Probability I

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Probability

- People talk loosely about *probability* all the time: "What are the chances the Hawkeyes will win this weekend?", "What's the chance of rain tomorrow?"
- For scientific purposes, we need to be more specific in terms of defining and using probabilities

Events

- A *random process* is a phenomenon whose outcome cannot be predicted with certainty
- An *event* is a collection of outcomes
- Examples:

Random process	Event
Flipping a coin	Obtaining heads
Child receives a vaccine	Child contracts polio
Patient undergoes surgery	Patient survives

Long-run frequency

- The probability of heads when flipping a coin is 50%
- The probability of rolling a 1 on a 6-sided die is $1/6$
- Everyone agrees with these statements, but what do they really mean?
- The probability of an event occurring is defined as the fraction of time that it would happen if the random process occurs over and over again under the same conditions
- Therefore, probabilities are always between 0 and 1

Long-run frequency (cont'd)

- Probabilities are denoted with a $P(\cdot)$, as in $P(\text{Heads})$ or $P(\text{Child develops polio})$ or "Let H be the event that the outcome of a coin flip is heads. Then $P(H) = 0.5$"
- Example:
  - The probability of being dealt a full house in poker is 0.0014
  - If you were dealt 100,000 poker hands, how many full houses should you expect?
  - $100,000(0.0014) = 140$
- Note: It is important to distinguish between a probability of .0014 and a probability of .0014% (which would be a probability of .000014)

Long-run frequency (cont'd)

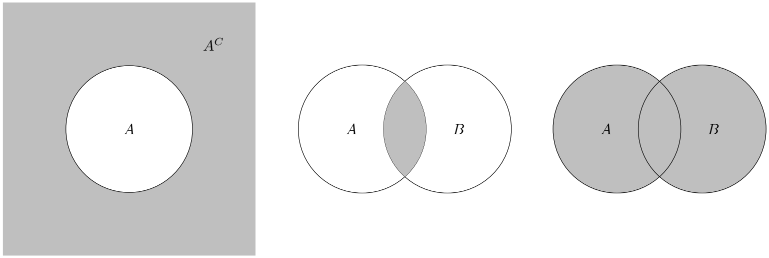
- This works both ways:
  - For the polio data, 28 per 100,000 children who got the vaccine developed polio
  - The probability that a child in our sample who got the vaccine developed polio is 28/100,000=.00028
- Of course, what we really want to know is not the probability of a child in our sample developing polio, but the probability of a child in the population developing polio – we're getting there

Intersections, unions, and complements

- We are often interested in events that are derived from other events:
  - Rolling a 2 or 3
  - Patient who receives a therapy is relieved of symptoms and suffers from no side effects
- The event that A does not occur is called the *complement* of A and is denoted A^C
- The event that both A and B occur is called the *intersection* and is denoted $A \cap B$
- The event that either A or B occurs is called the *union* and is denoted $A \cup B$

Venn diagrams

These relations between events can be represented visually using *Venn diagrams*:



Introduction

- Let event A denote rolling a 2 and event B denote rolling a 3
- What is the probability of rolling a 2 or a 3 ($A \cup B$)?
- It turns out to be

$$\frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

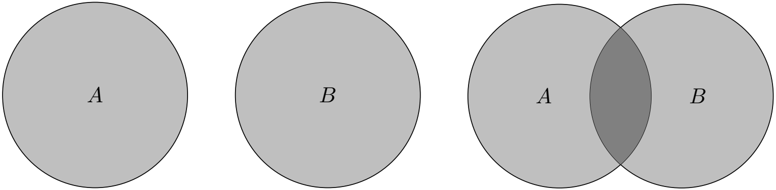
- On the surface, then, it would seem that $P(A \cup B) = P(A) + P(B)$
- However, this is not true in general

A counterexample

- Let A denote rolling a number 3 or less and B denote rolling an odd number
- $P(A) + P(B) = 0.5 + 0.5 = 1$
- Clearly, however, we could roll a 4 or a 6, which is neither A nor B
- What's wrong?

Double counting

- With a Venn diagram, we can get a visual idea of what is going wrong:



- When we add $P(A)$ and $P(B)$, we count $A \cap B$ twice
- Subtracting $P(A \cap B)$ from our answer corrects this problem

The addition rule

- In order to determine the probability of $A \cup B$, we need to know:
  - $P(A)$
  - $P(B)$
  - $P(A \cap B)$
- If we're given those three things, then we can use the *addition rule*:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- This rule is always true for any two events

Mutually exclusive events

- So why did $P(A \cup B) = P(A) + P(B)$ work when A was rolling a 2 and B was rolling a 3?
- Because $P(A \cap B) = 0$, so it didn't matter whether we subtracted it or not
- A special term is given to the situation when A and B cannot possibly occur at the same time: such events are called *mutually exclusive*

Mutually exclusive events, example

- According to the National Center for Health Statistics, the probability that a randomly selected woman who gave birth in 1992 was aged 20-24 was 0.263
- The probability that a randomly selected woman who gave birth in 1992 was aged 25-29 was 0.290
- Are these events mutually exclusive?
- Yes, a woman cannot be two ages at the same time
- Therefore, the probability that a randomly selected woman who gave birth in 1992 was aged 20-29 was $0.263+0.290=.553$

Example: Failing to use the addition rule

- In the 17th century, French gamblers used to bet on the event that in 4 rolls of the die, at least one "ace" would come up (an ace is rolling a one)
- In another game, they rolled a pair of dice 24 times and bet on the event that at least one double-ace would turn up
- The Chevalier de Méré, a French nobleman, thought that the two events were equally likely

Example: Failing to use the addition rule

- His reasoning was as follows: letting A_i denote the event of rolling an ace on roll i and AA_i denote the event of rolling a double-ace on roll i

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3 \cup A_4) &= P(A_1) + P(A_2) + P(A_3) + P(A_4) \\ &= \frac{4}{6} = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} P(AA_1 \cup AA_2 \cdots) &= P(AA_1) + P(AA_2) + \cdots \\ &= \frac{24}{36} = \frac{2}{3} \end{aligned}$$

Example: Failing to use the addition rule

- Is the Chevalier using the addition rule properly?
- Are A_1 and A_2 mutually exclusive?
- No; it is possible to get an ace on roll #1 and roll #2, so you have to subtract $P(A_1 \cap A_2)$, $P(A_1 \cap A_3)$, $\ldots$
- We'll calculate the real probabilities a little later

Using the addition rule correctly

- An article in the *American Journal of Public Health* reported that in a certain population, the probability that a child's gestational age is less than 37 weeks is 0.142
- The probability that his or her birth weight is less than 2500 grams is 0.051
- The probability of both is 0.031
- Can we figure out the probability that either event will occur?
- Yes: $0.142 + 0.051 - 0.031 = 0.162$

The complement rule

- Because an event must either occur or not occur, $P(A) + P(A^C) = 1$
- Thus, if we know the probability of an event, we can always determine the probability of its complement:

$$P(A^C) = 1 - P(A)$$

- This simple but useful rule is called the *complement rule*
- Example: If the probability of getting a full house is 0.0014, then the probability of not getting a full house must be $1 - 0.0014 = 0.9986$

Summary

- The probability of an event is the fraction of time that it happens (under identical repeated conditions)
- Know the meaning of complements (A^C), intersections ($A \cap B$), and unions ($A \cup B$)
- Addition rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If (and only if!) A and B are mutually exclusive, we can ignore $P(A \cap B)$ in the addition rule
- Complement rule: $P(A^C) = 1 - P(A)$