The Inventory Routing Problem [1]

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Abstract

Vendor managed resupply is an emerging trend in logistics and refers to a situation in which a supplier manages the inventory replenishment of its customers. Vendors save on distribution cost by being able to better coordinate deliveries to different customers, and customers do not have to dedicate resources to inventory management anymore. We present and discuss the inventory routing problem. The inventory routing problem captures the basic characteristics of situations where vendor managed resupply may be used, and methodologies developed for its solution could become building blocks for logistics planning systems.

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1 Introduction

The role of logistics management is changing. Many companies are realizing that value to a customer can, in part, be created through logistics management. Customer value can be created through product availability, timeliness and consistency of delivery, ease of placing orders, and other elements of customer service. Consequently, logistics service is becoming recognized as an essential element of customer satisfaction in a growing number of product markets today. The net effect is a shift in logistics from a reactive to proactive mode.

Vendor managed resupply is an example of value creating logistics. Vendor managed resupply is an emerging trend in logistics and refers to a situation in which a supplier manages the inventory replenishment of its customers. Vendor managed resupply creates value for both suppliers and customers, i.e., a win-win situation. Vendors save on

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distribution cost by being able to better coordinate deliveries to different customers, and customers do not have to dedicate resources to inventory management anymore.

Different industries are looking into the possibility of implementing vendor managed resupply. Traditionally, vendor managed resupply has been high on the wish list of logistics managers in the petrochemical and industrial gas industry. More recently, the automotive industry (parts distribution) and the soft drink industry (vending machines) have entered this arena.

One reason that vendor managed resupply is receiving a lot of attention is the rapidly decreasing cost of technology that allows monitoring customers' inventory. Vendor managed resupply requires accurate and timely information about the inventory status of customers.

If vendor managed resupply is a win-win situation for both supplier and customers, and relatively cheap monitoring technology is available, then why is vendor managed resupply not applied on a larger scale? The reason is, of course, that it is a complex task to develop a distribution strategy that minimizes the number of stockouts and at the same time realizes the potential savings in distribution costs.

In this paper, we present and discuss the inventory routing problem. The inventory routing problem captures the basic characteristics of situations where vendor managed resupply may be used, and methodologies developed for its solution could become building blocks for logistics planning systems.

The inventory routing problem is a challenging and intriguing problem that provides a good starting point for studying integration of different components of the logistics value chain, i.e., inventory management and transportation. Integration of production and transportation is another hot item on the wish list of logistics managers. Traditionally, production and transportation have been dealt with separately. However, it is intuitively clear that improvements may be obtained by coordinating production and transportation. It is less obvious how to do it.

The remainder of the paper is organized as follows. In Section 2, we formally introduce the inventory routing problem. In Section 3 and 4, we take a closer look at single and two-customer problems. In Section 5, we review and propose solution approaches. In Section 6, we address some practical issues. Finally, in Section 7, we discuss standard test problems.

2 The Inventory Routing Problem

The inventory routing problem (IRP) is concerned with the repeated distribution of a single product, from a single facility, to a set of n customers over a given planning horizon of length T, possibly infinity. Customers consume the product at a given rate u_i and have the capability to maintain a local inventory of the product up to a maximum of C_i. The inventory at customer i is I_i at time 0. A fleet of m homogeneous vehicles, with capacity Q, is available for the distribution of the product. The objective is to minimize the average distribution costs during the planning period without causing stockouts at any of the customers.

Three decisions have to be made:

- When to serve a customer?
- How much to deliver to a customer when it is served?
- Which delivery routes to use?

The IRP differs from traditional vehicle routing problems because it is based on customers' usage rather than customers' orders.

The IRP defined above is deterministic and static due to our assumption that usage rates are known and constant. Obviously, in real-life, the problem is stochastic and dynamic. Therefore, an important variant of the IRP is the stochastic inventory routing problem (SIRP). The SIRP differs from the IRP in that the future usage of a customer is uncertain. In the SIRP, we are given, for each customer i, the probability distributions of the usage amounts u_{it} between decision points t and $t+1$ for $t = 1, ..., T-1$. Because future usage is uncertain, there is often a positive probability that a customer runs out of stock, i.e, stockouts cannot always be prevented. Stockouts are discouraged with a penalty s_i per unit shortage per hour for customer i, where shortage is the usage between the time of stockout and the replenishment delivery. The objective is to choose a delivery policy that minimizes the average cost per unit time, or the expected total discounted cost, over the planning horizon.

To gain a better understanding of the IRP and SIRP, as well as the difference between them, it is worthwhile to spend some time analyzing single and two-customer problems.

3 The single customer problem

First, we consider the IRP. Let the usage rate of the customer be u, the tank capacity of the customer be C, the initial inventory level be I, the delivery cost to the customer be c, the vehicle capacity be Q, and the planning horizon be T.

It is easy to see that the optimal policy is to fill up the tank precisely at the time it becomes empty. Therefore the cost v_T for a planning period of length T is

$$v_T = \max(0, \lceil \frac{Tu - I}{\min(C, Q)} \rceil)c$$

Next, we consider the SIRP in which we decide daily whether to make a delivery to the customer or not. The usage amount U between consecutive decision points, i.e., the usage amount per day, is a random variable with known probability distribution.

We analyze the policy that makes a delivery to the customer every d days and delivers as much as possible, unless a stockout occurs earlier. When a stockout occurs earlier, the truck is sent right away which incurs a cost S. We assume that deliveries are instantaneous, so no additional stockout penalties are incurred. Let $v_T(d)$ be the expected total cost of this policy over a planning period of length T. Furthermore, let p_j be the probability that a stockout first occurs on day j $(1 \leq j \leq d-1)$. Then $p = p_1 + p_2 + \ldots + p_{d-1}$ is the probability that there is a stockout and $1-p$ is the probability that there is no stockout in the period $[1, ..., d-1]$. We now have for $d > T$

$$v_T(d) = \sum_{1 \leq j \leq T} p_j (v_{T-j}(d) + S)$$

and for $d \leq T$

$$v_T(d) = \sum_{1 \leq j \leq d-1} p_j (v_{T-j}(d) + S) + (1-p)(v_{T-d}(d) + c).$$

We have the following theorem [Bard et al. 1997].

Theorem 1 *The expected total cost of filling up a customer's tank every d days over a T-day period ($T \geq d$) is given by*

$$v_T(d) = f(T, d) + \alpha(d) + \beta(d)T$$

where

$$\beta(d) = \frac{pS + (1-p)c}{\sum_{1 \leq j \leq d} p_j}$$

and where $f(T, d)$ goes to zero exponentially fast as T goes to inf.

To find the best policy in the class, we need to minimize $v_T(d)$, which means finding a d for which $\beta(d)$ is minimum.

4 The two-customer problem

When more than one customer is served, the problem becomes significantly harder. Not only do we have to decide which customers to visit next, but also how to combine them into vehicle tours, and how much to deliver at each customer. Even if there are only two customers, these decisions may not be easy.

In a two-customer IRP, there are two extreme solutions: visit each customer by itself each time, and always visit both customers together. It is easy to express the cost associated with these solutions:

$$v_T = \lceil \frac{Tu_1 - I_1}{\min(C_1, Q)} \rceil c_1 + \lceil \frac{Tu_2 - I_2}{\min(C_2, Q)} \rceil c_2,$$

where we have implicitly assumed that we have two vehicles, and

$$v_T = \lceil \frac{T}{\min(\frac{C_1}{u_1}, \frac{C_2}{u_2}, \frac{Q}{u_1+u_2})} \rceil TSP(c_1, c_2),$$

where $TSP(c_1, c_2)$ denotes the optimal traveling salesman tour through c_1 and c_2.

Since traveling salesman problems on two customers are easy to solve, it is still easy to figure out which of these two extreme strategies is the best. However, there are other strategies possible: sometimes visit the customers together, and sometimes visit them by themselves. Intuitively, we expect that when one customer has a much higher usage rate or a much smaller tank size than the other, we visit that customer by itself several times and occasionally visit the two of them together. However, what if this customer cannot take a full truck load? Or, what if the two customers are close together? And, if we visit them together how much do we deliver to each of them? We soon realize that the answer is not so obvious.

When the two customers are visited together, it is intuitively clear that given the amount delivered at the first customer, it is optimal to deliver as much as possible at the second customer (determined by the remaining amount in the vehicle, and the remaining capacity at the second customer). Thus the problem of deciding how much to deliver at each customer involves a single decision. However, making that decision may not be easy, as the following two-customer instance of the SIRP shows.

We assume that product is delivered and consumed in discrete units. Each customer has a storage capacity of 20 units. The daily usage amounts of the customers are independent and identically distributed (across customers as well as across time), with $P[\text{usage amount} = 0] = 0.4$ and $P[\text{usage amount} = 10] = 0.6$. The shortage penalty is 1000 per unit shortage per day at customer 1 and 1005 per unit shortage per day at customer 2. The vehicle capacity is 10 units.

Every morning the inventory at the two customers is measured, and the decision maker decides how much should be delivered at each customer. There are three vehicle tours, namely tours exclusively to customers 1 and 2, with costs of 120 each, and a tour to both customers 1 and 2, with a cost of 180. Only one vehicle tour can be completed per day.

This situation can be modeled as an infinite horizon Markov decision process, with objective to minimize the expected total discounted cost, and, because of the small size, it is possible to compute the optimal expected value and the optimal policy.

Figure **??** shows the expected value (total discounted cost) as a function of the amount delivered at customer 1 (and therefore also at customer 2), when the inventory at each customer is 7, and both customers are to be visited in the next vehicle tour (which is the optimal decision in the given state). It shows that the objective function is not unimodal, with a local minimum at 3, and a global minimum at 7. Consequently, just to decide how much to deliver at each customer may require solving a nonlinear optimization problem with a nonunimodal objective function. This is a hard problem, for which improving search methods are not guaranteed to lead to an optimal solution.

It is also interesting to observe that it is optimal to deliver more at customer 1 than at customer 2, although the shortage penalty at customer 2 is higher than the shortage penalty at customer 1, and all other data, including demand probabilities, costs, and current inventories, are the same for the two customers. However, this decision starts to make sense when we look ahead at possible future scenarios. If in the next time period, customer 1 uses 10 units and customer 2 uses 0 units (w.p. 0.24), then at the next decision point the inventories will be 4 and 10 units respectively, and the vehicle will replenish 10 units at customer 1. In all other cases (w.p. 0.76), the vehicle will replenish 10 units at customer 2 in the next time period. Thus, in all cases, the vehicle will visit only one customer in the next time period, and it is more than three times as likely to be customer 2. Also, in all cases customer 2 will have 10 or more units in inventory after the delivery in the next time period, whereas customer 1 will have only 4 units in inventory with probability 0.36. This illustrates the importance of looking ahead more than one time period when choosing the best action.

5 Solution approaches

The inventory routing problem is a long-term planning problem. This long-term planning problem is already hard to formulate, it is almost impossible to solve. Therefore, almost all approaches that have been proposed and investigated up to now solve only a short-term planning problem. In early work, short-term was often just a single day, later short-term was expanded to a couple of days.

Two key issues need to be resolved with all of these approaches: how to model the long-term effect of short-term decisions, and which customers to include in the short-term planning period.

A short-term approach has the tendency to defer as many deliveries as possible to the next planning period, which may lead to an undesirable situation in the next planning period. Therefore, the proper projection of a long-term objective into a short-term planning problem is essential. The long-term effect of short-term decisions needs to capture the costs and benefits of delivering to a customer earlier than necessary. Delivering earlier than necessary leads to higher future distribution costs, but it reduces the risk of a stockout and may thus reduce future shortage costs.

We can distinguish two short-term approaches. In the first, it is assumed that all customers included in the short-term planning period have to be visited. In the second, it is assumed that customers included in the short-term planning period may be visited, but the decision whether or not to actually visit them still has to be made.

Decisions regarding who needs to be visited and how much should be delivered are usually guided by the following assumptions about what constitute good solutions:

- Always try to maximize the quantity delivered per visit.
- Always try to send out trucks with a full load.

When the short-term planning problem consists of a single day, the problem can be viewed as an extension of the vehicle routing problem (VRP) and solution techniques for the VRP can be adapted. Single day approaches usually base decisions on the latest inventory reading and maybe on a predicted usage for that day. Therefore, they avoid the difficulty of forecasting long-term usage. This makes the problem much simpler and may actually be quite reasonable when customers' usage is very unpredictable.

When the short-term planning problem consists of several days, the problem becomes harder, but has the potential to yield much better solutions. Typically the resulting short-term problems are formulated as mathematical programs and solved using decomposition techniques, such as Lagrangean relaxation.

5.1 Literature review

It is not our intention to provide a comprehensive review of the literature, but rather to discuss papers that are representative of the solution approaches that have been proposed and investigated.

Federgruen and Zipkin [11] approach the inventory routing problem as a single day problem and capitalize on many of the ideas from vehicle routing. Their version of the problem has a plant with a limited amount of available inventory and the usage amounts per day at a customer are assumed to be a random variable. For a given day, the problem is to allocate this inventory among the customers so as to minimize transportation costs plus inventory and shortage costs at the end of the day (after the day's usage and receipt of the day's delivery). Federgruen and Zipkin model the problem as a nonlinear integer program. Because of the inventory and shortage costs and the limited amount of inventory available, not every customer will be selected to be visited every day. This is handled in the model by the use of a dummy route that includes all the customers not receiving a delivery. The nonlinear integer program has the property that for any assignment of customers to routes, the problem decomposes into an inventory allocation problem which determines the inventory and shortage costs and a TSP for each vehicle which yields the transportation costs. This property is the key to the solution approach taken. The idea is to construct an initial feasible solution and iteratively improve the solution by exchanging customers between routes. Obviously, evaluating such exchanges is more computationally intensive than in standard vehicle routing algorithms. Each exchange defines a new customer to route assignment, which in turn defines a new inventory allocation problem and new TSPs.

Golden, Assad, and Dahl [14] develop a heuristic that tries to minimize costs on a single day while maintaining an "adequate" inventory at all customers. The heuristic starts with computing the 'urgency' of each customer. The urgency is determined by the ratio of tank inventory level to tank size. All customers with an urgency smaller than a certain threshold are excluded. Next, customers are selected to receive a delivery one at a time according to the highest ratio of urgency to extra time required to visit this customer. A large TSP tour is iteratively constructed. Initially, a time limit for the total travel time of the tour, say TMAX, is set to the number of vehicles multiplied by the length of a day. Customers are added until this limit is reached or there are no more customers left. The final tour is partitioned into a set of feasible routes by enforcing that each customer must be filled up when it receives a delivery. If this turns out to be impossible, the heuristic can be re-run with a smaller value for TMAX.

Chien, Balakrishnan, and Wong [7] also develop a single day approach, but theirs is distinctly different from that of Golden, Assad, and Dahl [14], because it does not treat each day as a completely separate entity. By passing information from one day to the next, the system simulates a multiple day planning model. Assuming that the maximum usage per day for each customer is known, we can define the daily profit in terms of a revenue per unit delivered and a penalty per unit of unsatisfied demand (lost revenue). Their heuristic tries to maximize the total profit on a single day. Once a solution for one day is found, the results are used to modify the revenues for the next day. Unsatisfied demand today is reflected by an increased revenue tomorrow. An integer program is created that handles the allocation of the limited inventory available at the plant to the customers, the customer to vehicle assignments, and the routing. A Lagrangean dual ascent method is used to solve the integer program.

Fisher et al. [12], [5] study the inventory routing problem at Air Products, an industrial gases producer. The objective considered is profit maximization from product distribution over several days. Rather than considering demand to be a random variable or completely deterministic, demand is given by upper and lower bounds on the amount to be delivered to each customer for every period in the planning horizon. An integer program is formulated that captures delivery volumes, assignment of customers to routes, assignments of vehicles to routes, and assignment of start times for routes. This integer program is again solved using a Lagrangean dual ascent approach.

In two companion papers, Dror and Ball [9, 8] propose a way to take into consideration what happens after the short-term planning period. Using the probability that a customer will run out on a specific day in the planning period, the average cost to deliver to the customer, and the anticipated cost of a stockout, the optimal replenishment day t^* minimizing the expected total cost can be determined for each customer. If t^* falls within the short-term planning period, the customer will definitely be visited, and a value c_t is computed for each of the days in the planning period that reflects the expected increase in future cost if the delivery is made on day t instead of on t^*. If t^* falls outside the short-term planning period a future benefit g_t can be computed for making a delivery to the customer on day t of the short-term planning period. These computed values reflect the long term effects of short term decisions. An integer program is then solved that assigns customers to a vehicle and a day, or just day, that minimizes the sum of these costs plus the transportation costs. This leaves either TSP or VRP problems to solve in the second stage.

Some of the ideas of Dror and Ball are extended and improved in Trudeau et al. [17]. Dror and Levy [10] use a similar analysis to yield a weekly schedule, but then apply node and arc exchanges to reduce costs in the planning period.

Jaillet et al. [15, 4, 3] discuss an extension of this idea. They take a rolling horizon approach to the problem by determining a schedule for two weeks, but only implementing the first week. The scenario includes a central depot and customers that need replenishing to prevent stockout, but also included is the idea of satellite facilities. Satellite facilities are location other than the depot where trucks can be refilled. An analysis similar to Dror and Ball's is done to determine an optimal replenishment day for each customer, which translates to a strategy for how often that customer should receive a delivery. A key difference is that only customers that have an optimal replenishment day within the next two weeks are included in the schedule. Incremental costs are computed that are the cost for changing the next visit to a customer to a different day but keeping the optimal schedule in the future. These costs are used in an assignment problem formulation that assigns each customer to a day in the two week planning horizon. This again yields a VRP for each day, but only the first week is actually routed. At the beginning of the next week, the problem will be solved again for the next two week horizon.

A slight variation on the inventory routing problem is the strategic inventory routing problem discussed by Larson and Webb [18] and is related to Larson's earlier work on scheduling ocean vessels [16]. For many companies, the fleet of vehicles needs to be purchased or leased months or even years before actual deliveries to customers start taking place. The strategic inventory routing problem seeks to find the minimum fleet size to service the customers from a single depot. This determination is based on information currently known about customers' usage rates. Consequently, this minimum fleet size must be able to handle a reasonable amount of variation in these usage rates. The fleet size estimate is determined by separating the customers into disjoint clusters and creating a route sequence for each cluster. A route sequence is a permanent set of repeating routes. Customers are allowed to be on more than one route in the sequence. The route sequences are created using a savings approach that minimizes vehicle utilization, which effectively minimizes the number of vehicles.

Anily and Federgruen [1, 2] look at minimizing long run average transportation and inventory costs by determining long term routing patterns. The routing patterns are determined using a modified circular partitioning scheme. After the customers are partitioned, customers within a partition are divided into regions so as to make the demand of each region roughly equal to a truck load. A customer may appear in more than one region, but then a certain percent of his demand is allocated to each region. When one customer in a region gets a visit, all customers in the region are visited. They also determine a lower bound for the long run average cost to be able to evaluate how good their routing patterns are.

Using ideas similar to those of Anily and Federgruen, Gallego and Simchi-Levi [13] evaluate the long run effectiveness of direct shipping (separate loads to each customer). They conclude that direct shipping is at least 94% effective over all inventory routing strategies whenever minimal economic lot size is at least 71% of truck capacity. This shows that direct shipping becomes a bad policy when many customers require significantly less than a truck load, making more complicated routing policies the appropriate choice.

Another adaptation of these ideas can be found in Bramel and Simchi-Levi [6]. They consider the variant of the IRP in which customers can hold an unlimited amount of inventory. To obtain a solution, they transform the problem to a capacitated concentrator location problem (CCLP), solve the CCLP, and transform the solution back into a solution to the IRP. The solution to the CCLP will partition the customers into disjoint sets, which in the inventory routing problem, will become the fixed partitions. These partitions are then served similar to the regions of Anily and Federgruen.

In the next two subsections, we propose two new solution approaches that we are currently investigating.

5.2 An integer programming approach for the IRP

Our approach is based on the assumption that periodic schedules constitute good solutions and that good periodic schedules are decomposable.

Define a cluster to be a group of customers that can be served cost effectively by a single vehicle for a long period. Note that the cost of serving a cluster does not only depend on the geographic locations of the customers in the cluster, but also on whether the customers in the cluster have *compatible inventory capacities and usage rates*.

Our approach will construct a periodic schedule consisting of several clusters, i.e., each customer will belong to precisely one cluster. Two important issues need to be addressed. How to compute the cost of serving a cluster? How do we partition the set of customers into clusters?

We have developed and evaluated several models to estimate the cost of serving a cluster.

The simplest estimate of the cost of serving a cluster is to assume that when we decide to make a delivery to one of the customers, we will make a delivery to all the customers. That means that the cost we incur each time we make deliveries is given by the length $l(S)$ of the traveling salesman tour through the set of customers S that makes up the cluster. To get an estimate on the cost of serving the cluster all that remains to be done is to compute how often we need to make deliveries. It is easy to see that this is number is approximately (since we ignore initial inventories)

$$\lfloor \frac{T}{\min(\min_i \frac{C_i}{u_i}, \frac{Q}{\sum_k u_k})} \rfloor,$$

where Q is the vehicle capacity, C_i is the customer tank capacity, u_i is the usage rate, and T is the length of the planning period.

This gives an overestimate of the cost of serving a cluster because we restrict ourselves to a policy in which we always visit all customers in the cluster together, potentially making unnecessary visits to some of them. The next estimate relaxes this condition.

Let c_r be the length of the optimal traveling salesman tour r through a subset of the customers in the cluster. Define the following variables. The total volume d_{ir} delivered to customer i on route r in the planning period and the route count x_r, and consider the following model

$$\min \sum_r c_r x_r$$

subject to

$$\sum_{i \in r} d_{ir} \leq \min(Q, \sum_{i \in r} C_i) x_r \qquad \forall r \tag{1}$$

$$d_{ir} \leq \min(Q, C_i) x_r \qquad \forall r, \forall i \in r \tag{2}$$

$$\sum_{r \ni i} d_{ir} = T u_i \qquad \forall i \tag{3}$$

$$x_r \text{ integer, } d_{ir} \geq 0.$$

Constraints (1) ensure that the total volume delivered on route r in the planning period is less than or equal to the minimum of the vehicle capacity and the total tank capacity times the number of times route r was executed. Constraints (2) ensure that we do not deliver more to a customer than the minimum of the vehicle capacity and its tank capacity times the number of times route r was executed. Constraints (3) ensure that the total volume delivered to a customer in the planning period is equal to its total usage during the planning period.

This gives an underestimate of the cost of serving the cluster since we have ignored the fact that we have only a single vehicle.

Next, we consider an integer programming model based on time discretization. Let the initial inventory be denoted by I_i, let l_i^t be a lower bound on the total amount that has to be delivered to customer i by day t, i.e., $l_i^t = \max(0, -I_i + tu_i)$, and let u_i^t be an upper bound on the total amount that can be delivered to customer i up to day t, i.e., $u_i^t = C_i - I_i + tu_i$. Define the following variables: route assignment x_r^t of route r to day t and delivery volume d_{ir}^t to customer i on route r on day t, and consider the following model

$$\min \sum_t \sum_r c_r x_r^t$$

subject to

$$l_i^t \leq \sum_{1 \leq s \leq t} \sum_{r \ni i} d_{ir}^s \leq u_i^t \qquad \forall i, \forall t \tag{4}$$

$$\sum_{i \in r} d_{ir} \leq Q x_r^t \qquad \forall r, \forall t \tag{5}$$

$$\sum_r x_r^t \leq 1 \qquad \forall t \tag{6}$$

$$x_r^t \text{ binary, } 0 \leq d_{ir}^t \leq \min(Q, C_i).$$

Constraints (4) ensure that the customers will not run out. Constraints (5) ensure that we do not deliver more than the truck capacity on a route. Constraints (6) ensure that we only use a single vehicle in each period.

Finally, we consider an integer programming model that uses continuous time. Let the route duration be denoted by D_r. Define the following variables: the start time t_k of trip k, the route assignment x_{rk} of route r to trip k, the delivery volume d_{ik} to customer i on trip k, the inventory level y_{ik} at customer i just prior to trip k, and consider the following model

$$\min \sum_k \sum_r c_r x_{rk}$$

subject to

$$\sum_r x_{rk} = 1 \qquad \forall k \tag{7}$$

$$y_{i1} = I_i - u_i t_1 \qquad \forall i \tag{8}$$

$$y_{i,k+1} = y_{ik} + d_{ik} - u_i(t_{k+1} - t_k) \quad \forall i, k = 1, ..., n-1 \tag{9}$$

$$y_{ik} + d_{ik} \leq C_i \quad \forall i, \forall k \tag{10}$$

$$\sum_i d_{ik} \leq Q \sum_r x_{rk} \quad \forall k \tag{11}$$

$$t_{k+1} - t_k \geq \sum_r D_r x_{rk} \quad \forall k \tag{12}$$

$$x_{rk} \text{ binary}, 0 \leq y_{ik} \leq \min(Q, C_i), t_k \geq 0.$$

Constraints (7) ensure that a route is assigned to each trip. (Note that this can be the empty route.) Constraints (8) and (9) balance the inventory from one period to the next. Constraints (10) ensure that the amount delivered fits in the customer's tank. Constraints (11) ensure that the amount delivered fits in the truck's tank. Constraints (12) ensure that the vehicle is executing one route at a time.

We now envision the following approach to identify good clusters:

1. Generate a large set of possible clusters
2. Estimate the cost of serving each cluster
3. Solve a set partitioning problem to select clusters

The choice of cost estimate obviously depends on the trade-off between speed of computation and quality of solution. Our experience indicates that the second cost estimate is both fast to compute and provides a good approximation of the cost of serving the cluster.

Now that we have determined a set of clusters, we still need to determine a long-term plan. We propose to do this using an extension of the third cost-estimate model for a cluster, in which

$$\sum_r x_r^t \leq 1 \quad \forall t$$

is replaced by

$$\sum_r x_r^t \leq m \quad \forall t$$

where m is the number of vehicles, and where we consider all customers instead of only the customers in a single cluster. Note that this problem has a block-angular structure, with a block for each cluster, which can be exploited computationally.

5.3 A value function approach for the SIRP

We model the SIRP as a discrete time Markov decision process (MDP). At the beginning of each time period, assumed to be a day from now on, the inventory at each customer is measured. Then a decision is made regarding which customers' inventories to replenish, how much to deliver at each customer, how to combine customers into vehicle tours, and which vehicle tours to assign to each of the vehicles. We call such a decision an itinerary. A vehicle can perform more than one tour per day, as long as all tours assigned to a vehicle together do not take more than a day to complete. Thus, all vehicles are available at the beginning of each day, when the tasks for that day are assigned. Although usage typically occurs throughout the day, and each customer's inventory therefore varies during the day, we assume that each customer's inventory is measured only at the beginning of the day, before decisions are made, and the state of the MDP is updated accordingly. The expected cost is computed taking into account the variation in inventory during the day, and the probability of stockout before the vehicle arrives at the customer's site.

We focus on the infinite horizon MDP; the finite horizon case can be treated in a similar way. The MDP has the following components:

1. The state x is the current inventory at each customer. Thus the state space $\mathcal{X}$ is $[0, C_1] \times [0, C_2] \times \cdots \times [0, C_n]$. Let $X_t \in \mathcal{X}$ denote the state at time t.

2. The action space $\mathcal{A}(x)$ for each state x is the set of all itineraries that satisfy the tour duration constraints, such that the vehicles' capacities are not exceeded, and the customers' storage capacities are not exceeded after deliveries. Let $\mathcal{A} \equiv \bigcup_{x \in \mathcal{X}} \mathcal{A}(x)$ denote the set of all itineraries. Let $A_t \in \mathcal{A}(X_t)$ denote the itinerary chosen at time t.

3. The known demand distribution gives a known Markov transition function Q, according to which transitions occur, i.e., for any state $x \in \mathcal{X}$, and any itinerary $a \in \mathcal{A}(x)$,

$$P\left[X_{t+1} \in B \mid X_t = x, A_t = a\right] = \int_B Q[dy \mid x, a]$$

4. Two costs are taken into account, namely transportation costs, which depend on the vehicle tours chosen, and a penalty when customers run out of inventory. Let $c(x, a)$ denote the expected daily cost incurred if the process is in state x at the beginning of the day, and itinerary $a \in \mathcal{A}(x)$ is implemented.

5. The objective is to minimize the expected total discounted cost over an infinite horizon ($T = \infty$). Let $\alpha \in [0, 1)$ denote the discount factor. Let $V^*(x)$ denote the

optimal expected cost given that the initial state is x, i.e.,

$$V^*(x) \equiv \inf_{\{A_t\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \alpha^t c\left(X_t, A_t\right) \middle| X_0 = x\right] \tag{13}$$

The actions A_t are restricted such that $A_t \in \mathcal{A}(X_t)$ for each t, and A_t has to depend only on the history $(X_0, A_0, X_1, \ldots, X_t)$ of the process up to time t, i.e., when we decide on an itinerary at time t, we are not allowed to know what is going to happen in the future.

Under certain conditions that are not very restrictive, the optimal expected cost in (13) is achieved by the class of stationary policies Π, which is the set of all functions that depend only on the current state and return an admissible itinerary for the current state. That is, a stationary policy $\pi \in \Pi$ is a function $\pi : \mathcal{X} \mapsto \mathcal{A}$, such that $\pi(x) \in \mathcal{A}(x)$ for all $x \in \mathcal{X}$. It follows that for any $x \in \mathcal{X}$,

$$\begin{aligned} V^*(x) &= \inf_{\pi \in \Pi} E\left[\sum_{t=0}^{\infty} \alpha^t c\left(X_t, \pi(X_t)\right) \middle| X_0 = x\right] \\ &= \inf_{a \in \mathcal{A}(x)} \left\{ c(x,a) + \alpha \int_{\mathcal{X}} V^*(y) Q[dy \mid x, a] \right\}. \end{aligned} \tag{14}$$

To determine an optimal policy, we need to solve the optimality equation (14). The three major computational requirements involved in solving (14) are the following.

1. Estimating the optimal cost function V^*.
2. Estimating the integral in (14).
3. Solving the minimization problem on the right hand side of (14) to determine the optimal itinerary for each state.

Rarely can these three computational tasks be completed sequentially. Usually an iterative procedure has to be used.

A number of algorithms has been developed to solve the optimality equation (to within a specified tolerance ε) if $\mathcal{X}$ is finite and the optimization problem on the right hand side can be solved in finite time (to within a specified tolerance δ). Examples are value iteration or successive approximation, policy iteration, and modified policy iteration. These algorithms are practical only if the state space $\mathcal{X}$ is small, and the optimization problem on the right hand side can be solved efficiently. None of these requirements are satisfied by practical instances of the SIRP, as the state space $\mathcal{X}$ is usually extremely large, even if customers' inventories are discretized, and the optimization problem on the right hand side has a vehicle routing problem as a special case, which is NP-hard. Our approach is therefore to develop approximation methods based on the MDP formulation above.

One approach is to approximate the optimal cost function $V^*(x)$ with a function $\hat{V}(x, \beta)$ that depends on a vector of parameters β. Some of the issues to be addressed when using this approximation method are the following.

1. The functional form of the approximating function $\hat{V}$. This may be the most important step in the approximation method, and also the one in which an intuitive understanding of the nature of the problem and the optimal value function plays the greatest role. A fair amount of experimentation is needed to develop and test different approximations. Functions $\hat{V}$ that are linear in β have the advantage that estimation algorithms for β with good theoretical properties have been developed, as discussed below.

2. Computational methods to estimate good values for the parameters β. Bertsekas and Tsitsiklis discuss a number of simulation based methods. They develop policy evaluation algorithms for which the parameter estimates β_t converge as $t \to \infty$, if $\hat{V}$ is linear in β, and the usual conditions for the convergence of many stochastic approximation methods hold. In addition, β_t converges to parameters β^π that give a best fit of the true expected value function V^π under stationary policy π, if the errors are weighted by the invariant distribution under policy π. However, many of the algorithms exhibit undesirable behavior, and many theoretical properties of these approximation methods remain to be established.

3. The integral in (14) can be computed explicitly only for some simple demand distributions. If the number of customers is small ($n \leq 8$), numerical integration can be used. If the demand distributions are more complex, and the number of customers is larger, simulation is usually the most efficient method to evaluate the integral.

4. Methods have to be developed to solve the minimization problem on the right hand side of (14). This optimization problem probably requires significant computational effort to solve to optimality, because it involves determining delivery quantities as well as vehicle routes. Therefore, it seems that heuristic methods have to be developed to find good solutions. Such a heuristic has to provide a good trade-off between computational speed and solution quality, as the optimization problem has to be solved thousands of times while estimating the parameters β, and the quality of the eventual approximation $\hat{V}$ and associated policy $\hat{\pi}$ may depend to a large extent on the quality of the heuristic solutions to the minimization problem.

6 Practical Issues

A number of important issues that occur in practice, and that have not been discussed above, are addressed in this section.

Usage rates are assumed to be constant in the IRP and probability distributions of the usage amounts between consecutive decision points are assumed to be known in the SIRP. In practice, the usage rates or the probability distributions of the usage amounts are typically not known, but have to be estimated from inventory measurements. Often these data are not collected at regular intervals, and thus it may not be easy to convert them to usage rates or probability distributions of usage amounts. The data are also subject to other sources of noise, such as measurement errors, which cause several statistical problems. These estimation problems have to be resolved before an IRP or SIRP can be solved in practice. Furthermore, the models ignore the typical time varying characteristics of usage, such as weekly and seasonal cycles, and any dependence between the usage on successive days.

Currently the costs involved in making inventory measurements are not insignificant, and these measurements are usually made at most once per day. One should be able to obtain fairly accurate estimates of the inventory levels at times between measurements based on the most recent measurements and past data of usage rates. Exactly how to do this estimation has to be addressed. A related problem may be to determine an optimal policy for making these costly measurements. However, it is expected that the technology will soon be available to continuously track customers' inventories at very low cost. Therefore, in the SIRP the inventories are modeled as known at the times that decisions are made, and customers' future usage amounts are modeled as random.

The models presented manage only a single resource "vehicles" to perform distribution tasks. In practice, other resources are required as well, for example drivers. The work rules that apply to drivers are quite different from those that apply to vehicles; for example, a vehicle can work more hours per day than a driver. The assignment of customers to tours in such a way that these tours can be performed by the available drivers and make the best use of the drivers' time, is therefore likely to be at least as important a consideration as the utilization of vehicles. If a sufficient number of vehicles are available, then driver considerations are the only constraints, and the objective should be to develop optimal driver itineraries.

It is not only the availability of drivers that restricts the set of feasible routes. Often deliveries at customers can only take place during specific time periods of the day.

Many companies operate a heterogeneous fleet of vehicles instead of a homogeneous fleet of vehicles.

We have considered the distribution of a product from a single plant. Often a company operates several plants that produce the same product, and distribution to some customers can occur from a number of plants. It may be optimal to distribute to a customer from different plants on different days, depending on how well the customer can be combined in a vehicle tour with the other customers that are to be visited on the particular day.

Frequently, a company produces and distributes several products, using the same fleet of vehicles to transport the different products. Examples are the transportation of different grades of oil in compartmentalized trucks, and the replenishment of beverages and snacks in vending machines and at restaurants. In this multi-product environment, besides deciding which customers to visit next and how to combine them into vehicle tours, we have to decide how much of each product to deliver at each visited customer.

We have assumed that a sufficient amount of the product is always available for distribution, and issues related to production capacity and scheduling are ignored. However, it is often necessary to coordinate production, storage, and transportation.

Inventory holding cost have not been addressed in the problem definition. In fact, this makes the problem more generic, because the treatment of inventory holding cost depends on the ownership and storage management of inventory at the plant and at the storage facilities of customers. For example, the distributor may be the same company that operates the production plant as well as the facilities at the next level of the distribution network (the "customers"), or the producer may distribute the product to and manage the inventory at independent customers (called vendor managed resupply), or an independent third party logistics provider may distribute the product from the producer to the customers, and manage their inventory. The treatment of inventory holding costs are different for the three cases above, but in all cases it can be incorporated relatively easily with the other costs.

System disruptions such as product shortages at the plant, vehicle breakdowns, work stoppages, and inventory measurement failures, are not incorporated. To address these issues, policies have to be developed to provide recourse actions when disruptions occur.

Travel times and costs are assumed to be known. A more realistic model may incorporate random travel times and costs. However, unless transportation occurs in heavily congested networks, a model assuming known travel times should give good results. If transportation networks are very congested, then the time of travel usually has a large impact on travel time besides the chosen route, and many other scheduling and routing issues have to be addressed. As the objective of the SIRP is to minimize the expected cost, only the expected travel costs need to be known, and not their distributions.

Many of the practical issues raised above can be easily incorporated in the models discussed and many of the solution approaches presented can be modified to handle them.

7 Test problems

Researchers need access to challenging instances of difficult routing problems. A standard set of instances allows the comparison of the performance of algorithms, but often it also provides an important stimulus for researchers. We have created a set of instances of the IRP that we hope will form such a test set. They have derived form real data from a company we work with. They are available on the web at http://www.tli.gatech.edu/research/casestudy/irp/irp.htm.

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