Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 9
Due: Wednesday, November 1

1. *Slutsky's extension.* Suppose $\mathbf{y}_n \xrightarrow{\text{d}} \mathbf{y}$, where $\mathbf{y}$ is a $d \times 1$ random vector, $\mathbf{A}_n \xrightarrow{\text{P}} \mathbf{A}$, where $\mathbf{A}$ is a positive definite matrix, and that $\mathbf{y}_n = \mathbf{A}_n \mathbf{x}_n$. Note: This problem would be fairly trivial if we knew that $\mathbf{A}_n$ were positive definite; the point of this problem is that you do *not* know this about $\mathbf{A}_n$, only that its limit $\mathbf{A}$ is positive definite.

   (a) Prove that $\mathbf{x}_n$ is bounded in probability. Hint: If $\mathbf{A}$ is positive definite, $\mathbf{A}^\top \mathbf{A}$ is positive definite (and its eigenvalues have certain helpful properties).

   (b) Prove that $\mathbf{x}_n \xrightarrow{\text{d}} \mathbf{A}^{-1}\mathbf{y}$.

2. *Bernstein-von Mises theorem.* Suppose the prior $p(\boldsymbol{\theta})$ is continuous with $p(\boldsymbol{\theta}) > 0$ for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, and that regularity conditions (A)-(C) as defined in the "Likelihood: Consistency" lecture are satisfied. Prove that

$$p(\hat{\boldsymbol{\theta}} + \boldsymbol{\delta}/\sqrt{n}|\mathbf{x})/p(\hat{\boldsymbol{\theta}}|\mathbf{x}) \xrightarrow{\text{P}} \exp\{-\tfrac{1}{2}\boldsymbol{\delta}^\top \boldsymbol{\mathscr{I}}(\boldsymbol{\theta}^*)\boldsymbol{\delta}\},$$

   where $\hat{\boldsymbol{\theta}}$ is the MLE and $\boldsymbol{\delta}$ is an arbitrary $d \times 1$ vector.

   Hint: Write the posteriors in terms of the likelihood, and take a Taylor series expansion of the log-likelihood.

   Note: I realize that the version we presented in class involved almost sure convergence; you are only asked to show convergence in probability here.