Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 6
Due: Monday, October 2

1. *Consistency of the OLS estimator.* Suppose that for a sequence of constant vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots$, the random variables $Y_1, Y_2, \ldots$ are independent with mean $\mathbf{x}_i^\top \boldsymbol{\beta}$ and variance σ^2. Prove that $\widehat{\boldsymbol{\beta}} = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{y}$ is a consistent estimator of $\boldsymbol{\beta}$. Do you need additional conditions on $\mathbf{X}$ in order to guarantee that $\widehat{\boldsymbol{\beta}}$ is consistent? If so, make any additional assumptions you wish, but clearly describe and label these additional conditions.

2. *A triangular array for binomial data.* Suppose $X_n \sim \text{Binom}(n, \pi_n)$; note in particular that the binomial probability π here is changing with n. Use the Lindeberg-Feller central limit theorem to show that

$$\frac{X_n - n\pi_n}{\sqrt{n\pi_n(1-\pi_n)}} \xrightarrow{\text{d}} \text{N}(0,1).$$

   As above, are there any additional conditions you need to require on the sequence π_n in order for this result to be true? If so, tell me what they are. In your proof, you will need to set up a triangular array; clearly describe what random variables you are using for this array and prove that it satisfies the requirements to be a triangular array as defined in the notes.

3. *Slutsky constant.* In Slutsky's theorem, we require that one of the two random variables is converging in probability to a constant. What happens if both $\mathbf{x}_n$ and $\mathbf{y}_n$ are converging in distribution? In other words, suppose $\mathbf{x}_n \xrightarrow{\text{d}} \mathbf{x}$ and $\mathbf{y}_n \xrightarrow{\text{d}} \mathbf{y}$; is it true that

$$\begin{bmatrix} \mathbf{x}_n \\ \mathbf{y}_n \end{bmatrix} \xrightarrow{\text{d}} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}?$$

   If this is not true, provide a counterexample.