H^*(M) \bigotimes H^*(F) \cong H^*(M \times F).$$

Let's clarify the expression $H^*(M) \bigotimes H^*(F) \cong H^*(M \times F)$ before moving on to the proof. If we have $V = \bigoplus_{n \in \mathbb{Z}} V_n$ and $W = \bigoplus_{n \in \mathbb{Z}} W_n$ two $\mathbb{Z}$-graded vector spaces, then the tensor product $V \bigotimes W$ has a $\mathbb{Z}$-grading that is compatible with the grading of V and W. Namely, we have $V \otimes W = \bigoplus_{n \in \mathbb{Z}} (V \bigotimes W)_n$ where $(V \bigotimes W)_n = \bigoplus_{i+j=n} V_i \bigotimes W_j$. Thus Künneth's theorem provides a collection of isomorphisms

$$H^n(M \times F) \cong \bigoplus_{i+j=n} H^i(M) \bigotimes H^j(F).$$

Notice that we can give $H^*(M) \bigotimes H^*(F)$ a chain complex structure by operating on simple tensors with $d(\omega \otimes \tau) = d\omega \otimes \tau + (-1)^{|\omega|} \omega \wedge d\tau$.

Exercise. Verify that the map $\psi : H^*(M) \bigotimes H^*(F) \longrightarrow H^*(M \times F)$ above is a chain map.

Proof. Let $\{U_p : p \in \Lambda\}$ be a finite good cover for M and let's proceed by induction on the cardinality of Λ. Suppose $|\Lambda| = 1$, then $M \cong \mathbb{R}^m$ for some m and so

$$H^i(M) \cong \begin{cases} \mathbb{R} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}.$$

Now, for any $i \neq 0$ and any j we have $H^i(M) \bigotimes H^j(F) = 0$ so

$$\bigoplus_{i+j=n} H^i(M) \bigotimes H^j(F) \cong \mathbb{R} \otimes H^n(F) \cong H^n(F) \cong H^n(M \times F)$$

since $\mathbb{R}$ is contractible and $M \times F$ is homotopic to F.

Now assume the statement of the theorem is true for $|\Lambda| < q$ and let $|\Lambda| = q$. Define $U = U_1 \cup \ldots \cup U_{q_1}$ and $V = U_q$, then $U \cap V = (U_1 \cap U_q) \cup \ldots \cup (U_{q_1} \cap U_q)$ and all of $U, V, U \cap V$ admit a good cover of at most $q-1$ open sets. By Mayer Vietoris we have the following long exact sequence.

$$\ldots \longleftarrow H^i(U\cap V) \longleftarrow H^i(U)\oplus H^i(V) \longleftarrow H^i(M) \xleftarrow{d^*} H^{i-1}(U\cap V) \longleftarrow \ldots$$

If we take tensor product with $H^j(F)$ at each step of the sequence, we obtain the following exact sequence.

$$\ldots \leftarrow H^i(U\cap V)\bigotimes H^j(F) \leftarrow (H^i(U)\oplus H^i(V))\bigotimes H^j(F) \leftarrow H^i(M)\bigotimes H^j(F) \xleftarrow{d^*} H^{i-1}(U\cap V)\bigotimes H^j(F) \leftarrow \ldots$$

Exercise. Prove that for any vector space W, the functor $\_\bigotimes W$ preserves exactness.

Exercise. Find an example of an exact sequence of abelian groups where exactness is not preserved by taking a tensor product.

Taking direct sums we obtain again an exact sequence as follows.

$$\ldots \leftarrow \bigoplus_{i+j=n} H^i(U\cap V)\otimes H^j(F) \leftarrow \bigoplus_{i+j=n} (H^i(U)\oplus H^i(V))\otimes H^j(F) \leftarrow \bigoplus_{i+j=n} H^i(M)\otimes H^j(F) \leftarrow \bigoplus_{i+j=n} H^{i-1}(U\cap V)\otimes H^j(F) \leftarrow \ldots$$

By the induction hypothesis we have isomorphisms $\psi|_{U\cap V}$ and $\psi|_U\otimes\psi|_V$ at each degree n, so we obtain the following diagram with exact rows (by Mayer Vietoris).

$$\begin{tikzcd}
\ldots & \bigoplus_{i+j=n} H^i(U\cap V)\otimes H^j(F) \arrow[l] \arrow[d, "\psi_{U\cap V}"] \arrow[r, phantom, "3"] & \bigoplus_{i+j=n} (H^i(U)\oplus H^i(V))\otimes H^j(F) \arrow[l] \arrow[d, "\psi_U\oplus\psi_V"] \arrow[r, phantom, "1"] & \bigoplus_{i+j=n} H^i(M)\otimes H^j(F) \arrow[l] \arrow[d, "\psi_M"] \arrow[r, phantom, "2"] & \bigoplus_{i+j=n} H^{i-1}(U\cap V)\otimes H^j(F) \arrow[l] \arrow[d, "\psi_{U\cap V}"] & \ldots \arrow[l] \\
\ldots & H^n((U\cap V)\times F) \arrow[l] & H^n(U\times F)\oplus H^n(V\times F) \arrow[l] & H^n(M\times F) \arrow[l] & H^{n-1}((U\cap V)\times F) \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

In order to obtain the isomorphism $\psi|_M$ we will apply the 5–lemma, so we need to verify that each of the squares 1, 2 and 3 commute. Let's look at them more closely.

$$\begin{tikzcd}
\bigoplus_{i+j=n} H^i(U\cap V)\bigotimes H^j(F) \arrow[d, "\psi_{U\cap V}"] \arrow[r, phantom, "3"] & (\bigoplus_{i+j=n} H^i(U)\bigotimes H^j(F))\oplus(\bigoplus_{i+j=n} H^i(V)\bigotimes H^j(F)) \arrow[l, "res_V-res_U"'] \arrow[d, "\psi_U\oplus\psi_V"] \\
H^n((U\cap V)\times F) & H^n(U\times F)\oplus H^n(V\times F) \arrow[l, "res_{V\times F}-res_{U\times F}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(\bigoplus_{i+j=n} H^i(U)\bigotimes H^j(F))\oplus(\bigoplus_{i+j=n} H^i(V)\bigotimes H^j(F)) \arrow[d, "\psi_U\oplus\psi_V"] \arrow[r, phantom, "1"] & \bigoplus_{i+j=n} H^i(M)\bigotimes H^j(F) \arrow[l, "(res_U{,}res_V)"'] \arrow[d, "\psi_M"] \\
H^n(U\times F)\oplus H^n(V\times F) & H^n(M\times F) \arrow[l, "(res_{V\times F}{,}res_{U\times F})"']
\end{tikzcd}$$

$$\begin{tikzcd}
\bigoplus_{i+j=n} H^i(M)\bigotimes H^j(F) \arrow[d, "\psi_M"] \arrow[r, phantom, "2"] & \bigoplus_{i+j=n} H^{i-1}(U\cap V)\bigotimes H^j(F) \arrow[l, "d^*"'] \arrow[d, "\psi_{U\cap V}"] \\
H^n(M\times F) & H^{n-1}((U\cap V)\times F) \arrow[l, "d^*"']
\end{tikzcd}$$

Notice that restriction commutes with pullback, thus the squares ① and ③ commute. Let $\{\rho_U, \rho_V\}$ be a partition of unity on M subordinate to $\{U, V\}$, then we obtain a partition of unity $\{\pi^*\rho_U, \pi^*\rho_V\}$ on $M \times F$ subordinate to $\{U \times F, V \times F\}$. With this we may describe the connecting homomorphism

$$d^*\omega = \begin{cases} -d(\rho_V \omega) & \text{on } U \\ d(\rho_U \omega) & \text{on } V \end{cases}.$$

Let $[\omega] \in H^{i-1}(U \cap V)$ and $[\tau] \in H^j(F)$ (we will omit the [ ] in the computations for simplicity), then

$$\begin{aligned} \psi d^*(\omega \otimes \tau) &= \psi(-d(\rho_U w) \otimes \tau) = -\pi^*(d(\rho_U w)) \wedge p^*\tau \\ d^*\psi(\omega \otimes \tau) &= d^*(\pi^*\omega \wedge p^*\tau) = -d(\pi^*\rho_U(\pi^*\omega \wedge p^*\tau) = -d(\pi^*(\rho_U\omega) \wedge p^*\tau) \\ &= -d(\pi^*(\rho_U\omega)) \wedge p^*\tau - (-1)^{|\pi^*\omega|}\pi^*(\rho_u\omega) \wedge d(p^*\tau) \\ &= -\pi^*(d(\rho_U\omega)) \wedge p^*\tau. \end{aligned}$$

Thus, the square ② also commutes.

□

11 Cech-deRham Cohomology

18 October 2021

Elise Askelsen

Suppose $\mathcal{U} = \{U_\alpha\}_{\alpha \in \lambda}$ is an open cover of a manifold, M where λ is countable and ordered. Let Ω be a deRham sheaf. Then $C(\mathcal{U}, \Omega)$ is a bicomplex.

Recall: In our proof that the sequence

$$\prod_{\alpha_0 \in \lambda} \Omega^*(U_{\alpha_0}) \xrightarrow{\ \delta\ } \prod_{\alpha_0 < \alpha_1} \Omega^*(U_{\alpha_0\alpha_1}) \xrightarrow{\ \delta\ } \prod_{\alpha_0 < \alpha_1 < \alpha_2} \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \longrightarrow \cdots$$

is exact, we obtain the below diagram where $U_{\alpha_0\alpha_1\ldots\alpha_n} = \bigcap_{k=0}^n U_{\alpha_0\ldots\alpha_k}$, d is the deRham differential, and $(\delta\omega)_{\alpha_0\ldots\alpha_n} := \sum_{i=0}^n (-1)^i \omega_{\alpha_0\ldots\hat{\alpha_i}\ldots\alpha_n}|_{U_{\alpha_0\ldots\alpha_n}}$

$$\begin{tikzcd}
{} & {} & {} \\
\prod_{\alpha_0 \in \lambda} \Omega^i(U_{\alpha_0}) \arrow[r, "\delta"] \arrow[u] & \prod_{\alpha_0 < \alpha_1} \Omega^i(U_{\alpha_0\alpha_1}) \arrow[r, "\delta"] \arrow[u] & \prod_{\alpha_0 < \alpha_1 < \alpha_2} \Omega^i(U_{\alpha_0\alpha_1\alpha_2}) \arrow[r] \arrow[u] & \cdots \\
\prod_{\alpha_0 \in \lambda} \Omega^{i-1}(U_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0 < \alpha_1} \Omega^{i-1}(U_{\alpha_0\alpha_1}) \arrow[r, "\delta"] \arrow[u, "(-1)^i d"] & \prod_{\alpha_0 < \alpha_1 < \alpha_2} \Omega^{i-1}(U_{\alpha_0\alpha_1\alpha_2}) \arrow[r] \arrow[u, "d"] & \cdots \\
{} \arrow[u] & {} \arrow[u] & {} \arrow[u]
\end{tikzcd}$$

Theorem 29. $r : H^*(M) \to H^*(C(\mathcal{U}, \Omega), D)$ is an isomorphism.

Note that our r is the map given in the following diagram from the map $\Omega^*(M) \to C^*(\mathcal{U}, \Omega)$

$$
\begin{tikzcd}
\Omega^{i+1}(M) \arrow[r, "r"] & \prod \Omega^{i+1}(U_\alpha) \arrow[r, "\delta"] & \cdots \\
\Omega^{i}(M) \arrow[u] \arrow[r, "r"] & \prod \Omega^{i}(U_\alpha) \arrow[u] \arrow[r, "\delta"] & \cdots \\
\Omega^{i-1}(M) \arrow[u] \arrow[r, "r"] & \prod \Omega^{i-1}(U_\alpha) \arrow[u] \arrow[r, "\delta"] & \cdots \\
{} \arrow[u] & {} \arrow[u] &
\end{tikzcd}
$$

First, we discuss an element in the collection $C(\mathcal{U}, \Omega)$. Visually, we have a collection

$$
\begin{tikzcd}
\bullet & & & & \\
\alpha_1 \arrow[u, "d"] \arrow[r, "\delta"] & \bullet & \bullet & \bullet & \bullet \\
\bullet & \alpha_2 \arrow[u, "d"] \arrow[r, "\delta"] & \bullet & \bullet & \bullet \\
\bullet & \bullet & \alpha_3 \arrow[u, "d"] \arrow[r, "\delta"] & \bullet & \bullet \\
\bullet & \bullet & \bullet & \alpha_4 \arrow[u, "d"] \arrow[r, "\delta"] & \bullet
\end{tikzcd}
$$

Note $\alpha_1 \in \prod_\alpha \Omega^i(U_\alpha)$, $\alpha_2 \in \prod_{\alpha_0 < \alpha_1} \Omega^i(U_{\alpha_0 \alpha_1})$.

Definition 11.1. The element α_i is a **cocycle** if $\delta\alpha_{i+1} = 0$

Definition 11.2. The element α_i is a **coboundary** of there exists a β_i such that $d\beta_i = \alpha_i$.

Next, we return to the proof of our statement.

Proof. **Claim 1: r is a chain map**First, we can note that by a prior lecture r is a chain map. This is specifically due to the fact that $r(d\omega) = \{d\omega|_{U_\alpha}\}_{\alpha \in \lambda}$ and $\delta(r\omega)) = 0$. This gives $Dr(\omega) = dr(\omega) = d(\{\omega_\alpha\}_{\alpha\in\lambda})$.

Claim 2: If $\phi \in C(\mathcal{U}, |Omega)$ is a D-cocycle, then $[\phi] = [\phi"]$ where $\phi"$ consists of elements in $\prod_\alpha \Omega(U_\alpha) \subset (\mathcal{U}, \Omega)$Next, suppose $\alpha \in C^i(\mathcal{U}, \Omega^*)$ is a cocycle where $\alpha = \{\alpha_{1,2}, ..., \alpha_n\}$. This gives that $D\alpha = 0$ so $\delta\alpha_n = 0$. (Visually this is represented in the grid as α_n has nothing to the right of it and hence must map to zero.)

In a formal argument, since δ is exact, there is some γ_n such that $\delta\gamma_n = \alpha_n$. Consider $\alpha - d\gamma_n$. First notice that in $C(\mathcal{U}, \Omega)$, $[\alpha - D\gamma_n] = [\alpha]$. Now $\alpha - D\gamma_n = \{\alpha_{1,2}, ..., \alpha_{n-1} \pm d\gamma_n\}$. This is once again a cocycle since implies that $D(\alpha - D\gamma_n) = D\alpha - D^2\gamma_n = 0 - 0 = 0$ so $D(\alpha - D\gamma_n) = 0$. Hence $\delta(\alpha^1_{n-1} + d\gamma_n) = 0$. From this, we can conclude that there exists some γ^1_{n-1} such that $\delta\gamma^1_{n-1} = \alpha_{n-1} + D\gamma_n$.

Then, continuing this process inductively, we obtain $\alpha - D_n - D\gamma^1_{n-1} - D\gamma^2_{n-2} - ... - D\gamma_0$. But this is in the same class as α and is an element in $\prod_\alpha \Omega(U_\alpha)$.

r is bijective For *surjectivity*: Let $\phi = \phi_1 + ... + \phi_n$. Then $D\phi = 0$ so $\delta\phi^1_1 = 0$. Hence $d\phi^1_1 = 0$ and $\delta\phi^1_1 = 0$. Therefore, $\phi^1_1 \in Ker(\delta) = Im(r)$ by the exactness of the rows. This shows that $r : H(M) \to H(C^*(\mathcal{U}, \Omega), D)$ is onto.

For *injectivity*: Suppose $r([\omega]) = 0$ with $[\omega] \in H(M)$. Hence $r(\omega) = D\phi$ where for this $\phi = \phi_1 + ... + \phi_n$ with $\delta\phi_n = 0$. Thus there exists α with $\delta\alpha = \phi_n$ giving $D(\phi - D\alpha) = D\phi = r(\omega)$

and $\phi^2 := \phi - D\alpha = \phi_1^2 + ... + \phi_{n-1}^2$. Continuing this process inductively, we find $\phi^n = \phi_1^n$ so $r(\omega) = D\phi^n = \pm\phi^n + \delta\phi^n = r(\omega) = 0$ since $\delta\phi^n = 0$. But $\delta\phi_1^n = 0$ gives that there exists $\beta \in \Omega^{|\omega|-1}$ with $r(\beta) = \phi_1^n$. Hence $d\beta = \omega$ since $r(d\beta - \omega) = 0$ and r is injective on forms so $d\beta - \omega = 0$. This gives $[\omega] = 0$ so r is injective.

This proves that $H(M) \cong H(C(\mathcal{U}, \Omega), D)$ via the map r. $\square$

11.1 A New Take on Homology

Now we reconsider the diagram with the following

$$\begin{tikzcd}
{} & {} & {} & \\
\Omega^2(M) \arrow[u] \arrow[r, "r"] & \prod \Omega^2(U_\alpha) \arrow[u] \arrow[r, "\delta"] & \prod \Omega^2(U_{\alpha_0\alpha_1}) \arrow[u] \arrow[r] & \cdots \\
\Omega^1(M) \arrow[u] \arrow[r, "r"] & \prod \Omega^1(U_\alpha) \arrow[u, "d"] \arrow[r, "\delta"] & \prod \Omega^1(U_{\alpha_0\alpha_1}) \arrow[u, "d"] \arrow[r] & \cdots \\
\Omega^0(M) \arrow[u] \arrow[r, "r"] & \prod \Omega^0(U_\alpha) \arrow[u, "d"] \arrow[r, "\delta"] & \prod \Omega^0(U_{\alpha_0\alpha_1}) \arrow[u, "d"] \arrow[r] & \cdots \\
 & Ker(d) \arrow[u] \arrow[r, "\delta"] & Ker(d) \arrow[u] \arrow[r] & \cdots \\
 & \prod F(U_{\alpha_0}) \arrow[u, "\mathcal{S}"] \arrow[r] & \prod F(U_{\alpha_0\alpha_1}) \arrow[u, "\mathcal{S}"] &
\end{tikzcd}$$

Note $Ker(d) \prod(\Omega^0(U_\alpha) = \prod C^\infty(U_\alpha) \to$ where $F(U_\alpha)$ are constant forms on U_α. Hence given a good cover, $U_\alpha \mathbb{R}^n$ so M is connected, then $F(U_\alpha)\mathbb{R}$.

We observe that a sequence is exact if its homology is zero.

Corollary 29.1. The vertical sequences (columns) are exact when $\mathcal{U}$ is a good cover.

Notice that this result follows from the Poincaré Lemma. This is because if $\mathcal{U}$ is a good cover, $U_{\alpha_0,...,\alpha_n} \begin{cases} \varnothing \\ \mathbb{R} \end{cases}$. Then, by the Poincaré Lemma, $H^i(U_{\alpha_0...\alpha_n}) = \begin{cases} \mathbb{R} & i = 0 \\ 0 & i \neq 0 \end{cases}$.

Therefore we have $H^i(\prod \Omega^i(U_{\alpha_0...\alpha_n})) = \prod H^i(U_{\alpha_0...\alpha_n}) = 0$ for $i > 0$. Note if $i = 0$, then the sequence is exact. Then, via the map δ,

$$C^1(\mathcal{U}, F) = \prod F(U_\alpha) \to \prod F(U_{\alpha_0\alpha_1} \to ...$$

Then $H(\mathcal{U}) := H(C(\mathcal{U}, F), \delta)$ is Cech Cohomology. Moreover, if $\mathcal{U}$ is a good cover, we have $H(\mathcal{U})H(C(\mathcal{U}, \Omega), D)H_{dR}(M)$ where $H(\mathcal{U})$ is the Cech Cohomology, $H(C(\mathcal{U}, \Omega), D)$ is the Cech-deRham Cohomology, and $H_{dR}(M)$ is the deRham Cohomology with the first isomorphism being from the map $\mathcal{S}$ while the second isomorphism is under the map r. Therefore, the cohomology theories agree.

Notice though that $H(\mathcal{U}$ is combiniatorial and if $\mathcal{U}$ is finite, $C(\mathcal{U}, F)$ is finite dimensional which implies that $H(\mathcal{U})$ and $H_{dR}(M)$ are also finite dimensional.

Moreover, if $\mathcal{U}$ and $\mathcal{U}'$ are good covers (even of homotopically equivalent manifolds), then $H^*(\mathcal{U})H_{dR}(M)H^*(\mathcal{U}')$.

12 Computations:

Next, we begin the practice of computations of the Cohomology of different spaces.

1. First, we compute $H^*(S^n)$.

   Recall from previously in class, we have found $H^n(S^1) = \begin{cases} \mathbb{R} & n = 0,1 \\ 0 & otherwise \end{cases}$

   Then, as a base case, consider when $n = 2$. Let $\{U, V\}$ be a good open cover of S^2 where U is an open set containing the north pole, V is an open set containing the south pole, and $U \cap V \neq \varnothing$. From this knowledge and by the Mayer Vietoris, we obtain the long exact sequence

$$\begin{tikzcd}
 & H^2(S^2) \arrow[r] & H^2(U) \oplus H^2(V) \arrow[r] & H^2(U \cap V) \arrow[r] & 0 \\
 & H^1(S^2) \arrow[r] & H^1(U) \oplus H^1(V) \arrow[r] & H^1(U \cap V) \arrow[ull] & \\
0 \arrow[r] & H^0(S^2) \arrow[r] & H^0(U) \oplus H^0(V) \arrow[r] & H^0(U \cap V) \arrow[ull] &
\end{tikzcd}$$

   Notice the following; $U \cap V S^1 \times (0,1)$ so by the Poincaré Lemma, we can conclude that $H^*(U) = H^*(S^1 \times (0,1)) H^*(S^1)$ which we have computed earlier. Moreover, $U \cong \mathbb{R}^2$ and $V \cong \mathbb{R}^2$. This gives that $H^*(U) \oplus H^*(V) = H^*(\mathbb{R}^2) \oplus H^*(\mathbb{R}^2)$.

   But we note that $H^*(\mathbb{R}^2) \oplus H^*(\mathbb{R}^2) = \begin{cases} \mathbb{R} \oplus \mathbb{R} & n = 0 \\ 0 & otherwise \end{cases}$. Hence $H^*(U) \oplus H^*(V) = \begin{cases} \mathbb{R} \oplus \mathbb{R} & n = 0 \\ 0 & otherwise \end{cases}$.

   Having gathered this information, we adapt our original long exact sequence to the following.

$$\begin{tikzcd}
 & H^2(S^2) \arrow[r, "h_1"] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
 & H^1(S^2) \arrow[r, "g_1"] & 0 \arrow[r, "g_2"'] & \mathbb{R} \arrow[ull, "\delta_1"'] & \\
0 \arrow[r] & H^0(S^2) \arrow[r, "f_1"'] & \mathbb{R}^2 \arrow[r, "f_2"'] & \mathbb{R} \arrow[ull, "\delta_0"'] &
\end{tikzcd}$$

   Now, by the exactness of the sequence, we know $Ker(f_1) = Im(f_2) = \mathbb{R}$. Therefore, $H^0(S^2) = \mathbb{R}$. Similarly, $Ker(0) = Im(f_1) = Ker(f_2) = Im(\delta_0) = Ker(g_1) = H^1(S^2) = 0$. Finally, $\mathbb{R} = Im(\delta_1) = Ker(h_1) = H^2(S^2)$ by the Short Five Lemma.

   Therefore, from this, we conclude that for $n = 2$,

$$H^*(S^2) = \begin{cases} \mathbb{R} & n = 0,2 \\ 0 & otherwise \end{cases}$$

$$\begin{tikzcd}
 & \mathbb{R} \arrow[r, "h_1"] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
 & 0 \arrow[r, "g_1"] & 0 \arrow[r, "g_2"'] & \mathbb{R} \arrow[ull, "\delta_1"'] & \\
0 \arrow[r] & \mathbb{R} \arrow[r, "f_1"'] & \mathbb{R}^2 \arrow[r, "f_2"'] & \mathbb{R} \arrow[ull, "\delta_0"'] &
\end{tikzcd}$$

Now, suppose for the sake of induction that for all $1 \leq k \leq n$,

$$H^*(S^k) = \begin{cases} \mathbb{R} & n = 0, k \\ 0 & otherwise \end{cases}$$

.

Let $\{U, V\}$ be a good open cover of S^{n+1} where U is an open set containing the north pole, V is an open set containing the south pole, and $U \cap V \neq \varnothing$. From this knowledge and by the Mayer Vietoris, we obtain the long exact sequence

$$\begin{tikzcd}
& H^n(S^n) \arrow[r] & H^n(U) \oplus H^n(V) \arrow[r] & H^n(U \cap V) \arrow[r] & 0 \\
& & & \cdots \arrow[ull] & \\
& \cdots \arrow[urr] & & & \\
& H^1(S^n) \arrow[r] & H^1(U) \oplus H^1(V) \arrow[r] & H^1(U \cap V) \arrow[ull] & \\
0 \arrow[r] & H^0(S^n) \arrow[r] & H^0(U) \oplus H^0(V) \arrow[r] & H^0(U \cap V) \arrow[ull] &
\end{tikzcd}$$

Notice the following; $U \cap V S^{n-1} \times (0,1)$ so by the Poincaré Lemma, we can conclude that $H^*(U) = H^*(S^{n-1} \times (0,1)) H^*(S^{n-1})$ which we have computed earlier. Moreover, $U \cong \mathbb{R}^2$ and $V \cong \mathbb{R}^2$. This gives that $H^*(U) \oplus H^*(V) = H^*(\mathbb{R}^2) \oplus H^*(\mathbb{R}^2)$.

But we note that $H^*(\mathbb{R}^2) \oplus H^*(\mathbb{R}^2) = \begin{cases} \mathbb{R} \oplus \mathbb{R} & n = 0 \\ 0 & otherwise \end{cases}$. Hence $H^*(U) \oplus H^*(V) = \begin{cases} \mathbb{R} \oplus \mathbb{R} & n = 0 \\ 0 & otherwise \end{cases}$.

Having gathered this information, we adapt our original long exact sequence to the following.

$$\begin{tikzcd}
& H^n(S^n) \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
& & & \cdots \arrow[ull] & \\
& \cdots \arrow[urr] & & & \\
& H^1(S^n) \arrow[r] & 0 \arrow[r] & 0 \arrow[ull] & \\
0 \arrow[r] & H^0(S^n) \arrow[r] & \mathbb{R}^2 \arrow[r] & \mathbb{R} \arrow[ull] &
\end{tikzcd}$$

Now, by the same argument as when $n = 2$, we can conclude that $H^0(S^n) = \mathbb{R}$ and $H^1(S^n) = 0$. Furthermore, for all $1 \leq j \leq n-1$, $H^j(S^n) = 0$ by the exactness of the sequence. Finally, $\mathbb{R} = Im(\delta_{n-1}) = Ker(f_{n_1}) = H^n(S^n)$ by the Short Five Lemma.

Therefore, from this, we conclude that for $n = 2$,

$$H^*(S^2) = \begin{cases} \mathbb{R} & * = 0, n \\ 0 & otherwise \end{cases}$$

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25 October

13 Cech Cohomology for S^1

27 October

Attempt 1 Let U be the open set covering the left half of the circle and V be the open set covering the right half of the circle.

We have the $U \cong \mathbb{R}, V \cong \mathbb{R}$, and $U \cap V \cong \mathbb{R} \oplus \mathbb{R}$.

Note that this cover from the Mayer-Vietoris argument is **not** a good cover.

Attempt 2 Now try covering S^1 with 3 open sets, U, V, and W. Since $U \cap V \cap W = \emptyset$ and $u \cap V \cong \mathbb{R}, V \cap W \cong \mathbb{R}, W \cap U \cong \mathbb{R}$, we can conclude that $\{U, V, W\}$ is a good cover.

By MV $H^*_{dR}(S^1) = \begin{cases} \mathbb{R}, & * = 0, 1 \\ 0, & * \neq 0, 1 \end{cases}$

Now consider the Cech Complex.

Cech Complex $(\star)$

$$\Pi_{\alpha \in \Lambda} C^{\text{const}}(U_\alpha) \xrightarrow{\delta} \Pi_{\alpha < \beta} C^{\text{const}}(U_{\alpha\ \beta}) \xrightarrow{\delta} \Pi_{\alpha < \beta < \delta} C^{\text{const}}(U_{\alpha\beta\delta})$$

Here we define C^{const} as follows:

Definition 13.1. $C^{\text{const}} = \{f \in C^\infty(U) \text{ and } \dfrac{\partial f}{\partial x^i} = 0\ \forall i\}$.

This is just a vector space. (Note that a constant function on a connected topological space is just a number).

For $U \subset \mathbb{R}^n$

$$\begin{aligned} C^{\text{const}}(U) &\to \mathbb{R} \\ f &\mapsto f(x) \text{for any } x \in \mathbb{R} \end{aligned}$$

Since $U = \{U, V, W\}$, $(\star)$ becomes

$$\begin{array}{cc} C^{\text{const}}(U) \oplus C^{\text{const}}(V) \oplus C^{\text{const}}(W) & (\text{deg } 0) \\ \downarrow \delta & \\ C^{\text{const}}(U \cap V) \oplus C^{\text{const}}(U \cap V) \oplus C^{\text{const}}(V \cap W) & (\text{deg0}) \\ \downarrow \delta & \\ 0 & \end{array}$$

Let ϕ_U, ϕ_V, ϕ_W be basis constant functions on open cover U, V, and W respectively i.e. $\phi_U(x) = 1, \ldots$ (or any constant value).

From the chain above, we have that $\mathbb{R}^3 \cong \mathbb{R} < \phi_{UW}, \phi_{UV}, \phi_{VW} >$

13.1 Computation

Definition 13.2. Let $(\delta\omega)_{\alpha_0,\ldots,\alpha_n} = \sum_{k=0}^{\infty} (-1)^* \omega_{\alpha_0,\ldots,\hat{\alpha_k},\ldots\alpha_n}$

Here $(\alpha_0, \ldots, \alpha_n)$ is the restriction.

$$\begin{array}{c} \omega \in \Pi\Omega^*(U_{\alpha_0,\ldots,\alpha_{n-1}}) \\ \Big\downarrow \\ \delta\omega \in \Pi\Omega^*(U_{\alpha_0,\ldots,\alpha_{n-1}}) \end{array}$$

Let $\phi = (\phi_U, \phi_V, \phi_W)$. Then, we have the following equations:

$$(\delta\phi)_{uV} = \phi_U|_{UV} - \phi_V|_{UV} \tag{20}$$
$$(\delta\phi)_{UW} = \phi_U|_W - \phi_W|_{UW} \tag{21}$$
$$(\delta\phi)_{VW} = \phi_V|_{UW} = \phi_W|_{VW} \tag{22}$$

Then, we have the following:

$$\vec{\phi_U} = (1 \cdot \phi_U, 0 \cdot \phi_V, 0 \cdot \phi_W)$$
$$\vec{\phi_V} = (0 \cdot \phi_U, 1 \cdot \phi_V, 0 \cdot \phi_W)$$
$$\vec{\phi_W} = (0 \cdot \phi_U, 0 \cdot \phi_V, 1 \cdot \phi_W)$$

$\vec{\phi_U}, \vec{\phi_v}, \vec{\phi_W} = C^{\vee\ 0}(U)$. Then, we have

$$\begin{aligned}\delta\vec{\phi_U} &= 1 \cdot \phi_{UV} + 1 \cdot \phi_{UW} + 0 \cdot \phi_W \\ \delta\vec{\phi_V} &= -1 \quad\ 0 \quad\ 1 \\ \delta\vec{\phi_W} &= 0 \quad -1 \quad -1\end{aligned}$$

Turning rows into columns, we have the following matrix for δ.

$$\delta = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

Now we do **row-reduction** to get rank of the δ-matrix.

$$0 \xrightarrow{\delta} C^{\vee\ 0} \underbrace{(U)}_{\mathbb{R}<\vec{\phi_U},\vec{\phi_V},\vec{\phi_W}>} \xrightarrow{\delta} C^{\vee\ 1}(U) \underbrace{\xrightarrow{\delta=0}}_{\ker(\delta)=\text{ everything}} 0$$

Set Up

$$H^0(U) = \frac{\ker\delta}{\text{im }\delta} = \frac{\ker}{0} - \ker(\delta) \subset C^{\vee 0}(U) = \text{nullspace}(\delta)$$

.

$$H^1(U) = \frac{\ker(\delta)}{\text{im }(\delta)} = \frac{C^{\vee 1}(U)}{(\delta)},\ \text{rk}H^1(U) = \ \text{rk}(C^{\vee\ 1}(U)) - \text{rk }(\text{im}(\delta)) = 3 - \text{rk}(\text{im}(\delta)) = \dim(\text{nullspace}\delta))$$

(The last equality follows by Poincare Duality).

Row Reduce

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ nullspace $(\delta)) = 1$. Note that $\phi_U + \phi_V - \phi_W$ generates the kernel.

Conclusion

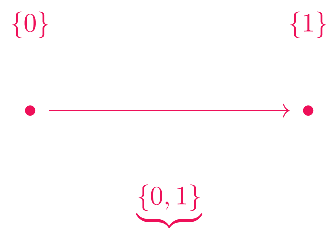
$$H^*_{\text{Cech}} = \begin{cases} \mathbb{R}, & * = 0, 1 \\ 0, & * \neq 0, 1 \end{cases}$$

14 Simplicial Homology

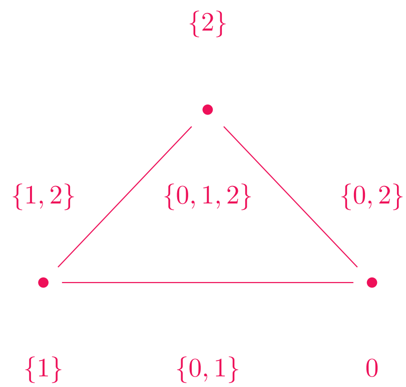
29 October

Definition 14.1. An **abstract simplicial complex** X is a collection of subsets of set S which satisfies:
If $Y \in X (\Rightarrow Y \subset S)$ and $Z \subset Y$, then $Z \subset X$ too.

Example 30. $S = \{0,1\}$, $X = \{\{0,1\},\{0\},\{1\},\emptyset\}$

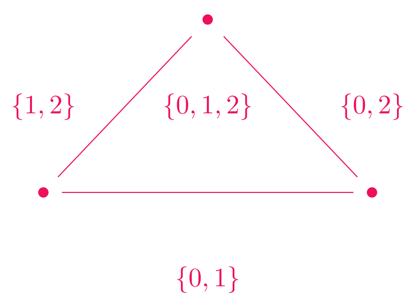


$S = \{0,1,2\}, X = P(S) = 2^S$. "2-simplex"
Example 31.

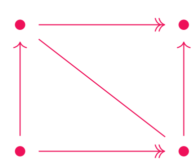


Example 32 (Simplicial Circle $P(\{0,1,2\} - \{0,1,2\})$).

$$\partial\triangle^2 = \{\{1,2\},\{1\},\{2\},\{0,2\},\{0\},\{2\},\{0,1\},\{0\},\{1\},\emptyset\}$$



Example 33 (CounterExample).



This does **NOT** come from an abstract simplicial complex.

Definition 14.2 (Geometric Realization)**.** $|X|$ geometric realization is defined as follwos:

$$|X| = \sqcup_{Y \in X} Y \times \triangle^{\#|Y|-1} / \sim$$

,
where Y is a point, $\triangle$ is a toplological space, and $\sim = \{\text{if} A \in X \text{and } A \subset Y, A \subset Y' \text{then} Y \times d_A \triangle^{\#Y-1} \sim Y' \times d_X \triangle^{\#Y-1}\}$
(In other words, we glue them along faces).

Definition 14.3.

$$\triangle^n = \{(x_0, x_1, ..., x_n) \in \mathbb{R}^n : nx_i \geq 0, \sum_{i=0} x_i = 1\}$$

Example 34. $\triangle^1$ is just a right triangle in 2D plane and ^{2} is a similar object in 3D plane with normal vector $< 1, 1, 1 >$.

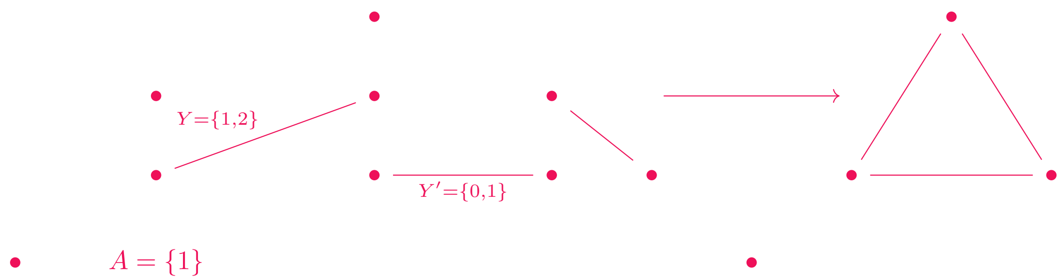
Let $v_i = \triangle^n \cap i^{th}$ axis. These have subspace topology.

Example 35. If $A \subset \{v_0, ..., v_n\}$ then there is a unique $\triangle^{\# n-1}$ simplex corresponding to A inside $\triangle^n$.

$$\triangle^{\# n-1} = d_A \triangle^n \subset \triangle^n$$

Example 36.

$$\begin{aligned} |\partial \triangle^2| &= \triangle^1_{\{1,2\}} \sqcup \triangle^1_{\{0,1\}} \sqcup \triangle^1_{\{0,2\}} \\ &= \sqcup \triangle^0_{\{1\}} \sqcup \triangle^0_{\{1\}} \sqcup \triangle^0_{\{2\}} / \sim \text{subset contained in 2 different sets} \end{aligned}$$



$$A \subset Y, A \subset Y'$$

If $U = \{U_\alpha\}_{\alpha \in \Lambda}$ is a good cover of M, then we can constant an abstract simplicial complex $N(U)$.

Definition 14.4 (The Nerve)**.** $N(U)$ called the **nerve** of U consists of the subsets of the indexing set Λ such that

$$Y \in N(U) \iff \cap_{\alpha \in Y} U_\alpha \neq \emptyset,$$

wherre $Y \subset \Lambda$.

Check That this is in fact an abstract simplicial complex.

If $z \subset Y$ and $Y \in N(U)$, then $\cap_{\alpha \in Y} V_\alpha \subset \cap_{\alpha \in z} U_\alpha$ and $\cap_{\alpha \in Y} V_\alpha \neq \emptyset \Rightarrow \cap_{\alpha \in z} U_\alpha \neq 0 \Rightarrow z \in N(U)$.

Definition 14.5 (Paracompact)**.** A topological space M is **paracompact** if every cover admits a **locally, finite subcover**.

A cover is locally finite if $\forall x \in M$, $\exists$ a $\{Uin\{U\} : B \subset U\} < \infty$ (this can be proven with partition of unity).

Theorem 37. If M paracompact and $\{U\}_{\alpha \in \Lambda}$ is a good dover then $|N(U)| \cong M$ (homotopy equivalent).

Theorem 38 (Folk Theorem). If X is an abstract simplicial space then there is a good cover of $|x|$ so $N(U) = X$.

1 November K abstract simplicial complex. $K \subset 2^{\Lambda}$ for some Λ.

For any simplicial complex, $K \rightsquigarrow |K|$ topological space.

$|K| = \{\alpha \in \mathrm{Map}(K\mathbb{R}_{\geq 0} : 1, 2\}$, where $1, 2$ are defined as follows:

1 If $\alpha \in |K|$ then $\{v | \alpha(v) \neq 0\} \in K$

2 If $\alpha \in |K|$ then $\sum_{v \in K^q} \alpha(v) = 1$.

Definition 14.6. If $\sigma \in K$ abstract simplicial complex. An orientation on $\sigma \in K, \sigma = \{v_0, ..., v_n\}$ is a choice of order $v_0 < v_1 < ... < v_n$.

Two such orientations are considered to be equivalent if they differ by an even permutation (permuation are either even or odd).

Let K be an abstract simplicial complex.

The set of oriented q-simplices is given by $OK^q = \{(\sigma, f) : r \to \{0, 1, .., q\} : \sigma \in K^q\}/\sim$, where the equivalence classes are given by

$(\sigma, f) \sim (\sigma, g)$ if $\exists\ \tau \in A_{q+1} \subset \mathrm{Sym}_{q+1}$ (Alternating group $\subset$ Symmetric Group).

So $\tau f = g$.

Definition 14.7. If K abstract simplicial complex, OK^q, $q = 0, 1, 2, ..$, then we have the free-module

$C_q(K, \mathbb{Z}) = \mathbb{Z} < (r, f) \in OK^q > ?(\sigma, f) = -\sigma(f))$

$C_q(K) = C_q(K, \mathbb{Z})$

$C_q(K) \xrightarrow{\partial_q} c_{q-1}(K)$

Definition 14.8. The **boundary map** (defined on generators) is given as follows: $((r, f)) = ([v_0, v_1, .., v_n])$ if $f(v_i) = i, \sigma = \{v_0, ..., v_n\}$. Then,

$$\partial_q([v_0, v_1, ..., v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, v_1, ..., \hat{v_i}, ..., v_n]$$

(Remove the ith vertex)

$C_q(K) \xrightarrow{\partial_q} C_{q-1}(K) \xrightarrow{\partial_{q-1}} C_{q-2}(K)$ with $C_q(K) \xrightarrow{0} C_{q-2}(K)$.

Note we still have $\partial_{q-1}\partial_q = 0\ \forall \mathbb{Z}_{\geq 1}$

So for every abstract simplicial complex K there is a chain complex:

$$H_i(K) := \frac{\ker \partial_i : C_i(K) \to C_{i-1}(K)}{\mathrm{im}\partial_{i+1} : C_{i+1}(K) \to C_i(K)}$$

Variation $C_0(K, R) := C(K; \mathbb{Z}) \otimes_{\mathbb{Z}} R$

$C_q(K; R) = R < OK^q / \sim$

$H_i(K, R) = \dfrac{\ker(\partial_i^R)}{\mathrm{im}(\partial_{i+1}^R)}$

This the "ith-homology of K with coefficients in the ring R).

Comment Different R can lead to different answers (this isn't obvious).

Remark $R = \mathbb{Z}/2$ has no signs!

3 November X abstract simplicial complex, $|X|$ topological space, $C_*(X, R)$ chain complex, $H_R(X, R)$ homology of spaces.

Theorem 39. $U = \{U_\alpha\}_{\alpha \subset \Lambda}$ is a good cover $\Rightarrow |N(U) \cong X$.

Definition 14.9. If Y is a space and X as above, then $|X| \cong Y$. Furture, $C_*(Y, R) = C_*(X, R)$ and $H_*(Y, R) = H_*(X, R)$. Next, recall

$C_q(X) := C_q(X, \mathbb{C})$

$... \to C_q(X) \xrightarrow{\partial_q} C_{q-1}(X) \xrightarrow{\partial_{q-1}} C_{q-2}(X) \to ...$
The Hom Functor comes into play.

$$d^{q-1} = \text{Hom}(\partial_q, A) = \partial^* q$$

$C^q(X, A) = \text{Hom}(C_q(X, \mathbb{Z}, A) = \text{Maps}(X^q, A)$
$f \in C^{q-1}(X, A), d^q(f) \in C^q(X, A), \sigma \in X^q$. Then, we have

$$\begin{aligned} d^q(f)(\sigma) &= f(\partial_q) \\ &= f(\sum_{i=1}^{q}(-1)^i d_i\sigma) \\ &= \sum_{i=0}^{q}(-1)^i f(d_i\sigma) \end{aligned}$$

If $\sigma \in X^q, \sigma = [v_0, .., v_q]$, then $d_i\sigma = [v_0, v_1, ..., \hat{v}_i, ..., v_q]$. Note the $d_i =$ ith face of σ.

Observe $d^q d^{q-1} = \partial^*_{q+1}\partial^*_q = (\partial_q \partial_{q+1})^* = 0^* = 0$.
So the dual of a chain complex (with decreasing differential) is the homology and the chain complex (with increasing differential) is the cohomology.

$H^*(X; A) = H_*(C^*(X; A), d^*)$ homology of the cochain complex is called the cohomology of X.

$H^q(X, A) = \dfrac{\ker d^{q+1} : C^q(X; A) \to C^{q+1}(X; A)}{\text{im} d^q : C^{q-1}(X; A) \to C^q(X; A)}$

If Y is a space, $|X| \cong Y$, then $H^*(Y; A) := H^*(X; A)$

If Y is a manifold, $U = \{U_\alpha\}_{\alpha \in \Lambda}$ is a good cover of $Y, N(U) = \{X \subset \Lambda : \cap_{\alpha \in X} U_{|alpha} \neq \emptyset\}$

Claim $H^*_{\text{simplicial}}(N(U), \mathbb{R}) \cong H^*_{\text{Cech}}(U)$

Know $H^{\vee\ *} \cong H^*_{dR}(Y)$. We also know that if Y paracompact, $|N(U)| \cong Y$ (geometric realization).
If cohomology is homotopy invariant, then $H^*(|N(U)| \cong H^*(Y)$.

Cech Complex

$$\begin{aligned} C^{\vee\ q}(U) &= \Pi F^{\text{constant}}(U_{\alpha_0,...,\alpha_q}) \\ C^q(N(U), \mathbb{R}) &= \text{Hom}(C_q(N(U)) < \mathbb{R}) \\ &= \text{Hom}(\oplus_{\sigma \in N(U)} \mathbb{Z}\sigma, \mathbb{R}) \\ &= \Pi_{\sigma \in N(U)^q} \text{Hom}(\mathbb{Z}_\sigma, \mathbb{R}) \\ &= \Pi_{\sigma \in N(U)^q} \mathbb{Z} < \text{Hom}(\sigma, \mathbb{R}) > \\ &= \Pi_{\alpha_0 < ... < \alpha_q} \mathbb{Z} < \text{Hom}(\sigma, \mathbb{R}) > \\ & \cap_{i=0}^q U_\alpha \neq 0 \end{aligned}$$

5 November

8 November

10 November

Tor	$\mathbb{Z}$	$\mathbb{Z}_n$
$\mathbb{Z}$	0	0
$\mathbb{Z}_m$	0	$\mathbb{Z}_{(m,n)}$

Ext	$\mathbb{Z}$	0
$\mathbb{Z}$	0	0
$\mathbb{Z}_m$	0	$\mathbb{Z}_{(m,n)}$

15 Summary of Homology Theories

Summary Here are the different types of homology theories we have studied.

1. de Rham $(H^*_{dR}(M)$ - Smooth Manifold
2. Compactly Supported $H^*_C(M)$
3. Cech Cohomology $H^\vee{}^*(M)$ (makes sense for any sheaf)
4. Simplicial homology
5. Cohomology

   If F is a presheaf $(U \to F(U)) \in$ Abelian Groups) and $U = \{U_\alpha\}_{\alpha\in\Lambda}$ any cover. Then $\exists$ a chain complex:

$$\Pi_\alpha F(U_\alpha) \xrightarrow{s} \Pi_{\alpha_0<\alpha_1} F(U_{\alpha_0\alpha_1} \xrightarrow{s}$$

   We have $\delta^2 = 0$ here as well.

   Homology is called Cech cohomology of U with coefficients in F for $H^\vee{}^*(U,F)$ if M is a space, F presheaf on M, then

$$H^\vee{}^* = \lim_U H^\vee{}^*(U;F)$$

   If U is a cover, V is a cover refining $M; V < U$, $\exists$ a map $H^\vee{}^* \to H^*(U,F)$.

FACT Good covers are final (otherwise, cofinal).

For any good cover, $H^\vee{}^*$

If F is a constant presheaf, then $U \mapsto F(U) = \mathbb{R}$.

$H^\vee{}^*(M,\mathbb{R})$ is the 1st version of Cech cohomology.

Relationship (1) Poincare Duality: When M is compact, $\dim M = m$, then

$$H^i_{dR}(M) \cong H^{m-i}_c(M)$$

(2) If U is a good cover, $H^\vee{}^*(U,F) \cong H^\vee{}^*(M;F)$

If $F = \mathbb{R}$ constant sheaf,

(combinatorial) $H^i_{dR}(M) \cong H^\vee{}^*(M;\mathbb{R}) \cong H^\vee{}^*(U;\mathbb{R})$.

Then,n $H^*(N(U),\mathbb{R}) \cong H^\vee{}^*(U,\mathbb{R})$ if U is a good cover of M and $|N(U)| = M$.

---

16 Universal Coefficients

$H^i(C;G) \cong \mathrm{Hom}(H_i(C), G) \oplus \mathrm{Ext}(H_{i-1}(C), G)$, where G is a free, Abelian group.
(This is NOT a natural map). This is a splitting of the short exact sequence i.e.
$... \to C_{i+1} \xrightarrow{d} C_i \xrightarrow{d} C_{i-1} \xrightarrow{d} ....$
$C_i \cong \mathbb{Z}^m$
Now let us look at the dual.

$$\leftarrow \mathrm{Hom}(C_{i+1}, G) \leftarrow \mathrm{Hom}(C_i, G) \leftarrow \mathrm{Hom}(C_{i-1}, G)$$

$$\downarrow$$

$$H^*(C, G)$$

Universal Coefficient $H^*(C, G) \iff \mathrm{Hom}(H_i(C), G)$
(Want both to be co or contravariant).

$\mathrm{Ext}(-,-)$: Input 2 Abelian groups $\to$ Get Abelian group:

$$\mathrm{Ext}(H \oplus H, G) = \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H, G)$$

Aside $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ SES and
$0 \leftarrow \mathrm{Hom}(A, G) \xleftarrow{\alpha^*} \mathrm{Hom}(B, G) \xleftarrow{\beta^*} \mathrm{Hom}(C, G) \leftarrow 0$

Question **Is this an Exact Functor?** NO. Hence we need the Ext. It is exactly that canonically defined functor that preserves exactness on the left.

$$\mathrm{Ext}(C, G)x \to \beta^*\mathrm{Ext}(B, G) \xrightarrow{\alpha^*} \mathrm{Ext}(A, G) \to 0$$

$\mathrm{Ext}^1(C, A) = \{0 \to A \to B \to C \to 0\}/\sim$
If $0 \to A \to B \to C \to 0$ then

$$\begin{array}{ccccccc}
\mathrm{Ext}^1(C, A) & \xleftarrow{\quad\delta\quad} & \mathrm{Hom}(A, A) & \longleftarrow & \mathrm{Hom}(B, A) & \longleftarrow & \mathrm{Hom}(C, A) \longleftarrow 0 \\
\downarrow & & & & & & \\
\mathrm{Ext}(B, A) & & & & & &
\end{array}$$

$\exists 1_A \in \mathrm{Hom}(A, A)$ and $\exists \Psi$ such that
$\Psi(0 \to A \to B \to C \to 0) := \delta(1_A \in \mathrm{Ext}(C, A))$.
"Measures extension"
Abelian Groups: $fg \cong \mathbb{Z}^m \oplus \mathbb{Z}_{m_1} \oplus ...$
$\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) \cong \mathbb{Z}/2$

Example 40. $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$$

$$\downarrow \Psi$$

$$\mathbb{Z}/2 = \{\Psi(1), \Psi(0))\}$$

12 November

$$\begin{tikzcd}
& C_i \arrow[r, "d_i"] \arrow[d] & C_{i-1} \arrow[r, "d_{i-1}"] & C_{i-2} \arrow[r] & {} \\
{} & \mathrm{Hom}(C_i, G) \arrow[l, "d^{i+1}"'] \arrow[d, "="'] & \mathrm{Hom}(C_{i+1}, G) \arrow[l, "d^i"'] \arrow[d, "="'] & G \arrow[l] & \\
{} & C^i \arrow[l, "d^{i+1}"] & C^{i+1} \arrow[l, "d^i"] & {} \arrow[l] &
\end{tikzcd}$$

$C^i \to H^i(C(G)) = \dfrac{\ker d^{i+1}}{\mathrm{im} d} \to \mathrm{Hom}(H_i(C), G)$
$f \in C^i = \mathrm{Hom}(C_i; G)$
$f : C_i \to G$ if $f \in H^i(C, G)$, then $f \in \ker(d^{i+1})$
$d^{i+1} f = 0 \Rightarrow d^{i+1} f(c) = 0 \Rightarrow f(d_{i+1})(c) = 0$
We want to use $f : C_i \to G$ to define a map:

$$\phi(f) : H_i(C) \to G$$
$$\frac{\ker(d_i)}{\mathrm{im}(d_i)} \to G$$

$c \in \ker d_i \subset c_i \iff \phi(f) = f(c)$.
If $c = d_{i+1}(c^i)$ then
$\phi(f)(c) = \phi(f)(d_{i+1}(c)) = f(d_{i+1}(c^i)) = 0$. So we have $\phi(f) : H_i(c) \to G$.

Exercise 1 Prove ϕ onto.

Exercise 2 Prove ϕ natural.

Exercise 3 Using Smith and our example below will give us a proofe of Universal Coefficient Theorem for cohomology.

Example 41.

$$\mathbb{Z} \to 0\mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0$$

$$H_i(C) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/m\mathbb{Z}, & i = 1 \\ 0, & i = 2\mathbb{Z}, , \\ \quad i = 3 \end{cases}$$

$$C^3 \leftarrow C^2 \leftarrow C^1 \leftarrow C^0 \leftarrow 0$$

$$0 \leftarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xleftarrow{0} \mathbb{Z} \xleftarrow{m} Z \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

$$H^i(C) = \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & i = 1 \\ \mathbb{Z}/m\mathbb{Z}, & i = 2 \\ \mathbb{Z}, & i = 3 \end{cases}$$

Cohomology is basically same as homology.
Torsion ($\mathbb{Z}/m\mathbb{Z}$) is shifted up by one from homology to cohomoloy.
Any chain complex of finite rank free abelian groups $C \cong \oplus\mathbb{Z}, \oplus_{i\in\mathbb{Z}}\mathbb{Z} \xrightarrow{m} \mathbb{Z}$
Example 42. $\cong \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z} \xrightarrow{>} \mathbb{Z})$
This follows from **Smith Normal Form**: $A : \mathbb{Z}^n \to\to Z^m$ is a $m \times n$ matrix with $a_1, ..., a_k$ among the first k diagonals and 0 everywhere else such that $a_i \in \mathbb{Z}$ and $a_i | a_{i+1}$.

16.1 Universal Coefficients Theorem For Cohomology

15 November

$$0 \to \mathrm{Ext}(H_{i-1}(X,G)) \xrightarrow{k} H^i(X,G) \to \mathrm{Hom}(H_i(X),G) \to 0$$

It i s natural.
If $f : X \to Y$ is a map of chain complexes, then there is a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{i-1}(X),G) \arrow[r] & H^i(X;G) \arrow[r] & \mathrm{Hom}(H_i(X),G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{i-1}(Y),G) \arrow[r] \arrow[u] & H^i(Y;G) \arrow[r] \arrow[u] & \mathrm{Hom}(H_i(Y),G) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

This SES is always split.
$H^i(X,G) \cong \mathrm{Hom}(H_i(X),G) \oplus \mathrm{Ext}(H_{i-1}(X),G)$
This isomorphism is NOT natural. i.e. the square consisting of $H^i(X,G), \mathrm{Hom}(H_i(X),G) \oplus \mathrm{Ext}(H_{i-1}(X))(G), H^i(Y,G), \mathrm{Hom}(H_i(Y),G) \oplus \mathrm{Ext}(H_{i-1}(Y))(G)$ is NOT going to commute.
Sometimes this behavior is called **non-canoncially split short exact sequences.**

Example 43. $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ (Not Split)
$\mathbb{Z} \ncong \mathbb{Z} \oplus \mathbb{Z}/2$

16.2 Universal Coefficients For Homology

There is a non-canonically split SES:
$0 \to H_n(X) \otimes G \to H_n(X;G) \to \mathrm{Tor}(H_n(X,G) \to 0$

1.

$\mathrm{Tor}(X,Y)$	$\mathbb{Z}$	$\mathbb{Z}/m$
$\mathbb{Z}$	0	0
$\mathbb{Z}_n$	0	$\mathbb{Z}_{(m,n)}$

2. Distributes over sums i.e. Tor $(\oplus X_i, \oplus Y_j) \cong \oplus \mathrm{Tor}(X_i, Y_j)$

Summarize Earlier, we had discussed Universal Coefficients for Cohomology: $H^n(X;G) \leftarrow \mathrm{Hom}(H_n(X;Z\mathbb{Z},G))$
Now, $H_n(X;\mathbb{Z}) \otimes G \to H_n(X;G)$
If we know $H_n(X;\mathbb{Z})$ for $n \neq n-1$ then we can compute $H_n(X;G)$ and $H^n(X;G)$.

17 Poincare Duality Simplicial Homology

From Bott's proof of Poincare Duality, we had

$$H_c^n(X) \cong H^{m-n}(X)^*$$

for X^m smooth, closed manifold where
$< \omega, \tau >$: $H_C^n(X) \otimes H^{m-n}(X) \to \mathbb{R}$. We looked at $\int_X \omega \wedge \tau$ for the proof earlier.
(We have discussed Combinatorial cohomology $\iff$ Cech cohomology. JNow let's add simplicial homology).

Definition 17.1. For X a closed simplicial manifold (equipped with a **triangulation**) i.e. equipped with a simplicial decomposition T, we have $|T| \cong X$.

There is a direct construction. If S is a simplex of T, then there is an $(m-i)$ simplex called DS of the dual triangulation:

$$DS = \cup_{S \le \triangle^m} DS \cap \triangle^m,$$

where ^{m} is the top dimensional simplex.
By $DS \cap \triangle^m$, we denoted the **complex hull of the barycenters of the subsets of vertices of that contain S.** (This is an important statement).

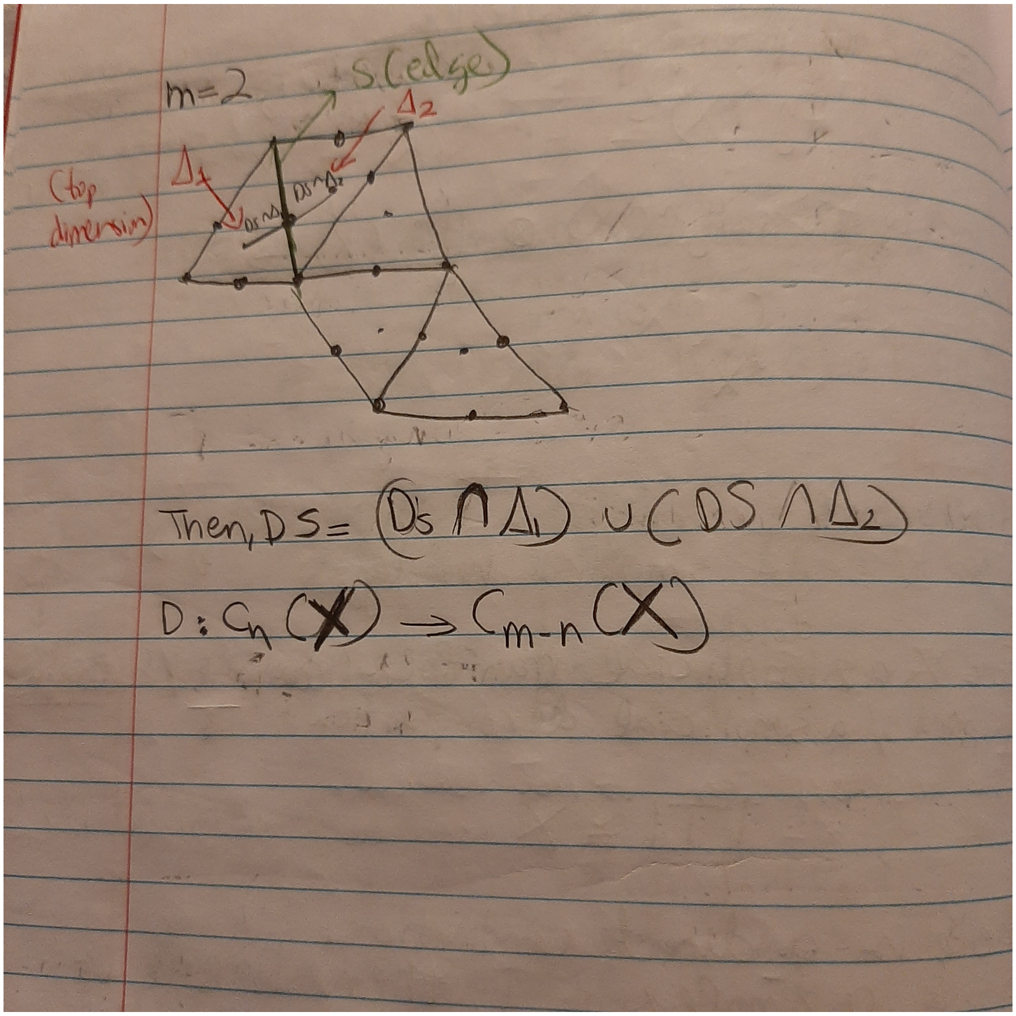
Definition 17.2. Recall an abstract simplicial complex is given by: Given (S, Λ) a collection $T \subset \Lambda$ (indexing set) such that if $T' \subset T \le S$, then $T' \in S$.
(BS, S)
0-simplex $BS^0 = S$
1-simplex $BS^1 = \{S_1, S_2\} \in (BS^0)^2 | S_1 \subset S_2\}$
2-simplex $BS^2 = \{(S_1, S_2, S_3\} \in (BS^0)^3 | S_1 \subset S_2 \subset S_3\}$
(This is the flag/filtration).

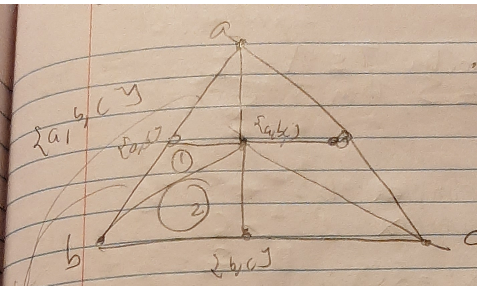
Proposition. If (S, Λ) is an abstract simplicial complex, then (BS, S) is an abstract simplicial complex.

Proof. If $T \subset BS^n$ for some $n \iff S_1 \subset S_2 \subset .... \subset S_n; S_i \in S, i = 1, ..., n$
Note that $T' \subset T$. Pick $1 \le i_1 \le i_2 \le ...i_k$ for $1 \le k \le n$.
T' is $S_{i_1} \subset S_{i_2} \subset ...S_{i_k}$.
By construction, $T' \in BS^k$. $\square$

17.1 Barycentric Subdivision

We will show this via images.





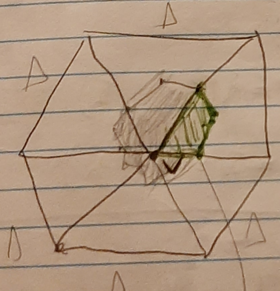
· S abstract simplicial complex

· new abstract simplicial complex

· barycentric subdivision

Issue w/ last Time: Triangulation Γ

duals to Simplices not a triangulation



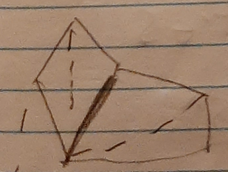
$$D_v = \bigcup_{v \in \Delta} (D_v \cap \Delta)$$

$= \Delta$

where $D_v \cap \Delta$ = "convex hull of all barycenters of simplices containing vertex v.

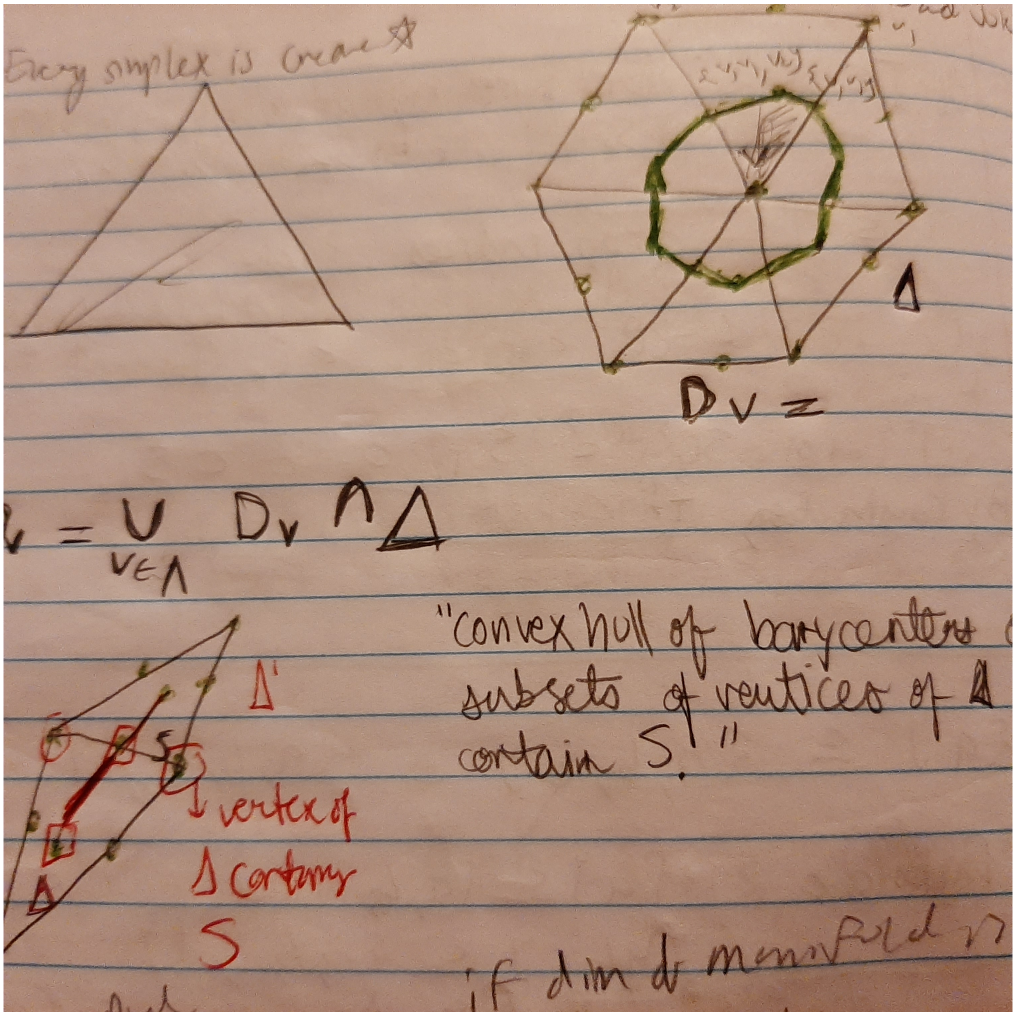
Not a 2-simplex

Want:



Will use:

Barycentric subdivision


Every simplex is
D v =
= ⋃ D v ∩ Δ
v∈Λ
"convex hull of barycenters
subsets of vertices of Δ
contain S! "
Δ'
S
vertex of
Δ containg
S
Δ
if dim d manifold

19 November Nitesh - Example on board - Poincare Duality Motivation via dual cell structures on Cube and Tetrahedron. You can watch the Youtube links below here for better description:

https://www.youtube.com/watch?v=aVHeWAJHRa0

https://www.youtube.com/watch?v=s_SIFNpOFcg

18 COMPUTATION Week 2

Recall that Poincare duality for abstract simplicial complexes states;

Theorem 44. Let M be an n-dimensional closed manifold equipped with a simplicial decomposition which we'll also call M. Then $H_i(M) \cong H^{n-i}(M)$ for all $0 \leq i \leq n$.

Notice that in order to prove this statement, we need to follow the given steps:

(a) Show that the intersection gives a chain map $\cap : C_*(M;R) \otimes C_*(M;R) \to R$. Note here $d(x \otimes y) == d(x) \otimes y + (-1)^{|x|} x \otimes d(y)$.

(b) Show that is $s \in C_i(M : R)$ then there is a canonical representative for $s, \overline{S} \in C_i(BM;R)$ in the barycentric subdivision. This defines an isomorphism of chain complexes

$$\bar{\cdot} : C_8(M;R) \to C_*(BM;R).$$

(c) Show that if $s \in C_i(M;R)$, then $Ds \in C_{n-i}(M;R)$ intersects $\overline{s}$ non-trivially: $s \cap Ds \neq 0$.

(d) Show that this implies the intersection is non-degenerate, and therefore induces an isomorphism $D : C_*(M) \to C^{n-*}(BM)$.

(e) Conclude, by combining the maps 2 and 4 to show that $H_i(M) H^{n-i}(M)$ for all $0 \leq i \leq n$.

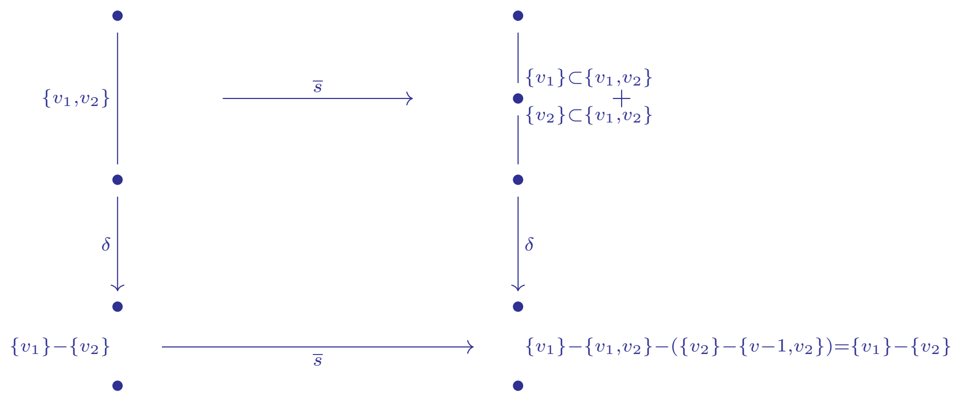
In class together, we focused mainly on the second item and will do so here. For simplicity, we will take $R = \mathbb{Z}/2$.

Let $s \in C_i(M;R)$ of degree n where we denote the vertices as $\{v_1, ..., v_n\}$. Now, consider the barycentric subdivision of M, BM. Let S_i denote the simplices in the barycentric subdivision. Intuitively, by considering a union of certain simplices in the barycentric subdivisions (the sum of flags in the barycentric subdivision), we can obtain our original simplex in M. Formally, we define that map $\bar{\cdot} : C_*(M;R) \to C_*(BM;R)$ where $s\overline{s} = \sum_{T=S} \sum S_0 \subset \cdots S_n \subset TS_0 \subset \cdots S_{n-1} \subset T$ where $|S_i| = |S_{i+1}| - 1$ for all $0 \leq i \leq n-1$. Then, we have to show that this is a chain map. To do this, recall that we have to show that $\bar{\cdot} \circ \delta = \delta \cdot \bar{\cdot}$ where δ is the boundary map $C_i(M;R) \to C_{i-1}(M;R)$. In order to show that $\bar{\cdot}$ is a chain map, we must show that the following diagram commutes:

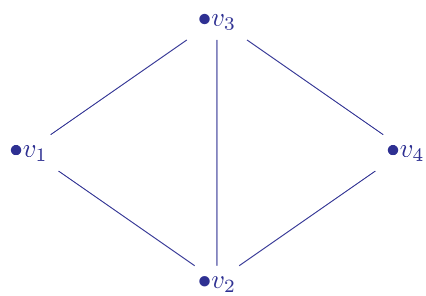
$$\begin{tikzcd}
C_i(M;R) \arrow[r, "\overline{s}"] \arrow[d, "\delta"] & C_i(BM;R) \arrow[d, "\delta"] \\
C_{i-1}(M;R) \arrow[r, "\overline{s}"] & C_{i-1}(BM;R)
\end{tikzcd}$$

To show this, we notice that for $s \in C_i(M;R)$, $\delta \circ \overline{s} = \delta(\sum_{T=S} \sum S_0 \subset \cdots S_n \subset TS_0 \subset \cdots S_{n-1} \subset T) = \sum_{j=1}^{n} (\sum_{T=S} \sum S_0 \subset \cdots S_n \subset TS_0 \subset \cdots \subset \hat{S_j} \subset \cdots \subset S_{n-1} \subset T)$. On the other hand, $\overline{s} \circ \delta(\sum_{j=1}^{n} \{v_1, ..., v_n\}) = \sum_{S_0 \subset \cdots S_n} (\{v_1, ..., \hat{v_j}, ...v_n\}) = \sum_{j=1}^{n} (\sum_{T=S} \sum S_0 \subset \cdots S_n \subset TS_0 \subset \cdots \subset \hat{S_j} \subset \cdots \subset S_{n-1} \subset T)$.

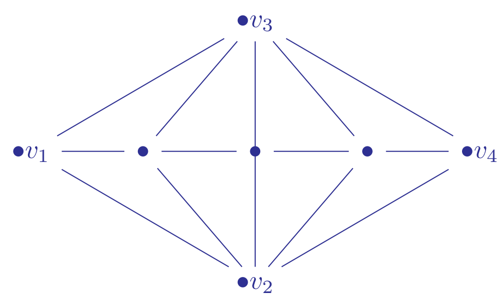
For the sake of understanding, we will do an example of making the diagram commute.



As a class, we also considered how to define the intersection of two simplicies towards the goal of finding a proof for d). In order to do so, consider the following example;



Taking the barycentric subdivision gives the following diagram;



Now, let $S = \{v_2, v_3\}$, i.e. the edge created by connecting v_2 and v_3. Then $\overline{s} = \{v_2\} \subseteq \{v_2, v_3\} + \{v_3\} \subseteq \{v_2, v_3\}$. Similarly, $Ds = \{v_2, v_3\} \subseteq \{v_2, v_3, v_4\} + \{v_2, v_3\} \subseteq \{v_2, v_3, v_1\}$. Pictorially, when we take the intersection, $Ds \cap \overline{s}$, we see that we in fact just get the vertex $\{v_2, v_3\}$ in the barycentric subdivision.

Next, consider the intersection of the following two flags which we can compute pictorially

$$\{v_2\} \subset \{v_2, v_3\} \subset \{v_2, v_3, v_4\} \cap \{v_3\} \subset \{v_2, v_3\} \subset \{v_2, v_3, v_4\} = \{v_2, v_3\} \subset \{v_2, v_3, v_4\}.$$

Another example, this time where the intersection is not the end of the flags can be

$$\{v_1\} \subset \{v_1, v_2\} \subset \{v_1, v_2, v_3\} \cap \{v_1\} \subset \{v_2, v_3\} \subset \{v_1, v_2, v_3\} = \{v_1\} \subset \{v_1, v_2, v_3\}.$$

Using this intuition, we can look at this intersection in more generality. First recall the following definitions;

$$\overline{s} = \sum_{T=S} \sum_{S_0 \subset \cdots S_{n-1} \subset T} S_0 \subset \cdots S_{n-1} T \text{ where } |S_i| = |S_{i+1}| - 1$$

$$Ds = \sum_{S\subset\Delta,\Delta\text{is max. dim.}} \sum_{S\subseteq\cdots\subset\Delta} S \subseteq \cdots \subset \Delta$$

Now we define the intersection of two flags in the following way;

Definition 18.1. Given two flags, $S_0 \subseteq \cdots \subset S_n$ and $T_0 \subseteq \cdots T_m$, their intersection is given by

$$(S_0 \subset \cdots \subset S_n) \cap (T_0 \subset \cdots T_m) = S_i \subset \cdots \subset S_j$$

where for all $i \leq r \leq j$, there exists some $T_k \in \{T_0, ..., T_m\}$ such that $S_r = T_k$.

We did note in class that this may work for the intersection of two chains, however in the case that we are taking the intersection of chains of flags, we may need to adapt this convention as when we let our ring $R = \mathbb{Z}/2$, then by how we have defined the intersection above, we find that the intersection, $\overline{s} \cap Ds = (\{v_2\} \subseteq \{v_2, v_3\} + \{v_3\} \subseteq \{v_2, v_3\}) \cap (\{v_2, v_3\} \subseteq \{v_2, v_3, v_4\} + \{v_2, v_3\} \subseteq \{v_2, v_3, v_1\}) = \{0\}$ i.e. is trivial. This would contradict both our conjecture that the intersection, $\overline{s} \cap Ds = S$ and the statement (which we know is true) that $\overline{s} \cap Ds \neq \varnothing$.

6 December 2021

19 Singular Homology/Cohomology

The are functions from Top $x \to H_*, H^*$ Graded Abelian Groups.
Let $f : X \to Y$ be ANY continuous map (may not be smooth), X, Y any topological space.
Then, H_* is a covariant functor defined from:

$$H_* : H_*(X) \to H_*(Y)$$

H^*, on the other hand, is a contravariant functor defined from:

$$H^* : H^*(Y) \to H^*(X)$$

These are big constructions because any continuous map or topological space is allowed.

Q. What is $S_*(X)$, the **singular chain complex of** X.

Definition 19.1.

$$S_*(X) = \oplus_{q=0}^{\infty} S_q(X)$$
$$S_q(X) = \mathbb{Z} < \sigma : \triangle^q \to X : \sigma \text{ continuous } >$$

Recall, the topological space $\triangle^q = \{(t_1, ..., t_{q+1}\} \in \mathbb{R}^{n+1}, t_i \geq 0, \sum t_i = 1\}$.
For example, $\triangle^2$ inherits the subspace topology from $\mathbb{R}^3$.
If $i = 1, ..., q+1$, ith face map, then

$$partialq^i : \triangle^{q-1} \to \triangle^q$$

where $\partial q^i(t_1, ..., t_q) = (t_1, ..., t_{i-1}, 0, t_i, ..., t_q$.

Definition 19.2. Given $\sigma : \triangle^q \to X \in S_q(X)$, then

$$\partial q\sigma = \sum_{i=1}^{q+1} (-1)^{i+1} \sigma \circ \partial q^i$$

(In general, $\sum_{i \in I} n_i\sigma_i \in S_q(X), \partial(\sum n_i\sigma_i) = \sum n_i \partial_q(\sigma_i)$
Proposition. $\partial^2 = 0$
$\partial_{q-1}\partial_q = 0 \ \forall q > 0$

Definition 19.3 (Singular Homology)**.**

$$H_q(X) = \frac{\ker(\partial q)}{\mathrm{im}(\partial_{q+1}}$$

abelian group (may not be free)

Exercise Prove the set $\{\sigma : \triangle^q \to X | \sigma \text{ continuous}\}$ is a simplicial complex.
Example 45. $S_q(X)$ is the simplicial chain complex.
If $f : X \to Y$ is a continuous map. Then,

$$f_* : S_{q*}(X) \to S_{q*}(Y)$$
$$f_*(\sigma) = f \circ \sigma \in S_{q*}(Y)$$

We can check that f_* is a chain map.

Observation 1 What is $H_0(X)$?

$$S_1(X) \xrightarrow{\partial} \xrightarrow{S}_0 (X) \xrightarrow{\partial} 0$$

given by

$$H_0(X) = \frac{\ker(\partial_0)}{\mathrm{im}\partial_1} = \frac{S_0(X)}{\mathrm{im}\partial_1}$$

Now we look at the elements of S_0.

$$\begin{aligned} S_0(X) &= \mathbb{Z} < \sigma : \triangle^0 \to X > \qquad (\mathbb{Z} \text{ linear combinations of points of } X). \\ &= \mathbb{Z} < x > \\ &= \oplus_{x \in X} \mathbb{Z} x \end{aligned}$$

Hence, $H_0(X) = \mathbb{Z} < x > / \mathrm{im}\partial_1$
Let $\tau - \triangle^1_{\text{interval}} \to X$. Then, $\partial\tau = \tau \circ \partial^1 - \tau_0 \partial^2$.
For example in a torus, $\partial\tau$ is the endpoints of interval. ("Any two points are equal if they are connected by a path")
Hence, we have

$$H_0(X) = \mathbb{Z} < X > / x_0 \sim x_1 \text{if } \exists \text{ an interval } \triangle^1 \xrightarrow{\tau} X \text{ and } \tau\partial^1 = x_0, \tau\partial^2 = x_1$$

i.e. rank $H_0(X) =$ number of path components of X.
$\star$ If a space X is path connected, then $H_0(X) = \mathbb{Z} < X >$

Observation 2 Let $X = \text{pt}$. Then,

$$S_0(\text{pt}) : \qquad \begin{array}{c} 0 \\ \uparrow \\ S_0(\text{pt}) = \mathbb{Z} \\ \uparrow \\ S_1(\text{pt}) = \mathbb{Z} \\ \uparrow \\ S_2(\text{pt}) = \mathbb{Z} \\ \uparrow \\ S_3(\text{pt}) = \mathbb{Z} \end{array}$$

because ther e is a unique continuous map
$\sigma_q :^q \to \text{pt}$.
$\sigma_q(t_1, ..., t_{q_n} = \text{pt}$.

$$\begin{aligned} \partial_q \sigma &= \sum_{i=1}^{n+1} (-1)^{i+1} \sigma_q \partial_q^i \\ &= \sum (-1)^{i+1} \sigma_{q+1} \\ &= \begin{cases} 0, & q \text{ even (everything cancels)} \\ \sigma_{q-1}, & q \text{ odd (only get one of them)} \end{cases} \end{aligned}$$

Chain Complex $\xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$
Then,

$$H_q \text{ pt } = \begin{cases} \mathbb{Z}, & q = 0 \\ 0, & q > 0 \end{cases}$$

In general,
(1) $\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z}$, we have $\dfrac{\ker 1}{\mathrm{im} 0} = 0$.
(2) $\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$, we have $\dfrac{\ker(0)}{\mathrm{im} 1} = \dfrac{\mathbb{Z}}{\mathbb{Z}} = 0$.

Poincare Lemma

$$H_q(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & q = 0 \\ 0, & q > 0 \end{cases}$$

$S_q(\mathbb{R}^n \to S_{q-1}(\mathbb{R}^n) \to ...$
The first one is $T = L\sigma$ and the second one is 0.
Next, we find a homotopy operator where $\partial L\sigma = \sigma$ and the boundary faces are $\dfrac{-L\partial\sigma}{\sigma}$

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8 December 2021 Let us sketch proof the Poincare Lemma from last time.
Let $\sigma : \triangle^q \to X \in S_q(X)$. Then, define $k\sigma : \triangle^{q+1} \to X \in S_{q+1}(X)$.
Then, $(\partial^i_{q+1} k_\sigma)(t_1, ..., t_q) = (1 - t_{q+1})\sigma(\frac{t_1}{x} ..., \frac{t_{i-1}}{x}, 0, \frac{t_{i+1}}{x}, ..., \frac{t_q}{x})$. Here $x = 1 - t_{q+1}$.
Then,

$$(k\partial^i_q \sigma)(t_1, ..., t_q) = (1 - t_q)\sigma(\frac{t_1}{x}, .., \frac{t_{i-1}}{q}, ..., \frac{t_q}{x})$$
$$\Rightarrow \partial k\sigma = \partial^1_{q+1} k\sigma - \partial^2_{q+1} k\sigma + ... + \partial^q_{q+1} k\sigma + \sigma$$
$$\Rightarrow k\partial\sigma = k\partial q^1\sigma - k\partial q^2\sigma + ... + \partial q^q\sigma + \sigma$$

$\Rightarrow$ " k is a homotopy." So $\partial k - k\partial 1_{S_q(\mathbb{R}^n)} - 0$
If the space is path connected, $H_0(\mathbb{R}^n) = \mathbb{Z}$.
If $[\sigma] \in H_q(\mathbb{R}^n)$
Pick σ such that $\partial\sigma = 0$.
$(\partial K - K\partial)\sigma = (1 - 0)\sigma$.
$\partial k\sigma = \sigma$.
$\exists$ an element $k_\sigma \in S_{q+1}(\mathbb{R}^k)$
So, $\partial(k_\sigma) = \sigma$.
So $\sigma \equiv 0$ in $H_q(\mathbb{R}^n), H_q(\mathbb{R}^n) = 0, q > 0$.
Corollary 45.1. If $f, g : X \to Y$ are homotopy equivalent maps of topological spaces then $f_* = g_* : H_q(X) \to H_q(Y)$ $\forall q \geq 0$.

M-V Let $U = \{U_\alpha\}_{\alpha \in \Lambda}$ open cover.
There is refinement $S^U_*(X)$ in terms of U. Then,
$S^U_*(X) = \oplus_{q \geq 0} S^U_q(X) = \mathbb{Z} < \sigma : \triangle^q \to X$ im $\sigma \subset U_\alpha$ for some $\alpha \in \Lambda¿$.

FACT (1)$i : S^U_*(X) \to S_*(X)$ is a chain map.
(2)i is a chain map.
Proof of (2) is tedious subdivision argument (via barycentric subdivision, estimate size, inverse image of open sets, repeat if not possible, get a linear bound).
Proposition. There is a chain complex:
$0 \leftarrow S^U_q(X) \xleftarrow{\epsilon} \oplus_\alpha S_q(U_{alpha}) \xleftarrow{\delta} \oplus_{\alpha_0 < \alpha_1}(U_{\alpha_0\alpha_1} \xleftarrow{\delta} S_q(U_{\alpha_0\alpha_1\alpha_2 \alpha_0 < \alpha_1 < \alpha_2}$
$\delta^2 = 0$ and this is exact.
$(\delta\sigma)_{\alpha_0, ..., \alpha_{p-1}} = \sum_{\alpha \in \Lambda} \sigma_{\alpha_0 ... \alpha_{p-1}}$
Exactness is proven by constructing a homotopy operator for δ.
If $|\Lambda| = 2$, then $U = \{U, V\}, X = U \cup V$
Then the chain complex becomes
$0 \leftarrow S^U_*(X) \leftarrow S_*(U) \oplus S_*(V) \leftarrow S_*(U \cap V) \leftarrow 0$ is a short exact sequence.
This gives us the M-V LES.
$H_i(U \cap V) \to H_i(U) \oplus H_i(V) \to H_i(X) \to H_{i-1}(U \cap V) \to H_{i-1}(U) \oplus H_{i-1}(V) \to ...$

10 December Let us look at another LES.
If $A \subset X$ is a subspace ($i : A \to X$ is continuous), denote this pair $A \subset X$ as (X, A).
Define SES as follows: $0 \to S_*(A) \xrightarrow{i_*} S_*(X) \xrightarrow{\pi}\to S_*(X)/S(A) = S_*(X, A) \to 0$

There is a LES: $H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{\pi_*} H_q(X, A) \to H_{q-1}(A) \to ...$
$H_q(X, A) = H_*(S_*(X, A))$ is a homotopy of quotient complex called “relative homology” (of pair (X, A)).
Ordinarily, $[\alpha] \neq 0 \in H_q(X)$
$\Rightarrow \partial\alpha = 0$ and $\nexists B$ such that $\partial B = \alpha$.
Example 46. Consider a torus with a path from α to β. Then,

$$\begin{aligned} \alpha &= \alpha_1 + \alpha + 2 \\ \partial\alpha &= \partial\alpha_1 + \partial\alpha_2 \\ &= b - a + a - b \\ &= 0 \\ \Rightarrow \partial B &= \alpha \end{aligned}$$

For $[\alpha] \in H_q(X, A), \alpha \in S_*(X), \partial\alpha \in S_*(A)$.
$\nexists \beta \in S_*(X)/S_n(X)$ such that $\partial B = \alpha$.
This is an idea relating to relative boundary and relative cycle.

FACTS (1) If $AsubsetX$ is **good** or (X, A) is a good pair i.e. $\exists U \subset X$ open, $A \subset U$ open such that
(i) $A \subset U$
(ii) U is a deformation retract to A with $r : U \to A, r^2 = r, r_i \equiv 1_A$.
Then, $H_q(X, A) = \begin{cases} H_q(X/A), & q > 0 \\ 0, & q = 0 \end{cases}$
(2) If $Z \subset A$, then $(X/Z)/(A/Z) = X/A$ (third isomorphism theorem)
Analogue Excision
$H_q(X, A) = H_q(X|Z, A|Z)$ (remove Z) when $Z \subset A$ is good.

There are analogues for everything here fo cohomology.

$$\begin{tikzcd} S_q(X) = \mathrm{Hom}(S_q(X), \mathbb{Z} \\ S^{q-1}(X) \arrow[u, "\partial^*"'] \end{tikzcd}$$

(Homology of this homology is cohomology)
$\partial^k \partial^k = (\partial^2)^* = 9$
$H^q(x) = H(S^*(X), \partial^*)$ (Universal Coefficients Theorem still holds).

There is more structure to cohomology. There is a product.
de Rham: wedge product ($H^*_{dR}(X)$ ring)
Singular homology: Cup Product
$X \xrightarrow{\triangle} X \times X$ given by $(x) = (x, x)$.

$$\begin{tikzcd} \triangle^* : H^*(X \times X) \arrow[r] \arrow[d, dash, "\cong(\text{via Kunneth})"'] & H^*(X) \\ H^*(X) \otimes H^*(X) \arrow[ur, "U"] & \end{tikzcd}$$

19.1 Eilenberg-Steenrod

$$H_* : \text{Top} \to \text{gr(Ab)} \text{ (graded Abelian groups)}$$
$$(X, A) \mapsto H_*(X, A), * > 0$$

uniquely characterized by

(1) $\exists\ \partial : H_q(X, A) \to H_{q-1}(A)$ giving a LES
$H_q(A) \to H_q(X) \to H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \to ...$

(2) $f \cong g \Rightarrow f_* = g_*$

(3) Excision Axiom.

$$Z \subset A \subset X \Rightarrow H_q(X, A) \cong H_q(X|Z, A|Z)$$

(4) Dimension Axiom.

$$H_q(\text{pt}, \emptyset) = \begin{cases} \mathbb{Z}, & q_* = 0 \\ 0, & q_* \neq 0 \end{cases}$$

What happens if we remove (4)? **Complex K-Theory** replaces (4).
$K(\text{pt}) = \mathbb{Z}[B, B^{-1}]$ (Laurent Polynomials $\cong \oplus\mathbb{Z}$

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20 Algebra Appendix

20.1 Chain Complexes

(Steven Un (started on 27 September 2021))

Definition 20.1 (Chain Complex)**.** A **chain complex** (E, d) of real vector spaces is a sequence $(E, d) = (E^i, d^i)_{i\in\mathbb{Z}}$, where for each integer index $i \in \mathbb{Z}$, E^i is vector space over $\mathbb{R}$ and $d^i : E^i \to E^{i+1}$ is an $\mathbb{R}$-linear map. such that $d^{i+1} \circ d^i = 0$ (that is, the composition of two successive maps is the zero map). We depict the chain complex $(E, d) = (E^i, d^i)_{i\in\mathbb{Z}}$ by the diagram:

... A^i A^{i+1} A^{i+2} ...

Remark 1. In Definition 6.1, the mathematical object in question is the sequence $(E^i, d^i)_{i\in\mathbb{Z}}$ of vector spaces and linear maps from each vector space into the next. The notation (E, d) is a shorthand notation, with the former sequence understood.

Remark 2. We choose the index set to be the set $\mathbb{Z}$ of all integers, to have the chain complex extend in both directions.

Remark 3. In the notation of Definition 6.1, the chain complex (E, d) is **bounded**, or **finite** if $E^i = \{0\}$ (that is, E^i is the zero vector space) for each $i \in \mathbb{Z} - \{1, 2 \ldots, n\}$ for some $n \in \mathbb{N}$. Of course this makes d^i the zero map for all $i < 1$ and all $i > n$.

Definition 20.2. Let $(E, d) = (E^i, d^i)_{i\in\mathbb{Z}}$ be a chain complex. For each $i \in \mathbb{Z}$, we have $\ker(d^i) = \{x \in E^i | d^i(x) = 0_{E^{i+1}}\}$ is a vector subspace of E^i. Also, $\mathrm{im}(d^{i-1})_=\{d^{i-1}(x) \in E^i | x \in E^{i-1}\}$ is a vector subspace of E^i.

The ith **cohomology group** of (E, d) is the quotient vector space

$$H^i(E) = \frac{\ker(d^i)}{\mathrm{im}(d^{i-1})}.$$

Definition 20.3. Let

$$(A, d_A) = (A^i, d^i_A)_{i\in\mathbb{Z}}$$

and

$$(A, d_A) = (E^i, d^i_B)_{i\in\mathbb{Z}}$$

be chain complexes. A **homomorphism** from (A, d_A) into (B, d_B) is a sequence of $\mathbb{R}$-linear maps $(f_i)_{i\in\mathbb{Z}}$, with $f_i : A^i \to B_i$ for each $i \in \mathbb{Z}$.

Definition 20.4. A homomorphism $(f_i)_{i\in\mathbb{Z}}$ from a chain complex (A, d_A) into a chain complex (B, d_B) is a **chain map** if for each $i \in \mathbb{Z}$, we have

$d^i_B \circ f_i = d^i_A \circ f_{i+1}$