Likelihood Theory and Extensions (BIOS:7110)
Breheny

Assignment 2
Due: Wednesday, September 6

1. *Vector norms.*
   (a) Show that $\|\cdot\|_2$ is a norm.
   (b) Show that $\|\cdot\|_\infty$ is a norm.
   (c) Let $\|\cdot\|_{1/2}$ denote the function of $\mathbf{x}$ you would obtain by using $p = 1/2$ in the definition of an L_p norm. Is $\|\cdot\|_{1/2}$ a norm? Why or why not?

2. *Uniform convergence.* For each of the following sequences, determine the pointwise limit of $\{f_n\}$ and decide whether $f_n \to f$ uniformly on the set given or not. You must justify your conclusion – don't just say uniform/not uniform.
   (a) $f_n(x) = \sqrt[n]{x}$ on $[0, 1]$.
   (b) $f_n(x) = e^x/x^n$ on $(1, \infty)$.
   (c) $f_n(\mathbf{x}) = n^{-1} \exp\{-\|\mathbf{x}\|^2\}$ on $\mathbb{R}^d$.

3. *Logistic regression.* The logistic regression model states that Y_i is equal to 1 with probability π_i and 0 otherwise, with π_i related to a set of linear predictors $\{\eta_i\}$ by the following model:

$$\log \frac{\pi_i}{1 - \pi_i} = \eta_i \qquad \text{for } i = 1, 2, \ldots, n$$
$$\boldsymbol{\eta} = \mathbf{X}\boldsymbol{\beta}$$

   where $\boldsymbol{\eta} \in \mathbb{R}^n$, $\boldsymbol{\beta} \in \mathbb{R}^d$, and $\mathbf{X}$ is an $n \times d$ matrix. For (b)-(d), express your answer in vector/matrix notation, not as a collection of scalar terms (i.e., something like $\mathbf{a} + \mathbf{b}$, not $z_1 = 1, z_2 = 3, \ldots$).
   (a) Let ℓ_i denote the contribution to the log-likelihood from observation i. Find the partial derivative of ℓ_i with respect to η_i. Simplify your answer as much as possible.
   (b) Let $\ell : \mathbb{R}^n \to \mathbb{R}$ denote the log-likelihood as a function of the linear predictors $\boldsymbol{\eta}$. Find $\nabla_{\boldsymbol{\eta}} \ell$.
   (c) Find $\nabla_{\boldsymbol{\beta}} \boldsymbol{\eta}$.
   (d) Find $\nabla_{\boldsymbol{\beta}} \ell$.

4. *The Riemann-Stieltjes Integral.* Suppose μ is a bounded, nondecreasing function on $[a, b]$ and that μ is continuous at a point $x_0 \in [a, b]$. Further suppose that $g(x_0) = 1$ and $g(x) = 0$ if $x \neq x_0$. Prove that $\int g d\mu = 0$. Hint: Show that for any $\epsilon > 0$, we can find a partition P such that $0 \leq U(P, g, \mu) < \epsilon$.