Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 3
Due: Wednesday, September 13

1. *O-notation proofs.* Prove the following results:
   (a) $O(1)o(1) = o(1)$.
   (b) $\{1 + o(1)\}^{-1} = O(1)$.
   (c) $o\{O(1)\} = o(1)$.

   Remarks:

   - Part (b) is trivial if you use properties of limits. For the purposes of this problem, however, prove the result using only the definition of o and O. I realize that a simpler proof exists, but I consider the longer proof quite instructive.
   - In part (c), you cannot use the result $o(r_n) = r_n o(1)$. This result is a consequence of the results in part (a) and part (c); using it would be circular logic.

2. *Exponential Taylor series.* Note: (b) and (c) are not trick questions; they take little effort to derive, but the results are useful to know.
   (a) Show that for any x,

   $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

   Note: one can partially prove this using the Poisson distribution, but this proof would only work for $x > 0$.
   (b) Starting with the result in (a), derive the infinite series for e^{ax}.
   (c) Starting with the result in (a), derive the infinite series for b^x.
   (d) Let $f : \mathbb{R}^d \to \mathbb{R}$. What is the second-order Taylor series for $f(\mathbf{x}) = \exp(\mathbf{a}^\top \mathbf{x})$ about $\mathbf{x} = \mathbf{0}$? Give both the o-notation and Lagrange forms.
   (e) Suppose $\mathbf{a} = [2 \quad -1]^\top$ and $\mathbf{x} = [1 \quad 1]^\top$. Find the point $\bar{\mathbf{x}}$ on the line segment connecting $\mathbf{x}$ and $\mathbf{0}$ that satisfies the Lagrange form of Taylor's theorem.

3. *Matrix square root.* Let $\mathbf{A}$ be a symmetric, positive definite matrix.
   (a) Suppose $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^\top$ is the eigendecomposition of $\mathbf{A}$. What is $\mathbf{A}^{1/2}$? Show that your answer satisfies $\mathbf{A}^{1/2}\mathbf{A}^{1/2} = \mathbf{A}$ (the definition of a matrix square root).
   (b) Show that $\mathbf{A}^{-1/2}\mathbf{A}\mathbf{A}^{-1/2} = \mathbf{I}$.
   (c) Suppose $\mathbf{A}$ was not symmetric. Would your derivation in (a) and (b) still work?
   (d) Suppose $\mathbf{A}$ was symmetric, but positive semidefinite. Would your derivation in (a) and (b) still work?