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Discrete Optimization

Extension of the 2-p-opt and 1-shift algorithms to the heterogeneous probabilistic traveling salesman problem

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Abstract

The probabilistic traveling salesman problem is a well known problem that is quite challenging to solve. It involves finding the tour with the lowest expected cost given that customers will require a visit with a given probability. There are several proposed algorithms for the homogeneous version of the problem, where all customers have identical probability of being realized. From the literature, the most successful approaches involve local search procedures, with the most famous being the 2-p-opt and 1-shift procedures proposed by Bertsimas [D.J. Bertsimas, L. Howell, Further results on the probabilistic traveling salesman problem, European Journal of Operational Research 65 (1) (1993) 68–95]. Recently, however, evidence has emerged that indicates the equations offered for these procedures are not correct, and even when corrected, the translation to the heterogeneous version of the problem is not simple. In this paper we extend the analysis and correction to the heterogeneous case. We derive new expressions for computing the cost of 2-p-opt and 1-shift local search moves, and we show that the neighborhood of a solution may be explored in $O(n^2)$ time, the same as for the homogeneous case, instead of $O(n^3)$ as first reported in the literature.

Keywords: Combinatorial optimization; Probabilistic traveling salesman; Heuristics; Local search; Stochastic vehicle routing

1. Introduction

For many delivery companies, only a subset of their customers require a pickup or delivery each day. Information may be not available far enough in advance to create optimal schedules each day for those

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customers that do require a visit or the cost to acquire sufficient computational power to find such solutions may be prohibitive. For these reasons, it is not unusual to design a distance minimizing tour containing all customers, and each day follow the ordering of this a priori tour to visit only the customers requiring a visit that day. The problem of finding an a priori tour of minimum expected cost, given a set of customers each with a given probability of requiring a visit, defines the Probabilistic Traveling Salesman Problem (PTSP).

Jaillet [9] was the first to look at how to evaluate the expected cost of an a priori tour. In [9,10], Jaillet points out that the optimal TSP tour through a set of customers is often not the optimal tour in an expected value sense which means that the PTSP should be solved separately from the TSP.

Due to the probabilistic nature of the problem, the cost of evaluating a proposed solution for the PTSP is expensive. This, combined with the fact that TSP problems are already hard to solve, makes it quite challenging to solve PTSP problems to optimality. An exact algorithm is described in [11], but it has been applied primarily to small instances of the problem. Most approaches in the PTSP literature focus on heuristics that efficiently find good, but not necessarily optimal, solutions (see for instance [3,4,6,8] and the references cited therein). One crucial ingredient in these heuristic approaches is the design of an effective local search algorithm. In the PTSP, the use of an expected value-based cost to evaluate a local search move, rather than a standard TSP local search procedure, grows increasingly important as the number of customers increases [3]. Thus, it is critical that the expected value-based costs in the local search procedures are quick to evaluate.

In the literature, there are two local search procedures created specifically for the PTSP that evaluate a change in terms of expected value: the 2-p-opt and the 1-shift. The 2-p-opt is the probabilistic version of the famous 2-opt procedure created for the TSP [12]. In 2-opt, the portion of the tour between two specified customers is reversed. The 2-p-opt and the 2-opt are identical in terms of local search neighborhoods, but greatly differ in the cost computation. The change in the TSP objective value (the tour length) can be easily computed in constant time, while the same cannot be said for the PTSP objective value. The 1-shift is the evaluation of the change in expected value associated with removing a customer from the tour and inserting it at another point in the tour.

For PTSP instances where each customer is present with the same probability (the homogeneous PTSP), Bertsimas proposed move evaluation expressions in [3] that explore the neighborhood of a solution (that is, that verify whether an improving 2-p-opt or 1-shift move exists) in $O(n^2)$ time. The intent of Bertsimas' equations is to provide a recursive means to quickly compute the exact change in expected value associated with either a 2-p-opt or 1-shift procedure. Evaluating the cost of a local move by computing the cost of two neighboring solutions and then evaluating their difference would require much more time ($O(n^4)$) than a recursive approach. Recently Bianchi et al. [5] re-analyzed and corrected Bertsimas' expressions, after evidence emerged that they did not exactly evaluate the cost of a 2-p-opt and 1-shift move. The correction of these equations confirms that it is possible to explore both the 2-p-opt and 1-shift neighborhood of a solution in $O(n^2)$ time, and does, as expected, create significant improvement in the already good results for the homogeneous PTSP.

The heterogeneous version, where probabilities at the various customers are allowed to vary, is actually a more important problem because it is clearly closer to real world applications. As delivery companies gather and retain more information about their customers, heterogeneous probabilities are becoming increasingly available in practice and represent large potential savings. Few of the results in the literature apply, though, when probabilities are not homogeneous. One paper, [1], provides a lower bound for the heterogeneous PTSP, and another paper, [2], reports computational results of 2-p-opt and 1-shift local search algorithms applied to some small heterogeneous PTSP instances. The results in [2] are based on the work of Chervi [7], who proposed recursive expressions for the cost of 2-p-opt and 1-shift moves for the heterogeneous PTSP. Chervi's expressions explore the 2-p-opt and 1-shift neighborhoods in $O(n^3)$ time, suggesting that it is not possible to retain the $O(n^2)$ complexity of the homogeneous PTSP. Moreover, Chervi's expressions reduce to the incorrect expressions for the homogeneous PTSP published in [3], when all customer probabilities are equal, and therefore are also not correct.

In this paper, we extend and generalize the analysis performed in [5] for the homogeneous case to the heterogeneous case and derive new expressions for computing the cost of 2-p-opt (Section 3) and 1-shift (Section 4) local search moves. We demonstrate that the neighborhood of a solution for this important problem may be explored in O(n^2) time, thus retaining the same complexity as the homogeneous case. This shows we can take advantage of important additional information without adding computational complexity.

2. Notation and objective function

Throughout the paper we use the following notation. $N = \{i \,|\, i = 1, 2, \ldots, n\}$ is a set of n customers. For each pair of customers $i, j \in N$, $d(i, j)$ represents the distance between i and j. Here, we assume that the distances are symmetric, that is, $d(i, j) = d(j, i)$. In the remainder of the paper, distances will also be referred to as costs. An a priori tour τ is a permutation over N, that is, a tour visiting all customers exactly once. Without loss of generality, we consider $\tau = (1, 2, \ldots, n)$. Given the independent probability p_i that customer i requires a visit, $q_i = 1 - p_i$ is the probability that i does not require a visit. In the remainder of the paper we will use the following convention for any customer index i:

$$i := \begin{cases} i(\bmod n) & \text{iff } i \neq 0 \text{ and } i \neq n, \\ n & \text{otherwise,} \end{cases} \tag{1}$$

where $i(\bmod n)$ is the remainder of the division of i by n. The expected length of a priori tour $\tau = (1, 2, \ldots, n)$ can be computed in O(n^2) time with the following expression [9]

$$E[L(\tau)] = \sum_{i=1}^{n} \sum_{r=1}^{n-1} d(i, i+r) p_i p_{i+r} \prod_{i+1}^{i+r-1} q. \tag{2}$$

We use the following notation for any $i, j \in \{1, 2, \ldots, n\}$

$$\prod_{i}^{j} q := \begin{cases} \prod_{t=i}^{j} q_t & \text{if } 0 \leqslant j - i < n - 1, \\ \prod_{t=i}^{n} q_t \prod_{u=1}^{j} q_u & \text{if } i - j > 1, \\ 1 & \text{otherwise.} \end{cases} \tag{3}$$

The expression for the objective function (Eq. (2)) has the following intuitive explanation: each term in the summation represents the distance between the ith customer and the $(i + r)$th customer weighted by the probability that the two customers require a visit ($p_i p_{i+r}$) while the $r - 1$ customers between them do not require a visit ($\prod_{i+1}^{i+r-1} q$).

It is convenient here to introduce also the following two dimensional matrices of partial sums A and B that will be used as building blocks of the 2-p-opt and 1-shift evaluation expressions

$$A_{i,k} = \sum_{r=k}^{n-1} d(i, i+r) p_i p_{i+r} \prod_{i+1}^{i+r-1} q, \tag{4}$$

$$B_{i,k} = \sum_{r=k}^{n-1} d(i-r, i) p_{i-r} p_i \prod_{i-r+1}^{i-1} q, \tag{5}$$

with i and k being positions in the original tour where $1 \leqslant k \leqslant n-1$ and $1 \leqslant i \leqslant n$. The matrices A and B are straightforward extensions of corresponding matrices defined for the homogeneous case to the heterogeneous PTSP (see [5]).

3. 2-p-opt: Derivation of an efficient cost evaluation expression

For an a priori tour τ, its 2-p-opt neighborhood is the set of tours obtained by reversing a section of τ (that is, a set of consecutive nodes) such as the example in Fig. 1. Denote by $\tau_{i,j}$ a tour obtained by reversing a section $(i, i+1, \ldots, j)$ of τ, where $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, n\}$, and $i \neq j$. Note that if $j < i$, the reversed section includes n. Let $\Delta E_{i,j}$ denote the change in the expected tour length $E[L(\tau_{i,j})] - E[L(\tau)]$. We will derive a set of recursive formulas for $\Delta E_{i,j}$ that can be used to efficiently evaluate a neighborhood of 2-p-opt moves. To describe this procedure, we first introduce a few definitions. Let $S, T \subseteq N$ be subsets of nodes, with λ representing any a priori tour, and $\lambda(i)$ representing the customer in the ith position on this tour such that $\lambda = (\lambda(1), \lambda(2), \ldots, \lambda(n))$. The product defined by Eq. (3) can be easily generalized by replacing q_t with $q_{\lambda(t)}$ and q_u with $q_{\lambda(u)}$.

Definition 1. $E[L(\lambda)]_{|T\to S} = \sum_{\lambda(i)\in S, \lambda(j)\in T, i\neq j} d(\lambda(i), \lambda(j)) p_{\lambda(i)} p_{\lambda(j)} \prod_{i+1}^{j-1} q_{\lambda}$, that is, the contribution to the expected cost of λ due to the arcs from the nodes in S to the nodes in T.

Note that $E[L(\lambda)]_{|T\to S} = E[L(\lambda)]$, when $T = S = N$.

Definition 2. $E[L(\lambda)]_{|T\leftrightarrow S} = E[L(\lambda)]_{|T\to S} + E[L(\lambda)]_{|S\to T}$.

For the two a priori tours τ and $\tau_{i,j}$ we introduce

Definition 3. $\Delta E_{i,j|T\leftrightarrow S} = E[L(\tau_{i,j})]_{|T\leftrightarrow S} - E[L(\tau)]_{|T\leftrightarrow S}$, that is, the contribution to $\Delta E_{i,j}$ due to the arcs from the nodes in S to the nodes in T and from the nodes in T to the nodes in S.

Unlike the TSP, the expected cost of an a priori tour involves the arcs between all of the nodes. The ordering of the nodes on the a priori tour simply affects the probability of an arc being used, and this probability determines the contribution this arc makes to the expected cost of the tour. The change in expected tour length, $\Delta E_{i,j}$, resulting from a reversal of a section is thus based on the change in probability, or weight, placed on certain arcs in the two tours τ and $\tau_{i,j}$. While computing $\Delta E_{i,j}$ it is thus necessary to evaluate the weight change of *each* arc. The change in weight on an arc is influenced by how many of its endpoints are included in the reversed section. Because of this, it is useful to consider the following partitions of the node set.

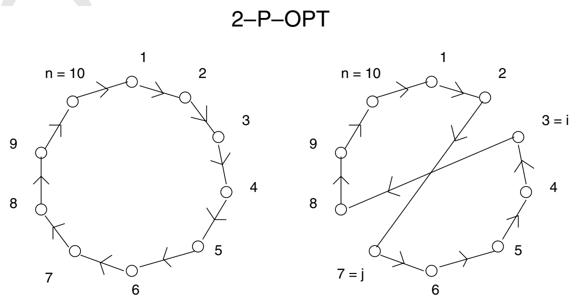

Fig. 1. Tour $\tau = (1, 2, \ldots, i, i+1, \ldots, j, j+1, \ldots, n)$ (left) and tour $\tau_{i,j} = (1, 2, \ldots, i-1, j, j-1, \ldots, i, j+1, \ldots, n)$ (right) obtained from τ by reversing the section $(i, i+1, \ldots, j)$, with $n = 10$, $i = 3$, $j = 7$.

Definition 4. $\text{inside}_{i,j} = \{i, \ldots, j\}$, that is, the section of τ that is reversed to obtain $\tau_{i,j}$.

Definition 5. $\text{outside}_{i,j} = N \setminus \text{inside}_{i,j}$.

Using the above definitions, $\Delta E_{i,j}$ may be expressed as

$$\Delta E_{i,j} = \Delta E_{i,j|\text{inside}_{i,j} \to \text{inside}_{i,j}} + \Delta E_{i,j|\text{outside}_{i,j} \to \text{outside}_{i,j}} + \Delta E_{i,j|\text{inside}_{i,j} \leftrightarrow \text{outside}_{i,j}}. \tag{6}$$

It is not difficult to verify that, as in the homogeneous PTSP [5], the contributions to $\Delta E_{i,j}$ due to $\Delta E_{i,j|\text{inside}_{i,j} \to \text{inside}_{i,j}}$ and to $\Delta E_{i,j|\text{outside}_{i,j} \to \text{outside}_{i,j}}$ are zero. The contribution to $\Delta E_{i,j}$ due to arcs between inside and outside (which is now equal to $\Delta E_{i,j}$) may be split into three components:

$$\Delta E_{i,j|\text{inside}_{i,j} \leftrightarrow \text{outside}_{i,j}} = E[L(\tau_{i,j})]_{|\text{inside}_{i,j} \to \text{outside}_{i,j}} + E[L(\tau_{i,j})]_{|\text{outside}_{i,j} \to \text{inside}_{i,j}} - E[L(\tau)]_{|\text{inside}_{i,j} \leftrightarrow \text{outside}_{i,j}}, \tag{7}$$

where the three terms on the right hand side of the last equation are, respectively, the contribution to $E[L(\tau_{i,j})]$ due to the arcs going from $\text{inside}_{i,j}$ to $\text{outside}_{i,j}$, the contribution to $E[L(\tau_{i,j})]$ due to the arcs going from $\text{outside}_{i,j}$ to $\text{inside}_{i,j}$, and the contribution to $E[L(\tau)]$ due to arcs joining the two customer sets in both directions. For compactness, these three components will be referenced hereafter by the notation:

$$E^{(1)}_{i,j} = E[L(\tau_{i,j})]_{|\text{inside}_{i,j} \to \text{outside}_{i,j}}, \tag{8}$$

$$E^{(2)}_{i,j} = E[L(\tau_{i,j})]_{|\text{outside}_{i,j} \to \text{inside}_{i,j}}, \tag{9}$$

$$E^{(3)}_{i,j} = E[L(\tau)]_{|\text{inside}_{i,j} \leftrightarrow \text{outside}_{i,j}}. \tag{10}$$

We may rewrite the expected tour length change $\Delta E_{i,j}$ as follows:

$$\Delta E_{i,j} = E^{(1)}_{i,j} + E^{(2)}_{i,j} - E^{(3)}_{i,j}. \tag{11}$$

Unlike the TSP, there is an expected cost associated with using an arc in a forward direction as well as a reverse direction, and these costs are usually not the same. The expected costs are based on which customers would have to be "skipped" in order for the arc to be needed in the particular direction. For example, the weight on arc $(1, 2)$ is based only on the probability of nodes 1 and 2 requiring a visit, whereas the weight on arc $(2, 1)$ is also based on the probability of nodes $(3, 4, \ldots, n)$ not requiring a visit. (The tour will travel directly from the 2 to 1 only if none of the rest of the customers on the tour are realized.) For the homogeneous PTSP, the equations are much simpler since the expected cost is based on the number of nodes that are skipped, not which nodes are skipped. This difference dictates the new set of equations we present here.

We will now derive recursive expressions for $E^{(1)}_{i,j}, E^{(2)}_{i,j}, E^{(3)}_{i,j}$, respectively, in terms of $E^{(1)}_{i+1,j-1}, E^{(2)}_{i+1,j-1}$ and $E^{(3)}_{i+1,j-1}$. These recursions are initialized with the expressions corresponding to entries (i, i) and $(i, i+1)$ for all i. We will derive these expressions quite easily later. First, let us focus on the case where $j = i + k$ and $2 \leqslant k \leqslant n - 2$. The case $k = n - 1$ may be neglected because it would lead to a tour that is reversed with respect to τ, and, due to the symmetry of distances, this reversed tour would have the same expected length as τ. Let us consider the tour $\tau_{i+1,j-1} = (1, 2, \ldots, i-1, i, j-1, j-2, \ldots, i+1, j, j+1, \ldots, n)$ obtained by reversing section $(i+1, \ldots, j-1)$ of τ. We can make three important observations. The first one is that the partitioning of customers with respect to $\tau_{i,j}$ is related to the partitioning with respect to $\tau_{i+1,j-1}$ in the following way:

$$\text{inside}_{i,j} = \text{inside}_{i+1,j-1} \cup \{i, j\}, \tag{12}$$

$$\text{outside}_{i,j} = \text{outside}_{i+1,j-1} \setminus \{i, j\}. \tag{13}$$

The second observation is that for any arc (l, r), with $l \in \text{inside}_{i+1,j-1}$ and $r \in \text{outside}_{i,j}$, the weight on the arc in the expected total cost equation for $\tau_{i,j}$ can be obtained by multiplying the weight that the arc has in $\tau_{i+1,j-1}$ by q_i and dividing it by q_j. One way to see this is to compare the set of skipped customers in $\tau_{i,j}$ with the set of skipped customers in $\tau_{i+1,j-1}$. In $\tau_{i,j}$, the fact that arc (l, r) is used implies that customers $(l-1, l-2, \ldots, i+1, i, j+1, \ldots, r-1)$ are skipped, while in $\tau_{i+1,j-1}$ using arc (l, r) implies that customers $(l-1, l-2, \ldots, i+1, j, j+1, \ldots, r-1)$ are skipped. Therefore, the set of skipped customers in $\tau_{i+1,j-1}$ is equal to the set of skipped customers in $\tau_{i,j}$ except for customer j in $\tau_{i+1,j-1}$, which is replaced by customer i in $\tau_{i,j}$. In terms of probabilities, our second observation can be expressed as

$$E[L(\tau_{i,j})]|_{\text{inside}_{i+1,j-1}\to\text{outside}_{i,j}} = \frac{q_i}{q_j} E[L(\tau_{i+1,j-1})]|_{\text{inside}_{i+1,j-1}\to\text{outside}_{i,j}}. \tag{14}$$

The third important observation is similar to the previous one, but it refers to arcs going in the opposite direction. More precisely, for any arc (r, l), with $r \in \text{outside}_{i,j}$ and $l \in \text{inside}_{i+1,j-1}$, the weight of the arc in $\tau_{i,j}$ can be obtained by multiplying the weight that the arc has in $\tau_{i+1,j-1}$ by q_j and by dividing it by q_i. It is not difficult to verify this using the same argument as in the previous observation. Similar to the second observation, the third observation can be expressed as

$$E[L(\tau_{i,j})]|_{\text{outside}_{i,j}\to\text{inside}_{i+1,j-1}} = \frac{q_j}{q_i} E[L(\tau_{i+1,j-1})]|_{\text{outside}_{i,j}\to\text{inside}_{i+1,j-1}}. \tag{15}$$

Now, by Eqs. (8), (9) and (12) we can write

$$E^{(1)}_{i,j} = E[L(\tau_{i,j})]|_{\text{inside}_{i+1,j-1}\to\text{outside}_{i,j}} + E[L(\tau_{i,j})]|_{\{i,j\}\to\text{outside}_{i,j}}, \tag{16}$$

$$E^{(2)}_{i,j} = E[L(\tau_{i,j})]|_{\text{outside}_{i,j}\to\text{inside}_{i+1,j-1}} + E[L(\tau_{i,j})]|_{\text{outside}_{i,j}\to\{i,j\}}. \tag{17}$$

By combining Eq. (14) with Eq. (16), we obtain

$$E^{(1)}_{i,j} = \frac{q_i}{q_j} E[L(\tau_{i+1,j-1})]|_{\text{inside}_{i+1,j-1}\to\text{outside}_{i,j}} + E[L(\tau_{i,j})]|_{\{i,j\}\to\text{outside}_{i,j}}, \tag{18}$$

which, by Eq. (13), becomes

$$E^{(1)}_{i,j} = \frac{q_i}{q_j} E[L(\tau_{i+1,j-1})]|_{\text{inside}_{i+1,j-1}\to\text{outside}_{i+1,j-1}} - \frac{q_i}{q_j} E[L(\tau_{i+1,j-1})]|_{\text{inside}_{i+1,j-1}\to\{i,j\}} + E[L(\tau_{i,j})]|_{\{i,j\}\to\text{outside}_{i,j}}. \tag{19}$$

We can rewrite this to obtain the following recursion:

$$E^{(1)}_{i,j} = \frac{q_i}{q_j} E^{(1)}_{i+1,j-1} - \frac{q_i}{q_j} E[L(\tau_{i+1,j-1})]|_{\text{inside}_{i+1,j-1}\to\{i,j\}} + E[L(\tau_{i,j})]|_{\{i,j\}\to\text{outside}_{i,j}}. \tag{20}$$

In an analogous way, we can create a recursive expression for $E^{(2)}_{i,j}$. By first combining Eq. (14) with Eq. (17), and then applying (13), we obtain

$$E^{(2)}_{i,j} = \frac{q_j}{q_i} E^{(2)}_{i+1,j-1} - \frac{q_j}{q_i} E[L(\tau_{i+1,j-1})]|_{\{i,j\}\to\text{inside}_{i+1,j-1}} + E[L(\tau_{i,j})]|_{\text{outside}_{i,j}\to\{i,j\}}. \tag{21}$$

Let us now focus on $E^{(3)}_{i,j}$. This term refers to the original tour τ. Therefore, in order to get a recursive expression in terms of $E^{(3)}_{i+1,j-1}$, we must isolate the contribution to $E^{(3)}_{i,j}$ due to arcs going from $\text{inside}_{i+1,j-1}$ to $\text{outside}_{i+1,j-1}$ and vice versa. Thus, by combining Eq. (10) with both (12) and (13) we obtain

$$E^{(3)}_{i,j} = E^{(3)}_{i+1,j-1} - E[L(\tau)]|_{\{i,j\}\leftrightarrow\text{inside}_{i+1,j-1}} + E[L(\tau)]|_{\{i,j\}\leftrightarrow\text{outside}_{i,j}}. \tag{22}$$

In Appendix A, we complete the derivation by showing that it is possible to express the 'residual' terms on the right hand side of $E^{(s)}_{i,j}$, $s = 1, 2, 3$ in Eqs. (20)–(22) in terms of the already defined matrices A, B , and Q, $\overline{Q}$, defined as follows

$$Q_{i,j} = \prod_{i}^{j} q, \ \overline{Q}_{i,j} = \prod_{j+1}^{i+n-1} q. \tag{23}$$

Expressing the $E_{i,j}^{(s)}$ expressions in terms of these defined matrices allows to minimize the number of calculations necessary in evaluating a neighborhood of local search moves. By substituting in Eqs. (20)–(22) the appropriate terms from Eqs. (44), (46), (48), (50) and (52), (54) from Appendix A, we obtain the following final recursive equations for the 2-p-opt local search for $j = i + k$ and $k \geqslant 2$:

$$\Delta E_{i,j} = E_{i,j}^{(1)} + E_{i,j}^{(2)} - E_{i,j}^{(3)}, \tag{24}$$

$$E_{i,j}^{(1)} = \frac{q_i}{q_j} E_{i+1,j-1}^{(1)} + q_i \frac{1}{Q_{i,j}} A_{i,k+1} - q_i \overline{Q}_{i,j}(A_{i,1} - A_{i,k}) - \frac{1}{q_j}\frac{1}{\overline{Q}_{i,j}} A_{j,n-k+1} + \frac{1}{q_j} Q_{i,j}(A_{j,1} - A_{j,n-k}), \tag{25}$$

$$E_{i,j}^{(2)} = \frac{q_j}{q_i} E_{i+1,j-1}^{(2)} - \frac{1}{q_i}\frac{1}{\overline{Q}_{i,j}} B_{i,n-k+1} + \frac{1}{q_i} Q_{i,j}(B_{i,1} - B_{i,n-k}) + q_j \frac{1}{Q_{i,j}} B_{j,k+1} - q_j \overline{Q}_{i,j}(B_{j,1} - B_{j,k}), \tag{26}$$

$$\begin{aligned} E_{i,j}^{(3)} = {} & E_{i+1,j-1}^{(3)} - A_{i,1} + A_{i,k} + A_{i,k+1} + A_{j,1} - A_{j,n-k} - A_{j,n-k+1} \\ & + B_{i,1} - B_{i,n-k} - B_{i,n-k+1} - B_{j,1} + B_{j,k} + B_{j,k+1}. \end{aligned} \tag{27}$$

For $k = 1$, we can express the three components of $\Delta E_{i,i+1}$ (8)–(10) in terms of A and B and obtain the following equations

$$E_{i,i+1}^{(1)} = \frac{1}{q_{i+1}} A_{i,2} + q_i(A_{i+1,1} - A_{i+1,n-1}), \tag{28}$$

$$E_{i,i+1}^{(2)} = q_{i+1}(B_{i,1} - B_{i,n-1}) + \frac{1}{q_i} B_{i+1,2}, \tag{29}$$

$$E_{i,i+1}^{(3)} = A_{i,2} + A_{i+1,1} - A_{i+1,n-1} + B_{i,1} - B_{i,n-1} + B_{i+1,2}. \tag{30}$$

For $j = i$, $\Delta E_{i,i} = 0$ since $\tau_{i,i} = \tau$. It is still necessary, though, to compute the three components $E_{i,i}^{(s)}$, $s = 1, 2, 3$ separately, in order to initiate the recursion $E_{i-1,i+1}^{(s)}$, $s = 1, 2, 3$. By expressing (8)–(10) in terms of A and B, we obtain

$$E_{i,i}^{(1)} = A_{i,1}, \tag{31}$$

$$E_{i,i}^{(2)} = B_{i,1}, \tag{32}$$

$$E_{i,i}^{(3)} = A_{i,1} + B_{i,1}. \tag{33}$$

Note that $\Delta E_{i,i} = E_{i,i}^{(1)} + E_{i,i}^{(2)} - E_{i,i}^{(3)} = 0$, as expected. It is possible to verify that when $p_i = p$ and $q_i = q = 1 - p$, we obtain the same recursive $\Delta E_{i,j}$ expressions as for the homogeneous PTSP in [5].

The 2-p-opt local search procedure for the heterogeneous PTSP is similar to the one for the homogeneous PTSP, with the main difference being that the three components of $\Delta E_{i,j}$ ($E_{i,j}^{(s)}, s = 1, 2, 3$) must now be computed separately. In both cases, the recursive calculations allow us to evaluate all possible 2-p-opt moves from the current solution in O(n^2) time. Such a procedure can be used to find the shift that creates the largest possible improvement in the current solution or to make shifts as improving moves (negative values) are encountered. The local search proceeds in two phases. The first phase consists of computing $\Delta E_{i,i+1}$ for every value of i (by means of Eqs. (24) and (28)–(30)). Note that for this purpose it is only necessary to compute two rows of the matrices A and B. Each time a negative $\Delta E_{i,i+1}$ value is encountered, the two nodes should immediately be switched. The n calculations of phase one require O(n) time apiece, or O(n^2) time in all. At the end of this phase, an a priori tour is reached for which every $\Delta E_{i,i+1}$ value is positive. Additionally, at the end of the first phase, the matrices A and B are re-computed, and Q and $\overline{Q}$ are computed (in O(n^2) time), so they can be used in the second phase (in O(1) time). The second phase of the local search consists of computing $\Delta E_{i,j}$ recursively by means of Eqs. (24)–(27). Since each $\Delta E_{i,j}$ in phase two is computed in O(1) time, this phase, and thus the entire 2-p-opt checking sequence, is performed in O(n^2). With a straightforward implementation that does not utilize recursion, evaluating the 2-p-opt neighborhood would require O(n^4) time instead of O(n^2). Since a local search procedure involves many iterations, these savings can lead to much better solutions.

The expression for $\Delta E_{i,j}$ derived by Chervi in [7] is of the form $\Delta E_{i,j} = \Delta E_{i+1,j-1} + \xi$. This greatly differs from our set of recursive equations. First, the ξ term, as derived in [7], is not computable in O(1) time but is O(n). Second, the expression derived in [7] is incorrect since it reduces to the incorrect 2-p-opt expression for the homogeneous PTSP published in [3] when all customer probabilities are equal [5].

4. 1-shift: Derivation of an efficient cost evaluation expression

Given an a priori tour τ, its 1-shift neighborhood is the set of tours obtained by moving a node which is at position i to position j of the tour, with the intervening nodes being shifted backwards one space accordingly, as in Fig. 2. Denote by $\tau_{i,j}$ a tour obtained from τ by moving node i to the position of node j and shifting backwards the nodes $(i+1,\ldots,j)$, where $i \in \{1,2,\ldots,n\}$, $j \in \{1,2,\ldots,n\}$, and $i \neq j$. Note that the shifted section may include n. Let $\Delta' E_{i,j}$ denote the change in the expected tour length $E[L(\tau_{i,j})] - E[L(\tau)]$. In the following, the correct recursive formula for $\Delta' E_{i,j}$ is derived for the 1-shift neighborhood. We will again focus on the features of the derivation of the 1-shift equations that are necessary to incorporate heterogeneous probabilities. A detailed derivation for the homogeneous version can be found in [5].

Let $j = i + k$. For $k = 1$, the tour $\tau_{i,i+1}$ obtained by 1-shift is the same as the one obtained by 2-p-opt, and the expression for $\Delta' E_{i,i+1}$ may be derived by applying the equations derived for the 2-p-opt. By summing Eqs. (28), (29) and by subtracting Eqs. (30) we find

$$\Delta' E_{i,i+1} = \left(\frac{1}{q_{i+1}} - 1\right) A_{i,2} + (q_{i+1} - 1)(B_{i,1} - B_{i,n-1}) + (q_i - 1)(A_{i+1,1} - A_{i+1,n-1}) + \left(\frac{1}{q_i} - 1\right) B_{i+1,2}. \quad (34)$$

We will now focus on the more general case where $2 \leqslant k \leqslant n-2$. Again, the case where $k = n-1$ can be neglected because it does not produce any change to the tour τ. We re-define the notions of *inside*, *outside* and the contributions to the change in expected tour length adapting them for the 1-shift.

Definition 6. $\Delta' E_{i,j|S \leftrightarrow T} = E[L(\tau_{i,j})]_{|T \leftrightarrow S} - E[L(\tau)]_{|T \leftrightarrow S}$. This is similar to Definition 3, the only difference being the meaning of the a priori tour $\tau_{i,j}$, that here is obtained from τ by a 1-shift move.

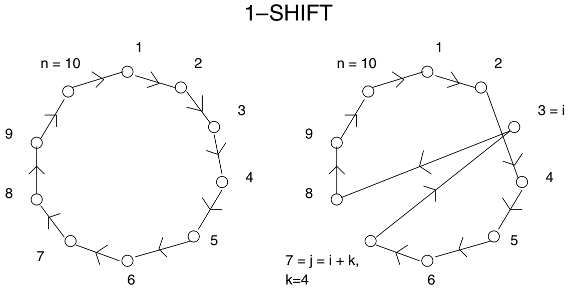

Fig. 2. Tour $\tau = (1,2,\ldots,i,\ i+1,\ldots,j,\ j+1,\ldots,n)$ (left) and tour $\tau_{i,j} = (1,2,\ldots,i-1,\ i+1,\ i+2,\ldots,j,\ i,\ j+1,\ldots,n)$ (right) obtained from τ by moving node i to position j and shifting backwards the nodes $(i+1,\ldots,j)$, with $n = 10$, $i = 3$, $j = 7$.

Definition 7. $\text{inside}_{i,j} = \{i+1, \ldots, j\}$, that is, the section of τ that is shifted to obtain $\tau_{i,j}$.

Definition 8. $\text{outside}_{i,j} = N \setminus (\text{inside}_{i,j} \cup \{i\})$.

It is not difficult to verify that the weights on arcs between outside nodes and arcs between inside nodes again do not change as a result of the shift. Therefore, the only contribution to $\Delta' E_{i,j}$ is given by the change in weight placed on arcs between $\text{inside}_{i,j} \cup \{i\}$ nodes and $\text{outside}_{i,j}$ nodes, and on arcs between node $\{i\}$ and $\text{inside}_{i,j}$ nodes, that is

$$\Delta' E_{i,j} = \Delta' E_{i,j|(\text{inside}_{i,j} \cup \{i\}) \leftrightarrow \text{outside}_{i,j}} + \Delta' E_{i,j|\{i\} \leftrightarrow \text{inside}_{i,j}}. \tag{35}$$

In the following, we derive a recursive expression for each of the two components of $\Delta' E_{i,j}$. Let

$$\Delta' E_{i,j|(\text{inside}_{i,j} \cup \{i\}) \leftrightarrow \text{outside}_{i,j}} = \Delta' E_{i,j-1|(\text{inside}_{i,j-1} \cup \{i\}) \leftrightarrow \text{outside}_{i,j-1}} + \delta, \tag{36}$$

and

$$\Delta' E_{i,j|\{i\} \leftrightarrow \text{inside}_{i,j}} = \Delta' E_{i,j-1|\{i\} \leftrightarrow \text{inside}_{i,j-1}} + \gamma. \tag{37}$$

Then, by Eq. (35), we can write the following recursive expression

$$\Delta' E_{i,j} = \Delta' E_{i,j-1} + \delta + \gamma. \tag{38}$$

In Appendix B, we complete the derivation by showing that it is possible to express the 'residual' terms δ and γ of Eq. (38) in terms of the already defined matrices A, B, and Q', $\overline{Q'}$, defined as follows

$$Q'_{i,j} = \prod_{i+1}^{j} q, \qquad \overline{Q'}_{i,j} = \prod_{j+1}^{i+n-1} q. \tag{39}$$

Expressing the $\Delta' E_{i,j}$ expressions in terms of these defined matrices allows to minimize the number of calculations necessary in evaluating a neighborhood of local search moves. By substituting the expression for δ (Eq. (60)) and γ (Eq. (63)) from Appendix B in Eq. (38), we obtain the following final recursive equations for the 1-shift local search for $j = i + k$ and $k \geqslant 2$:

$$\begin{aligned}
\Delta' E_{i,j} = {} & \Delta' E_{i,j-1} + \left(\overline{Q'}_{i,j} - \frac{1}{Q'_{i,j}}\right)(q_j A_{i,k} - A_{i,k+1}) + \left(\frac{1}{\overline{Q'}_{i,j}} - Q'_{i,j}\right)\left(B_{i,n-k} - \frac{1}{q_j} B_{i,n-k+1}\right) \\
& + \left(1 - \frac{1}{q_j}\right) Q'_{i,j} B_{i,1} + \left(\frac{1}{q_i} - 1\right) B_{j,k+1} + (1 - q_j)\overline{Q'}_{i,j} A_{i,1} + \left(1 - \frac{1}{q_i}\right) A_{j,n-k+1} \\
& + (q_i - 1)(A_{j,1} - A_{j,n-k}) + (1 - q_i)(B_{j,1} - B_{j,k}).
\end{aligned} \tag{40}$$

It is possible to verify that when $p_i = p$ and $q_i = q = 1 - p$, we obtain the same recursive $\Delta' E_{i,j}$ expressions as for the homogeneous PTSP in [5].

The 1-shift algorithm for the heterogeneous PTSP proceeds similarly as for the 2-p-opt. In the first phase of computation, $\Delta' E_{i,i+1}$ values for every i are computed by means of Eq. (34), while only the required rows of the matrices A and B are computed. At the end of the first phase, the matrices A and B are re-computed, and Q' and $\overline{Q'}$ are computed. This phase requires $O(n^2)$ time, the same as with 2-p-opt. The second phase of the local search consists of computing $\Delta' E_{i,j}$ values recursively by means of Eq. (40). Like 2-p-opt, since each $\Delta' E_{i,j}$ in phase two is computed in $O(1)$ time, this phase, and thus the entire 1-shift checking sequence, may be performed in $O(n^2)$. With a straightforward implementation that does not utilize recursion, evaluating the 1-shift neighborhood would require $O(n^4)$ time instead of $O(n^2)$.

The expression for $\Delta' E_{i,j}$ derived by Chervi in [7] is of the form $\Delta' E_{i,j} = \Delta' E_{i,j-1} + \xi'$. Again, the ξ' term, as derived in [7], is not computable in $O(1)$ time but requires $O(n)$. This expression is also incorrect since it reduces to the incorrect 1-shift expression for the homogeneous PTSP published in [3] when all customer probabilities are equal [5].

5. Conclusions

In this paper, we focused on the general PTSP problem where no assumption is made on the value of customer probabilities (heterogeneous PTSP). We have derived new expressions for the efficient computation of the expected cost of 2-p-opt and 1-shift local search moves. These derivations imply that it is possible to compute the cost evaluations of the entire neighborhood of a solution in $O(n^2)$ time, as in the homogeneous PTSP. Moreover, this result corrects the methods known in the literature and improves them by an $O(n)$ time factor. Future work will evaluate and compare alternate solution techniques for the heterogeneous PTSP. As this problem becomes increasingly important, so will the need for efficient, successful solution techniques.

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Appendix A

We first re-define the matrices A (Eq. (4)) and B (Eq. (5)) in terms of the matrix Q (Eq. (23))

$$A_{i,k} = \sum_{r=k}^{n-1} d(i, i+r) p_i p_{i+r} Q_{i+1,i+r-1}, \tag{41}$$

$$B_{i,k} = \sum_{r=k}^{n-1} d(i-r, i) p_{i-r} p_i Q_{i-r+1,i-1}. \tag{42}$$

Let us now focus on the 'residual' terms on the right hand side of $E^{(s)}_{i,j}$, $s = 1, 2, 3$ in Eqs. (20)–(22). Recalling that $j = i + k$, the second term on the right hand side of Eq. (20) is the following

$$\begin{aligned} -\frac{q_i}{q_j} E[L(\tau_{i+1,j-1})]|_{\text{inside}_{i+1,j-1} \to \{i,j\}} = & -\frac{q_i}{q_j} \sum_{t=1}^{k-1} d(i+t, i) p_{i+t} p_i Q_{i+1,i+t-1} \overline{Q}_{i,j-1} \\ & -\frac{q_i}{q_j} \sum_{t=1}^{k-1} d(i+t, i+k) p_{i+t} p_{i+k} Q_{i+1,i+t-1}. \end{aligned} \tag{43}$$

The right hand side of the above equation is in two pieces. In the first piece, the factor $\overline{Q}_{i,j-1}$ may be taken out from the sum and, by applying the definition of A from Eq. (41) to the remaining terms in the sum, we get $-\frac{q_i}{q_j}\overline{Q}_{i,j-1}(A_{i,1} - A_{i,k})$. Also the second piece can be expressed in terms of the A matrix, but it requires a bit more work. First, we substitute $(q_i/q_j)Q_{i+1,i+t-1}$ with $Q_{i,i+t-1}/q_j$. Then, we multiply and divide it by the product $q_j q_{j+1} \cdots q_{i+n-1}$, and we obtain the term $Q_{j+1,i+t-1}/Q_{j,i+n-1}$, whose denominator (which is equivalent to $\overline{Q}_{i,j-1}$) may be taken out from the sum. Finally by replacing $i + t$ with $j + n - k + t$, and by applying the definition of A to the remaining terms in the sum, the second piece of the right hand side of Eq. (43) becomes $\frac{1}{\overline{Q}_{i,j-1}} A_{j,n-k+1}$, and the whole Eq. (43) may be rewritten as

$$-\frac{q_i}{q_j}E[L(\tau_{i+1,j-1})]|_{\text{inside}_{i+1,j-1}\to\{i,j\}} = -\frac{q_i}{q_j}\overline{Q}_{i,j-1}(A_{i,1}-A_{i,k}) - \frac{1}{\overline{Q}_{i,j-1}}A_{j,n-k+1}. \tag{44}$$

The rightmost term of Eq. (20) may be written as

$$\begin{aligned}E[L(\tau_{i,j})]|_{\{i,j\}\to\text{outside}_{i,j}} = {} & \sum_{r=1}^{n-k-1} d(i,i+k+r)p_i p_{i+k+r}Q_{i+k+1,i+k+r-1}\\ & + \sum_{r=1}^{n-k-1} d(i+k,i+k+r)p_{i+k}p_{i+k+r}Q_{i+k+1,i+k+r-1}Q_{i,i+k-1}.\end{aligned} \tag{45}$$

By applying the definition of A from Eq. (41) to the right hand side of the last equation we obtain

$$E[L(\tau_{i,j})]|_{\{i,j\}\to\text{outside}_{i,j}} = \frac{q_i}{q_j}\frac{1}{Q_{i,j-1}}A_{i,k+1} + Q_{i,j-1}(A_{j,1}-A_{j,n-k}). \tag{46}$$

The second term on the right hand side of Eq. (21) is the following

$$\begin{aligned}-\frac{q_j}{q_i}E[L(\tau_{i+1,j-1})]|_{\{i,j\}\to\text{inside}_{i+1,j-1}} = {} & -\frac{q_j}{q_i}\sum_{t=1}^{k-1} d(i,i+k-t)p_i p_{i+k-t}Q_{i+k-t+1,i+k-1}\\ & -\frac{q_j}{q_i}\sum_{t=1}^{k-1} d(i+k,i+k-t)p_{i+k}p_{i+k-t}Q_{i+k-t+1,i+k-1}\overline{Q}_{i+1,j},\end{aligned} \tag{47}$$

which, by applying the definition of B from Eq. (42), becomes

$$-\frac{q_j}{q_i}E[L(\tau_{i+1,j-1})]|_{\{i,j\}\to\text{inside}_{i+1,j-1}} = -\frac{q_j}{q_i}\frac{1}{\overline{Q}_{i,j-1}}B_{i,n-k+1} - \overline{Q}_{i,j-1}(B_{j,1}-B_{j,k}). \tag{48}$$

The rightmost term of Eq. (21) may be written as

$$E[L(\tau_{i,j})]|_{\text{outside}_{i,j}\to\{i,j\}} = \sum_{r=1}^{n-k-1} d(i-r,i)p_{i-r}p_i Q_{i-r+1,i-1}Q_{i+1,i+k} + \sum_{r=1}^{n-k-1} d(i-r,i+k)p_{i-r}p_{i+k}Q_{i-r+1,i-1}. \tag{49}$$

By applying the definition of B from Eq. (42) to the right hand side of the last equation we obtain

$$E[L(\tau_{i,j})]|_{\text{outside}_{i,j}\to\{i,j\}} = \frac{q_j}{q_i}Q_{i,j-1}(B_{i,1}-B_{i,n-k}) + \frac{1}{Q_{i,j-1}}B_{j,k+1}. \tag{50}$$

The second term on the right hand side of Eq. (22) is the following

$$\begin{aligned}-E[L(\tau)]|_{\{i,j\}\leftrightarrow\text{inside}_{i+1,j-1}} = {} & -\sum_{t=1}^{k-1} d(i,i+t)p_i p_{i+t}Q_{i+1,i+t-1} - \sum_{t=1}^{k-1} d(i+k,i+t)p_{i+k}p_{i+t}Q_{i-n+k+1,i+t-1}\\ & -\sum_{t=1}^{k-1} d(i+k-t,i)p_{i+k-t}p_i Q_{i+k-t+1,i+k-1}Q_{i+k,i+n-1}\\ & -\sum_{t=1}^{k-1} d(i+k-t,i+k)p_{i+k-t}p_{i+k}Q_{i+k-t+1,i+k-1},\end{aligned} \tag{51}$$

which, by applying the definition of A (Eq. (41)) and B (Eq. (42)), becomes

$$-E[L(\tau)]|_{\{i,j\}\leftrightarrow\text{inside}_{i+1,j-1}} = -\left(A_{i,1}-A_{i,k}+A_{j,n-k+1}+B_{i,n-k+1}+B_{j,1}-B_{j,k}\right). \tag{52}$$

The rightmost term of Eq. (22) is the following

$$
\begin{aligned}
E[L(\tau)]_{|\{i,j\}\leftrightarrow \mathrm{outside}_{i,j}} &= \sum_{r=1}^{n-k-1} d(i-r,i)p_{i-r}p_iQ_{i-r+1,i-1} + \sum_{r=1}^{n-k-1} d(i-r,i+k)p_{i-r}p_{i+k}Q_{i-r+1,i+k-1} \\
&\quad + \sum_{r=1}^{n-k-1} d(i,i+k+r)p_ip_{i+k+r}Q_{i+1,i+k+r-1} + \sum_{r=1}^{n-k-1} d(i+k,i+k+r)p_{i+k}p_{i+k+r}Q_{i+k+1,i+k+r-1},
\end{aligned}
\tag{53}
$$

which, by applying the definition of A (Eq. (41)) and B (Eq. (42)), becomes

$$
E[L(\tau)]_{|\{i,j\}\leftrightarrow \mathrm{outside}_{i,j}} = B_{i,1} - B_{i,n-k} + B_{j,k+1} + A_{i,k+1} + A_{j,1} - A_{j,n-k}. \tag{54}
$$

Appendix B

We first re-define the matrices A (Eq. (4)) and B (Eq. (5)) in terms of the matrix Q' (Eq. (39))

$$
A_{i,k} = \sum_{r=k}^{n-1} d(i,i+r)p_ip_{i+r}Q'_{i,i+r-1}, \tag{55}
$$

$$
B_{i,k} = \sum_{r=k}^{n-1} d(i-r,i)p_{i-r}p_iQ'_{i-r,i-1}. \tag{56}
$$

Let us now focus on the ‘residual’ term δ from (36). The contribution to $\Delta' E_{i,j}$ due to arcs between inside$_{i,j} \cup \{i\}$ and outside$_{i,j}$ for $\tau_{i,j}$ is the following:

$$
\begin{aligned}
\Delta' E_{i,j|(\mathrm{inside}_{i,j}\cup\{i\})\leftrightarrow \mathrm{outside}_{i,j}} &= \sum_{r=1}^{n-k-1} \left[d(i-r,i)p_{i-r}p_iQ'_{i-r,i-1}\left(Q'_{i,j}-1\right) + d(i,i+k+r)p_ip_{i+k+r}Q'_{i+k,i+k+r-1}\left(1-Q'_{i,j}\right)\right] \\
&\quad + \sum_{t=1}^{k}\sum_{r=1}^{n-k-1}\left[d(i-r,i+t)p_{i-r}p_{i+t}Q'_{i-r,i-1}Q'_{i,i+t-1}(1-q_i)\right. \\
&\quad \left. + d(i+k-t+1,i+k+r)p_{i+k-t+1}p_{i+k+r}Q'_{i+k-t+1,i+k}Q'_{i+k,i+k+r-1}(q_i-1)\right],
\end{aligned}
\tag{57}
$$

while the contribution to $\Delta' E_{i,j-1}$ due to arcs between inside$_{i,j-1} \cup \{i\}$ and outside$_{i,j-1}$ for $\tau_{i,j-1}$ is

$$
\begin{aligned}
&\Delta' E_{i,j-1|(\mathrm{inside}_{i,j-1}\cup\{i\})\leftrightarrow \mathrm{outside}_{i,j-1}} \\
&\quad = \sum_{r=1}^{n-k}\left[d(i-r,i)p_{i-r}p_iQ'_{i-r,i-1}\left(Q'_{i,j-1}-1\right) + d(i,i+k-1+r)p_ip_{i+k-1+r}Q'_{i+k-1,i+k+r-2}\left(1-Q'_{i,j-1}\right)\right] \\
&\quad + \sum_{t=1}^{k-1}\sum_{r=1}^{n-k}\left[d(i-r,i+t)p_{i-r}p_{i+t}Q'_{i-r,i-1}Q'_{i,i+t-1}(1-q_i)\right. \\
&\quad \left. + d(i+k-t,i+k+r-1)p_{i+k-t}p_{i+k+r-1}Q'_{i+k-t,i+k-1}Q'_{i+k-1,i+k+r-2}(q_i-1)\right].
\end{aligned}
\tag{58}
$$

Observe that here, exactly like in the homogeneous PTSP [5], the difference between $\Delta' E_{i,j|(\mathrm{inside}_{i,j}\cup\{i\})\leftrightarrow \mathrm{outside} i,j}$ and $\Delta' E_{i,j-1|(\mathrm{inside}_{i,j-1}\cup\{i\})\leftrightarrow \mathrm{outside} i,j-1}$ will only involve arcs which are connected to nodes i and j, that is,

$$\begin{aligned}\delta = {} & \text{terms with } i \text{ and } j \text{ in } \Delta' E_{i,j|(\mathrm{inside}_{i,j} \cup \{i\}) \leftrightarrow \mathrm{outside}_{i,j}} \\ & - \text{terms with } i \text{ and } j \text{ in } \Delta' E_{i,j-1|(\mathrm{inside}_{i,j-1} \cup \{i\}) \leftrightarrow \mathrm{outside}_{i,j-1}}.\end{aligned} \tag{59}$$

So, by extracting the terms which contain the appropriate arcs from Eqs. (57) and (58) and by expressing them in terms of the matrices A (Eq. (55)) and B (Eq. (56)) we obtain the following expression for δ

$$\begin{aligned}\delta = {} & (1 - Q'_{i,j})\left[\frac{1}{Q'_{i,j}} A_{i,k+1} - (B_{i,1} - B_{i,n-k})\right] + (q_i - 1)\left[(A_{j,1} - A_{j,n-k}) - q_i^{-1} B_{j,k+1}\right] \\ & - \left(1 - Q'_{i,j-1}\right)\left[\frac{1}{Q'_{i,j-1}} A_{i,k} - (B_{i,1} - B_{i,n-k+1})\right] - (q_i - 1)\left[(B_{j,1} - B_{j,k}) - q_i^{-1} A_{j,n-k+1}\right],\end{aligned} \tag{60}$$

which completes the recursive expression of Eq. (36). Let us now focus on the 'residual' term γ from Eq. (37). The contribution to $\Delta' E_{i,j}$ due to arcs between $\{i\}$ and inside is the following:

$$\Delta' E_{i,j|\{i\} \leftrightarrow \mathrm{inside}_{i,j}} = (\overline{Q'}_{i,j} - 1) \sum_{t=1}^{k} [d(i, i+t) p_i p_{i+t} Q'_{i,i+t-1} - d(i+k-t+1, i) p_{i+k-t+1} p_i Q'_{i+k-t+1,i+k}], \tag{61}$$

while the contribution to $\Delta' E_{i,j-1}$ due to arcs between $\{i\}$ and $\mathrm{inside}_{i,j-1}$ for $\tau_{i,j-1}$ is

$$\Delta' E_{i,j-1|\{i\} \leftrightarrow \mathrm{inside}_{i,j-1}} = (\overline{Q'}_{i,j-1} - 1) \sum_{t=1}^{k-1} [d(i, i+t) p_i p_{i+t} Q'_{i,i+t-1} - d(i+k-t, i) p_{i+k-t} p_i Q'_{i+k-t,i+k-1}]. \tag{62}$$

Now, by subtracting Eq. (62) from Eq. (61) and by applying the definition of A (Eq. (55)), and B (Eq. (56)), we obtain the following expression for γ

$$\gamma = (\overline{Q'}_{i,j} - 1)\left[A_{i,1} - A_{i,k+1} - \frac{1}{\overline{Q'}_{i,j-1}} B_{i,n-k})\right] + \left(1 - \overline{Q'}_{i,j-1}\right)\left[A_{i,1} - A_{i,k} - \frac{1}{\overline{Q'}_{i,j-1}} B_{i,n-k+1}\right], \tag{63}$$

which completes the recursive expression of Eq. (37).

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