Enforcement Frictions and Optimal Lending Contracts[*]
(Job Market Paper)

Latchezar Popov [†]

University of Iowa

November 2008

Abstract

I consider an environment in which contract enforcement is a decision variable for the principal. I construct a model in which entrepreneurs cannot commit to repaying investors for the capital advanced, but investors can force repayment by spending resources. The principal uses enforcement to reduce the resources available to the agent after a default, thus providing incentives for the agent to stay in the relationship. She also ensures contract compliance by *backloading* the payments to the agent: expected utility rises over time, preventing a default. I consider two applications of the framework developed in the paper. The first is in firm dynamics. I show that enforcement and backloading are always used jointly. Firm size (measured by capital) grows with time and each firm converges to the efficient size. Enforcement is generally nonmonotone in capital and it is used more heavily for medium-sized firms. The second is in the field of economic development. I compare the stationary equilibria in two economies with differing enforcement costs. Higher enforcement cost eliminates investment in high risk - high return enterprises and distorts the distribution of firm sizes. This can explain observed differences in total factor productivity between poor and rich countries.

JEL classification numbers: D87, O16, O43

Keywords: Costly enforcement, lending contracts, economic development, firm dynamics

---

[*] I am grateful to B. Ravikumar for his support, encouragement and useful discussions. I also thank Gabriele Camera, Yuzhe Zhang, Erwan Quintin and seminar participants at the University of Iowa for helpful comments. I acknowledge help from The University of Iowa Seashore-Ballard Dissertation Year Fellowship.

[†] E-mail: latchezar-popov@uiowa.edu

1 Introduction

In many economic relationships that involve agents taking actions at different points in time, a decision rule that is ex ante efficient and satisfies the participation constraints of all the agents may not be time consistent. One of the agents may increase her ex post utility by choosing an action contrary to the original plan. For example a borrower may refuse to repay a loan. Another example would be producing a specialized product: if the product is less valuable for outside parties, then the buyer would be able to renegotiate the contract after delivery, lowering the price.

There are two approaches to this problem in the literature. First, we may assume that the agents have access to some commitment technology (for example contract law) that allows them to make credible promises at the beginning of the game. After all the parties have entered into binding agreement, the specified strategies are played out without making any further decisions. In this case, the agents must bargain to choose an allocation from the set of all Pareto-efficient allocations. Second, we concentrate on strategies that are optimal at every decision node - Perfect Bayesian Equilibria.

However, in most commercial relationships contract enforcement is neither impossible nor perfect. In this paper I consider an intermediate case: one of the parties in the relationship can force a contractually specified action by paying a cost. For example if we consider enforcement by a court action, the cost would be out-of-pocket legal expenses, the probability of an adverse outcome and the cost of delays. Agents in less developed countries may use informal or even illegal enforcement mechanisms, but these are still costly. Djankov et. al. (2008) document significant differences in enforcement cost and quality around the world which suggest that the use of enforcement versus other methods to provide contract compliance varies significantly between countries and industries. In order to explore these issues, I construct a model in which enforcement is a decision variable and the cost of enforcement is exogenous.

I consider the relationship between a financial intermediary (principal) and an entrepreneur (agent) in an environment without commitment. The agent can operate a stochastic production technology, but is born without capital. The intermediary can borrow any amount from capital markets at the risk-free rate (or, equivalently, has deep pockets). I assume that the project, operated at the optimal scale, yields output greater than the total cost of capital. There is no private information, nor exogenous contract incompleteness. Therefore the two parties can create some surplus together. However, the agent cannot commit to repaying the principal. The principal can use dynamic incentives to ensure payment: the share of the project revenues that go to the agent grows over time, which reduces the benefit of defaulting. As the value of the agent's participation in the relationship rises, the principal can increase the project size.[1] Additionally, the principal can enforce payment. However enforcement is costly: to force a payment of e units, the principal must pay a cost of $\gamma(e)$.

Costly enforcement is a deadweight loss and leads to incentives for renegotiation. The cost of enforcement affects the bargaining power of the two parties in the case of a renegotiation, but actual enforcement will never be observed in equilibrium. Krasa and Villamil (2000) show that enforcement will be used in a model with private information. The intuition behind it is simple: if the intermediary cannot commit to enforcing the contract, in equilibrium she must be indifferent between enforcement and no enforcement. Their model is essentially static and does not allow for a choice of investment level. I follow Koeppl (2007) in assuming that the principal must invest $\gamma(e)$ units of the consumption good in order to ensure enforcement capacity of e at the end of the period. (Or, we can simply assume that the principal can commit to enforcement.) This gives us a tractable framework in which enforcement, investment and entrepreneur's equity change over time. Increasing investment increases the total surplus, but also increases the incentives for default. To prevent default, the principal can decrease investment from the first-best level, use enforcement or dynamic incentives. This model allows us to study the interactions between these 3 tools.

If the agent defaults, the principal uses the enforcement option, collects a payment of e and breaks off the contract. The agent receives some outside option which is a function of the net resources available after the default. The contract is incentive-compatible only if the utility of staying in the relationship is equal to or exceeds the outside option. Thus if we fix the utility the agent receives from the contract, the principal ensures that the contract is incentive-compatible by reducing the agent's resources after a default by reducing investment in the project and obtaining enforce-

---

[1] Albuquerque and Hopenhayn (2004) consider a similar setup with risk-neutral entrepreneurs and endogenous firm liquidation.

ment capacity.

Expected output is strictly concave and satisfies the Inada properties, which implies that there is an optimal scale of production. I show that investment never exceeds the optimal level. Since there is an optimal scale to production and the productivity shock has compact support, the resources available after a default are bounded from above by some number $\overline{y}$ almost surely. I prove that if the agent's promised utility is above the outside option of $\overline{y}$, the first-best allocation (optimal investment and consumption in each period, no enforcement) is incentive-compatible.

Increasing the agent's promised utility relaxes the incentive constraints. I prove that capital increases monotonically with the agent's promised utility and converges to the first-best. The agent has limited liability - there is no outside source of payment - which implies that the principal will reduce enforcement (to minimize costs) if investment is low. Therefore capital and enforcement are complements: enforcement tends to rise with investment. Thus increasing promised utility has ambiguous effects on enforcement: it would increase capital, which would increase enforcement, but on the other hand it relaxes the incentive constraints and reduces enforcement. I find an example in which enforcement is nonmonotone.

Commitment frictions become less severe as promised utility rises. Therefore, the principal will tend to increase promised utility over time. I prove that there is an upward trend in promised utility: expected continuation utility is always higher than current utility if the incentive constraints are binding. (That happens when the promised utility is lower than the outside option of the highest output realization $\overline{y}$). Therefore expected continuation utility is strictly larger than current promised utility. Promised utility is a submartingale and by the submartingale convergence theorem it converges almost surely to some value v_∞ at which the incentive constraints no longer bind. Since the principal and agent have the same discount factor, the first-best allocation requires constant consumption. Giving the agent a rising consumption profile is costly - it increases average consumption, required to deliver the initial promised utility. However, it allows a more efficient scale of production in subsequent periods. The former effect is second order and is dominated by the latter.

I consider two applications of the principal-agent model, developed in the paper.

The first one is in firm dynamics. There is an extensive literature[2], that reports that financing constraints are important for firm growth and survival. For any starting initial promised utility v_0 the contract induces some process for promised utility v_t and therefore capital. Thus, if we pin down initial utility, the model delivers an equilibrium process of firm sizes. If the intermediation sector is competitive, then the agent's initial utility is determined by a break-even condition: the principal's expected profit at the start of the relationship must be equal to the firm set-up cost. I assume exogenous firm destruction probability as in Cooley and Quadrini (2001) which implies that there is a stationary distribution of firm sizes (measured by capital). If the enforcement cost is high, enforcement frictions are severe and dynamic incentives are used heavily. If the enforcement cost is lower, new firms are less constrained and start producing on a larger scale. On the other hand, they will grow slower. The average size of the firm will be larger if the enforcement cost is lower. If the model is extended to include endogenous firm exit, the effect will be even more pronounced.

The second application I consider is in the field of Economic Development. This paper provides a mechanism to explain two widely accepted stylized facts. First, the distribution of firm sizes is skewed towards smaller sizes in developing countries. Second, more than 50% of per-capita income differences are accounted for by differences in observed total factor productivity. Financial frictions and problems with enforcement have been advanced as explanations to both of these facts. In that literature there is some exogenously imposed limit on enforcement, which affects the equilibrium lending contracts. (For example, Buera et. al. (2008), Castro et al. (2004) and Quintin (2007)). I endogenize the degree of enforcement.[3] I compare the stationary equilibria in a pair of countries with identical technology and preferences, but different enforcement costs.

I showed above that in an economy with higher enforcement costs firms will be smaller on average, which accounts for the first fact.

---

[2] For example, Cooley and Quadrini (2001)

[3] Koeppl, Monnet and Quintin (2008) have a model in which enforcement *institutions* are endogenized, but individuals do not choose enforcement as a decision variable. I assume exogenous differences in enforcement costs and derive the optimal choice of agents how much enforcement to use.

To address the second question, I allow the entrepreneur to choose the project type from a set of different possible types with differing risk and return characteristics. When enforcement is more costly, high variance projects pose severe incentive problems, and the initial utility offered to the entrepreneur is low. On the other hand, increasing the mean of the productivity process increases the agent's initial utility for sure. Thus differences in enforcement costs change the nature of the mean-variance trade-off. As enforcement cost declines, variance becomes less important and in the limit of perfect enforcement, only the mean of the productivity process is relevant. (I assume idisyncratic shocks.) Then differences in enforcement costs lead to endogenous choice of technology with different means and volatility. In particular, if enforcement is costly it is possible (but not necessary) that entrepreneurs will choose low mean, low volatility firm type.

The rest of the paper is organized as follows. In the next section I describe the model. First I show a static example that develops some intuition. Then I show the link between sequential and recursive formulation of the relationship. Section 2.4. presents the optimal allocations. I consider two applications in section 3 and briefly conclude in section 4.

2 Principal-agent model

Technology Time is discrete and runs forever. There is one undifferentiated good that can either be consumed or converted to capital. If a firm uses k units of capital and is subject to productivity shock s, it produces

$$y = F(k, s). \tag{1}$$

where $F(k, s)$ is a production function that is strictly increasing in both k and s, strictly concave in k for each s, differentiable and satisfies the Inada conditions: $\lim_{k\to\infty} F'(k, s) < 1$, $\forall s \in S$, $\lim_{k\to 0} F'(k, s) = \infty$, $\forall s \in S$. Capital is necessary for production: $F(0, s) = 0$. Every active firm needs a manager to operate. s is an idiosyncratic i.i.d. productivity shock that can take values in a finite set $S = \{s_1, s_2, \cdots, s_N\}$. Finally, $\sup_{k>0} F(k, s_N)/F(k, s_N) < \infty$. The timing of events in the production process is as follows:

1. Capital k is invested at the beginning of the period.
2. Productivity shock $s \sim \pi(s)$ is realized and publicly observed.
3. Output $F(k, s)$ is produced and the invested capital fully depreciates.

Note that in this specification $F(k, s)$ is gross output. For example, we can have

$$F(k, s) = sk^{\theta} + (1 - \delta)k.$$

It is easy to incorporate labor into the model. Assume that the firm produces output using capital and labor with production function $G(k, l, s)$. Then F is the profit function: $F(k, s) = \max_l G(k, l, s) - wl$. The reason that F displays decreasing returns to scale can be explained by the span of control model, introduced by Lucas (1978), or by assuming monopolistically competitive firms facing a downward sloping demand curve.

Preferences The entrepreneurs are risk-averse, and maximize discounted expected utility.

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t). \tag{2}$$

where $\beta \in (0, 1)$ and u is a utility function with standard properties.

The other type of economic agents are investors. They are risk-neutral and maximize expected discounted net cash flows with the same discount factor β. Equivalently, they face gross interest rate $R = 1/\beta$.

Principal-agent relationship Before specifying the equilibrium in the model, I will consider an exclusive relationship between one agent and one investor (or principal).

The total surplus that can be created by operating the technology is the expected period profit of investing k:

$$p(k) = \sum_{i=1}^{N} \pi(s_i) F(k, s_i) - Rk. \tag{3}$$

The assumptions on $F(k, s)$ ensure that the profit function $p(k)$ is strictly concave and that it attains a maximum for some $k^* > 0$.

$$\sum_{i=1}^{N} \pi(s_i) F_k'(k^*, s_i) = R. \tag{4}$$

The optimal expected profit $p^* = p(k^*) > 0$. However, the agent is financially constrained and needs outside investment. The principal cannot produce but she can credibly commit to any contract, which gives her access to the capital markets. Together the principal and agent can create some surplus.

If contracts can be costlessly enforced, the Modigliani-Miller theorems apply. Capital structure would be irrelevant and the optimal amount of capital k^* will be invested.

The friction is that the agent cannot commit to repaying the investor. At any point, the agent has an option to default and end the relationship. The value to the agent after a default is a function of available resources $O(y)$. $O(.)$ is some function to be specified. It is assumed that it is increasing, differentiable and strictly concave. $\inf_{y>0} O(y) \geq \inf_{c>0} u(c)/[1-\beta]$ and $\sup_{y>0} O(y) \leq \sup_{c>0} u(c)/[1-\beta]$. In other words $O()$ corresponds to the utility of some possible consumption sequence. Different possible outside options can be captured by specifying different function $O(y)$.

1. Autarchy. The agent can operate the technology without the principal, but has no access to capital markets after defaulting.

$$\begin{aligned} O(y) &= \max E_0 \sum_{j=0}^{\infty} \beta^j u(c_j(s^j)) \\ c_0 + k_1 &\leq y \\ c_j(s^j) + k_{j+1}(s^j) &\leq F(k_j(s^{j-1}), s_j) \end{aligned} \tag{5}$$

2. Agent has access to storage technology

$$\begin{aligned} O(y) &= \max \sum_{j=0}^{\infty} \beta^j u(c_j) \\ c_0 + x_1 &\leq y \\ c_j + x_{j+1} &\leq R_s x_j \\ R_s &\leq R \end{aligned} \tag{6}$$

3. Recontracting. The agent may enter into a new relationship in the future.

$$O(y) = \max_c u(c) + \beta \underline{v}(y - c), \tag{7}$$

   where $\underline{v}$ is the utility of the new relationship as a function of the resources the agent brings and can be determined by some equilibrium condition.

4. Home production. If the agent can work in the labor market or has access to some home production technology, then her value after a default would be:

$$O(y) = u(y) + \frac{\beta}{1-\beta} u(w) \tag{8}$$

In the applications I concentrate on the first two cases, but cases 3 and 4 show that the agent need not consume zero if she defaults without any residual resources.

There are two ways for the principal to ensure payment: promise benefits in the future if the agent stays in the contract or reduce the resources available after a default. The second option is through formal enforcement.

Enforcement The principal has the option to use an enforcement technology. She can ensure a payment of e, by spending $\gamma(e)$ before production. An example of such kind of investment would be a legal department which an intermediary would maintain at all times. $\gamma(e)$ is assumed to be strictly convex, differentiable, $\gamma(0) = 0$ and $\lim_{e\to 0} \gamma'(e) = 0$, $\lim_{e\to\infty} \gamma'(e) > 1$. Since γ is continuous everywhere, $\lim_{e\to 0} \gamma(e) = \gamma(0) = 0$, that is there are no fixed costs.

The timing of events in the relationship is as follows:

1. The principal spends $\gamma(e)$ to ensure enforcement capacity e.

2. The principal invests k in the productive technology. (In the model, the agent cannot expropriate the capital to be invested in the project. This assumption can be relaxed.)

3. The productivity shock s is realized and output is $F(k, s)$.

4. The agent decides what payment to make to the principal.

5. The principal can enforce payment up to e units.

2.1 Static Example

If the interaction between the principal and the agent is repeated, expectations for the future utility of staying in the contract can provide the right incentive for the agent to perform her contractual obligations. On the other hand, if the relationship is one-period enforcement is the only way to ensure contract compliance. Thus to highlight the role of enforcement I will look at a simplified static version of the model first.

Let $F(k, s)$ be given by

$$F(k, s) = sk^{\theta}$$

and s is a continuous random variable with a cumulative distribution function $G(s)$. Assume that s has compact support: $G(\overline{s}) = 1$ for some $\overline{s} < \infty$. The agent has some exogenous outside option v. Individual rationality imposes the constraint that the agent's utility from participating in the contract is at least v. The principal chooses capital investment k, payment schedule from the agent to the principal $b(s)$ and enforcement capacity e. It is possible that $b(s)$ is negative, that is the principal subsidizes the agent after production.

Consumption is nonnegative and the agent has no other resources, thus we get the following constraint:

$$b(s) \leq sk^{\theta} \tag{9}$$

Let's impose the constraint that the agent receives utility of exactly v:

$$\int_0^{\overline{s}} u(sk^{\theta} - b(s))dG(s) = v \tag{10}$$

The relationship is terminated when the period ends, therefore the principal can never extract more than e units from the agent:

$$b(s) \leq e, a.s. \tag{11}$$

Since the principal can commit, there is no corresponding incentive constraint on the side of the principal.

The principal maximizes expected payments net of interest costs and enforcement costs.

$$P(v) = \max_{(k,b(s),e)} \int_0^{\overline{s}} b(s)dG(s) - Rk - R\gamma(e) \tag{12}$$
$$\text{subject to } (9), (10), (11)$$

If there is no incentive problem, the principal will choose the optimal level of investment $k^* = (\theta Es/R)^{\frac{1}{1-\theta}}$ and consumption would be constant: $b(s) = sk^{*\theta} - u^{-1}(v)$. There is no incentive problem if prescribed payments are from the principal to the agent. Therefore the first-best solution would be optimal if $v \geq v^* \equiv u(\bar{s}k^{*\theta})$.

Fix some k and e. The lowest utility that can be delivered, consistent with incentive constraints, is:

$$\underline{v} = \int_0^{\bar{s}} u(\max\{sk^\theta - e, 0\})dG(s)$$

If $v \geq \underline{v}$ then there is a payment schedule that delivers the promised utility. Since the agent is risk-averse, the optimal consumption schedule is as flat as possible and in states in which the incentive constraints don't bind consumption is constant.

Proposition 1 *The optimal payment schedule is characterized by a cutoff level $\tilde{s}$ such that:*

$$b(s) = \begin{cases} e + (s - \tilde{s})k^\theta & \text{if } s < \tilde{s} \\ e & \text{if } s \geq \tilde{s} \end{cases} \tag{13}$$

The cutoff level is determined by the promise-keeping constraints. Thus the cutoff level $\tilde{s}(v, k, e)$ is an implicit function of promised utility v and k and e that satisfies the following constraint:

$$G(\tilde{s})u(\tilde{s}k^\theta - e) + \int_{\tilde{s}}^{\bar{s}} u(sk^\theta - e)dG(s) = v \tag{14}$$

The implicit function theorem shows that the cutoff level is decreasing in capital invested (at each productivity shock s there is now more output available to the agent, so to keep the promised value transfers to the agent must be decreased) and is increasing in promised value v and enforcement capacity e. For given k and e, the cutoff level summarizes the incentive and promise-keeping constraints. Straightforward algebra shows that the principal's value is given by

$$P(v) = \max_{(k,e)} k^\theta \int_0^{\tilde{s}(v,k,e)} (s - \tilde{s}(v, k, e))dG(s) + e - Rk - R\gamma(e) \tag{15}$$

For example, if the agent's utility function is CARA: $u(c) = -\exp(-c)$ and the productivity shock is uniformly distributed on $[0, 1]$, the cutoff level is determined by:

$$\frac{v}{\exp(e)} - \frac{\exp(-k^\theta)}{k^\theta} = -\exp(-\tilde{s}k^\theta)\left[\frac{1}{k^\theta} + \tilde{s}\right] \tag{16}$$

Using the cutoff characterization, we can analyze the the relationship between investment, enforcement and promised value.

Proposition 2 *If* $v < v^*$*, invested capital is suboptimal* $k < k^*$ *and is increasing in* v *and converges to* k^**. Enforcement capacity is nonzero* $e > 0$ *and converges to zero as* v *converges to* v^**.*

The intuition is clear: in the presence of incentive constraints, increasing capital at the margin distorts the consumption allocation. Therefore expected consumption must increase, so the marginal benefit to the principal is lower than the expected marginal product of capital. Therefore the first-best capital level would be suboptimal. When the promised value is higher, the measure of the set of states in which the incentive constraints do not bind is larger, therefore increasing capital brings higher marginal benefits to the principal (in terms of lowering subsidies to the agent). Thus increasing the promised value increases average consumption but increases expected output, thus the principal's profit converges to the first best as v increases. It is possible to construct an example in which the principal's profit is increasing in the promised value.

From equation (15) we can see the role of enforcement for the principal. Increasing enforcement allows for a more even consumption profile, which lower the average consumption required for delivering the promised utility. Also, since it allows the principal to capture a larger fraction of output it allows the principal to choose a more efficient scale of production. This simple model shows that enforcement and capital investment are complements. Since investment rises with promised utility, there are benefits to the principal from increasing enforcement. On the other hand, a higher promised utility implies that a larger share of output is given to the agent, reducing the need for enforcement. Thus it is possible that enforcement is nonmonotone.

2.2 Dynamic Model

In this subsection I consider the dynamic model. The static relationship is repeated indefinitely.

Contract There is no exogenous contract incompleteness and no private information, so I allow the contract to be conditioned on all the available relevant information.

This means that the contractible variables can depend on the history of productivity shocks s^t. Thus the contract consists of: capital to be invested $k_t(s^{t-1})$, enforcement capacity $e_t(s^{t-1})$ and repayment $b_t(s^t)$. This is the most general formulation consistent with the technology and information in the model: capital and enforcement must be invested before the period productivity shock is observed, but the payment from the agent (and hence consumption) can be conditioned on it.

Note that the contract specified here is very general and nests a lot of possibilities. For example, if the agent stops producing after finite time and the expected cash flows to the principal are nonpositive after every history (discounted at the appropriate factor), then the contract is an annuity. If the expected payments are zero at the beginning of the game, then the annuity is actuarially fair to the principal. Agent's consumption is provided by retaining part of the output or receiving a subsidy from the principal. Therefore there is no restriction that the transfer be positive.

The contract induces the following total output in every period:

$$y_t(s^t) = F(k_t(s^{t-1}), s_t) \tag{17}$$

It will be more convenient to work directly with consumption induced by the contract $c_t(s^t) = y_t(s^t) - b_t(s^t)$.

The expected utility to the agent after a history s^{j-1} is given by the conditional expectation:

$$v_j(s^{j-1}) = E\left[\sum_{t=j}^{\infty} \beta^{t-j} u[c_t(s^t)] | s^{j-1}\right] \tag{18}$$

.

The principal is risk-neutral and wants to maximize expected discounted net cash flows. Her payoff after a history of shocks s^{j-1} is:

$$\begin{aligned} P(s^{j-1}) &= E\left[\sum_{t=j}^{\infty} \beta^{t-j}[b_t(s^t) - Rk_t(s^{t-1}) - R\gamma(e_t(s^{t-1}))] | s^{j-1}\right] \\ &= E\left[\sum_{t=j}^{\infty} \beta^{t-j}[y_t(s^t) - Rk_t(s^{t-1}) - R\gamma(e_t(s^{t-1})) - c_t(s^t)] | s^{j-1}\right]. \end{aligned} \tag{19}$$

So, we can think that the principal's objective function includes the total surplus of the relationship net of enforcement costs and consumption for the agent.

The agent has some (unmodelled) outside option v and the contract that the principal offers must give the agent expected utility of at least v. For the time being, I will assume that the principal is constrained to deliver utility *exactly* equal to v. Therefore the promise-keeping constraint is:

$$v_0(\emptyset) = v \tag{20}$$

That is the utility of the agent with no history must be exactly equal to the outside option.

Default The only decision the agent makes is how much to pay to the principal. Since the principal can commit, obviously there is no incentive problem in the states in which the principal makes a payment to the agent ($c_t(s^t) \geq y_t(s^t)$). I assume that in the event of a default (payment less than the contracted) the principal exercises the enforcement option, collects up to $e_t(s^{t-1})$ units from the agent and breaks off the relationship.[4] If the agent chooses to pay $b' < b_t(s^t)$, she will leave the relationship with $\max\{y_t(s^t) - b' - e, 0\}$. (It is assumed that the agent has no resources outside this contract and the maximum the principal can extract is output net of the previous payment.) If the agent decides to default, her utility would be $O(\max\{y_t(s^t) - b' - e, 0\})$. Clearly the best option in this case is not to pay anything at all: $b' = 0$. Then the contract is incentive-compatible if the agent's utility from following the contract is higher than the value of defaulting:[5]

$$u(c_t(s^t)) + \beta v_{t+1}(s^t) \geq O(\max\{y_t(s^t) - e_t(s^{t-1}), 0\}), \quad \forall s^t \in S^t \tag{21}$$

The principal can credibly commit to a contract, and she would choose the contract to maximize time-zero expected profit:

$$P^*(v) = \sup_{\{k,c,e\}} E\left[\sum_{t=0}^{\infty} \beta^t [y_t(s^t) - Rk_t(s^{t-1}) - R\gamma(e_t(s^{t-1})) - c_t(s^t)]\right] \tag{22}$$
$$\text{subject to } k_t(s^{t-1}) \geq 0, c_t(s^t) \geq 0, e_t(s^{t-1}) \geq 0, (20) \text{ and } (21)$$

---

[4] This is not an arbitrary assumption. It is possible to construct a game that in which the principal's actions are dependent on productivity shocks and the payments by the agent. If the principal can commit, then the optimal strategy is as described above.

[5] This constraint may appear to be too strong: if the outside option is good enough in some state, it may be more efficient for the principal to let the agent leave the contract. However, as long the agent does not have access to better technology outside the contract, the principal can replicate the autarchy allocation. Therefore this assumption is without loss of generality.

For any contract, the agent can default in the first period and get an outside option of at least $O(0)$. Therefore there is no feasible contract that delivers utility less than $O(0)$. This implies that the principal's value function is defined only for $v \geq O(0)$. If the agent has some unalienable asset (for example human capital) the contract must deliver at least the reservation utility, associated with it. Obviously, promised utility is also defined only if $v < \sup u(c)/(1-\beta)$. Therefore the principal's value function is defined only on the set $\mathcal{V} = (O(0), \sup u(c)/(1-\beta))$.

2.3 Recursive formulation

The incentive constraints have an obvious recursive structure: they include current consumption and expected continuation utility to provide incentives for payment. Similarly, expected utility after each history can be decomposed into utility from current consumption and from continuation utility. This recursive structure implies that expected continuation utility is a sufficient statistic that encapsulates all the information in the history.

In particular, we can rewrite the expected continuation utility as:

$$\begin{aligned} v_t(s^{t-1}) &= E\left[\sum_{j=0}^{\infty} \beta^j u(c_{t+j})(s^{t+j}))|s^{t-1}\right] \\ &= \sum_{i=1}^{N} \pi_i \left\{ u(c_t(s^{t-1}, s_i)) + \beta E\left[\sum_{j=1}^{\infty} \beta^{j-1} u(c_{t+j}(s^{t+j}))|(s^{t-1}, s_i)\right]\right\} \\ &= \sum_{i=1}^{N} \pi_i \left\{ u(c_t(s^{t-1}, s_i)) + \beta v_{t+1}(s^{t-1}, s_i)\right\} \end{aligned}$$

Similarly, we can rewrite the incentive constraint as follows:

$$u(c_t(s^t) + \beta v_{t+1}(s^t) \geq O(F(k_t(s^{t-1}), s_t) - e_t(s^{t-1}))$$

In this section, we conjecture that $c_t(s^t)$, $k_t(s^{t-1})$, $e_t(s^{t-1})$ and $v_{t+1}(s^t)$ can be expressed as a function of $v_t(s^{t-1})$ and s_t. I will derive a recursive version of the problem and then I will show that the conjecture is correct: the recursively generated contract is the optimal one.

If an agent has some promised expected utility v, the contract would consist of capital k, enforcement capacity e, consumption $c(s)$ as a function of the productivity shock s and continuation utility $v'(s)$ as a function of the shock. The recursive contract must satisfy the following constraints:

$$\sum_{i=1}^{N} \pi_i[u(c(s_i)) + \beta v'(s_i)] = v \tag{23}$$

$$u(c(s)) + \beta v(s) \geq O(\max\{F(k,s) - e, 0\}), \quad \forall s \in S \tag{24}$$

$$c(s) \geq 0,\ k \geq 0,\ e \geq 0,\ v'(s_i) \in \mathcal{V} \tag{25}$$

Equation (23) is the promise-keeping constraint and (24) is the incentive constraint. For every state the utility the agent gets from following the contract must be greater than the outside option of resources $\max\{F(k,s) - e, 0\}$.

Then the principal's value function solves the following Bellman equation:

$$P(v) = \sup_{(k,c,e,v')} \sum_{i=1}^{N} \pi_i[F(k,s_i) - Rk - R\gamma(e) - c(s_i) + \beta P(v'_i)] \tag{26}$$

$$\text{subject to } (23) - (25)$$

Let's denote the operator defined above by T. In this section I will show that it is a contraction on an appropriately defined function space. Then I will prove that the supremum function P^* is a fixed point of T and lies in that function space, therefore the solution to the Bellman equation is the supremum function. Finally, I prove two technical results: the contract generated recursively attains the maximum and satisfies all constraints and second, the value function is concave.

If we return to the sequence problem, we can find an upper and lower bound for the supremum function.

First, for any v, set $k_t(s^{t-1}) = 0$, $c_t(s^t) = u^{-1}((1-\beta)v)$ and $e_t(s^{t-1}) = 0$ for all histories. This allocation gives the principal:

$$P_1(v) = \frac{-u^{-1}((1-\beta)v)}{1-\beta} \tag{27}$$

This allocation satisfies all the constraints and is therefore a lower bound for the principal's value, $P_1(v) \leq P(v)$.

On the other hand, if the incentive constraints are dropped, the optimal allocation would be: $k_t(s^t) = k^*$, $c_t(s^t) = u^{-1}((1-\beta)v)$ and $e_t(s^{t-1}) = 0$ for all histories. In this case, the value of the allocation for the principal would be:

$$P_2(v) = \frac{-u^{-1}((1-\beta)v) + E(F(k^*,s)) - Rk^*}{1-\beta} = P_1(v) + \frac{p^*}{1-\beta} \tag{28}$$

where $p^* = E(F(k^*, s)) - Rk^* > 0$.

$P_2(v)$ is an upper bound for the principal's value: $P(v) \leq P_2(v)$. Then the principal's value function must lie in the space of all functions defined on $\mathcal{V}$ and within these two bounds. Let the function space be given by: $D = \{f : \mathcal{V} \to \mathbb{R} : P_1(v) \leq f(v) \leq P_2(v)\}$. We can define a metric on this space in the usual way: $\rho(f_1, f_2) = ||f_1 - f_2||_\infty$. This is a complete metric space and by construction it is bounded: $||f_1 - f_2||_\infty \leq p^*/(1-\beta)$. This implies that the Blackwell's theorem applies and the operator is a contraction if it maps that space to itself and it satisfies monotonicity and discounting.

Proposition 3 *The operator T is a contraction on D and the supremum function P^* is the unique fixed point on D.*

Proof. In the appendix. ■

The reasoning is standard: the operator satisfies monotonicity and discounting. Then it is only necessary to show that the operator maps D into itself. For any $f \in D$ it is feasible to invest nothing, and set consumption equal to $u^{-1}((1-\beta)v)$and continuation utility to be v. By the way we constructed the bounds, the value for the principal of this contract is at least P_1. On the other hand, since the operator is monotone, Tf is less than TP_2, which in turn is less than P_2. Then T is a contraction on D. Finally since the principal's objective function and constraint set have a recursive structure, we can separate an optimal sequence contract into current consumption, investment and enforcement and continuation utility. This new contract solves the Bellman equation. To summarize, this implies that the supremum function P^* is a fixed point of the operator. Moreover, P^* is the unique fixed point since T is a contraction.

The result above shows that we can analyze the problem with standard recursive techniques. To complete the characterization I need to show to additional technical results. First, the principal's value function is continuous and the optimum is attained. Second, the allocation generated recursively from the optimal policies satisfies the sequential constraints and attains the optimum in the original sequence problem.

Lemma 1 *The principal's value function P^* is continuous and for each $v \in \mathcal{V}$ there exists some contract (k, c, e, v') that solves the Bellman equation.*

Proof. In the appendix. ■

The operator is a contraction therefore it is sufficient to show that the operator maps the space of continuous functions in D to itself. The maximum theorem cannot be applied directly, however, since the feasibility correspondence is not compact-valued. In the appendix I show that we can use the fact that Tf is in D for every $f \in D$ to impose some additional nonbinding constraints that make the correspondence continuous and compact-valued.

I have shown that the maximum is attained. The maximum theorem implies that the maximizer is an upper hemi-continuous correspondence. It is possible to choose some policy functions such that $k(v), c_i(v), e(v), v'(v)$ is a maximizer. (Since the shocks take finitely many values, any such selection will be a measurable function). After proving some preliminary results, I show that the maximizer is a single-valued correspondence, and therefore a continuous function.

With this characterization in hand, I can start to show a few obvious results.

Lemma 2 *For any $v \in \mathcal{V}$, $k(v) \leq k^*$.*

Proof. Assume that $k > k^*$. Then decreasing k to k^* increases $EF(k, s) - Rk$ and leaves all the constraints satisfied. Therefore this is a superior contract, which gives a cotradiction. ■

This implies that for any (even suboptimal) process for promised utility, the resources available to the agent in the event of a default are bounded from above by $F(k^*, s_N)$ after every history. Then if promised utility $v \geq v^* \equiv O(F(k^*, s_N))$ the incentive constraints will not bind in any state and the first-best solution is feasible.

Lemma 3 *If $v \geq v^*$, $P^*(v) = P^{FB}(v)$.*

Proof. Note that $k = k^*$, $c_i = u^{-1}((1-\beta)v)$, $v_i' = v$ and $e = 0$ is a feasible contract. Assume that it is not the optimal contract. Then:

$$P^*(v) > p^* - u^{-1}((1-\beta)v) + \beta P^*(v)$$

$$P^*(v) > \frac{p^*}{1-\beta} - \frac{u^{-1}((1-\beta)v)}{1-\beta} = P^{FB}(v)$$

Which is a contradiction since $P^*(v) \leq P^{FB}(v)$. The result follows directly. ■

This lemma also implies that if $v \geq v^*$, $v_i'(v) = v$ and $k(v) = k^*$.

A second technical issue is whether the recursively generated allocation satisfies the constraints in the original problem and whether that allocation actually attains the maximum in the supremum problem. The incentive constraints will be satisfied if the allocation generated recursively delivers the utility promised at some initial date.[6]

Let $\tilde{v}_t(s^{t-1})$ be the process for promised utility the policy functions induce: $\tilde{v}_0 = v_0$ and $\tilde{v}_t(s^{t-1}) = v'(\tilde{v}_{t-1}(s^{t-2}), s_{t-1})$. Then the policy functions induce the following process for consumption, investment and enforcement: $c_t(s^t) = c(\tilde{v}_t(s^{t-1}), s_t)$, $k_t(s^{t-1}) = k(\tilde{v}_t(s^{t-1}))$, $e_t(s^{t-1}) = e(\tilde{v}_t(s^{t-1}))$.

Applying the promise-keeping constraint inductively, we get:

$$\begin{aligned} v_0 &= \sum_{j=0}^{t} \beta^j E u(c_j(s^j)) + \beta^{t+1} E \tilde{v}_{t+1}(s^t) \\ &= \sum_{j=0}^{\infty} \beta^j E u(c_j(s^j)) + \lim_{t\to\infty} \beta^{t+1} E \tilde{v}_{t+1}(s^t) \qquad (29) \end{aligned}$$

Similarly, applying the Bellman equation successively, we get:

$$\begin{aligned} P^*(v_0) &= \sum_{j=0}^{t} \beta^j E\{F(k_h(s^{h-1}), s_j) - c_j(s^j) - R(k_j(s^{j-1}) + e_j(s^{j-1}))\} + \beta^{t+1} E P^*(\tilde{v}_{t+1}(s^t)) \\ &= \sum_{j=0}^{\infty} \beta^j E\{F(k_j(s^{j-1}), s_j) - c_j(s^j) - R(k_t(s^{j-1}) + e_t(s^{j-1}))\} + \lim_{t\to\infty} \beta^{t+1} E P^*(\tilde{v}_{t+1}(s^t)) \qquad (30) \end{aligned}$$

Therefore to show that the contract satisfies the constraints I need to show that $\beta^t E \tilde{v}_{t+1}(s^t)$ and $\beta^t E P^*(\tilde{v}_{t+1}(s^t))$ converge to zero. Both of these inequality follow from the bounds on the principal's value function.

Theorem 1 *The allocation generated recursively from the policy functions satisfies all the constraints in the sequence problem and attains the maximum of the principal's problem.*

---

[6] This is in contrast with private information environments in which sequential promise-keeping and the recursive incentive constraint do not imply the sequential incentive constraint directly. The reason for that is that with hidden information, the agent can deviate from the prescribed behavior and stay in the relationship. This implies the possibility of deviations in infinite number of periods. In my model, the relationship ends after the first deviation, thus infinite deviations are not possible.

Proof. In the appendix. ■

This result allows to switch between sequential and recursive contracts, when characterizing the dynamics of the relationship. In particular any constraints on the optimal sequential contract, place the same constraint on the recursive policy functions.

2.4 Optimal allocations

I have shown the equivalency between the sequential and the recursive formulations of the problem and can now start to analyze the optimal allocations in the contract.

First, let us revisit the promise-keeping constraint. In the recursive formulation, I have constrained the principal to deliver *exactly* the promised utility v. If the interpretation of the promise-keeping constraint is that it is a participation constraint, then the promise-keeping constraint need to keep with inequality. In this model increasing the agent's promised utility does not affect the incentives (it can loosen some of the constraints), therefore the principal can choose to deliver more than v_0 at date 0, but the promise-keeping constraint will hold with equality in all subsequent periods.

Let A be defined as the set of promised utilities such that increasing the agent's promised utility lowers the principal's profit. $A = \{v \in \mathcal{V} : \forall v' > v, v' \in \mathcal{V} : P^*(v) \geq P^*(v')\}$. This set is nonempty since P^* is strictly decreasing if $v > v^*$. If the initial utility $v_0 \in A$, then expected utility $v_t(s^{t-1}) \in A$. In the beginning of the contract, the principal may offer the agent some starting utility higher than her outside option (to increase their own profit), but in all subsequent periods the constraint will bind.

Lemma 4 *If $v_0 \in A$, then for any history of shocks $v_t(s^{t-1}) \in A$.*

Proof. It is sufficient to show that if $v \in A$, then $v_i'(v) \in A$ for all i. Assume not. Then for some j, $v_j'(v) \notin A$, which implies that for some $\tilde{v}_j > v_j'(v)$, $P^*(\tilde{v}_j) > P^*(v_j'(v))$. Then if we modify the contract to change $v_j'(v)$ to $\tilde{v}_j$, we deliver utility $\tilde{v} = v + \beta\pi_j(\tilde{v}_j - v_j'(v))$ at an increased profit of $P^*(v) + \beta\pi_j[P^*(\tilde{v}_j) - P^*(v_j'(v))]$ which contradicts the assumption that $v \in A$. ■

This lemma implies that only the decreasing part of the principal's value function is relevant. In the rest of this paper, I will only consider optimal allocations for $v \in A$. Moreover, if a contract can be modified to increase the agent's utility without decreasing the principal's profit, the original contract cannot be optimal.

Using this result, I show that the maximizing correspondence is single-valued. The theorem of the maximum guarantees that the correspondence is upper hemicontinuous, therefore the policy functions are continuous. I make two assumptions: on the outside option and on the productive technology, that are sufficient (but not necessary) conditions.

Assumption 1 *For any $y_1 \geq 0$, $y_2 \geq 0$, $\lambda \in [0,1]$*

$$O(\lambda y_1 + (1-\lambda)y_2) \leq \frac{1}{1-\beta} u\left[\lambda u^{-1}((1-\beta)O(y_1)) + (1-\lambda)u^{-1}((1-\beta)O(y_2))\right]$$

For example, this assumption would be satisfied if the agent uses a storage technology after defaulting.

Assumption 2 *$F(k,s) = sf(k) + (1-\delta)k$, where f is strictly concave, differentiable and satisfies the Inada conditions.*

Lemma 5 *For any $v \in A$ and O that satisfies the assumptions above, the maximizing correspondence is single-valued.*

Proof. In the appendix. ■

If the policy correspondences are not single-valued, for some v_0 there will be two allocations that give the same profit to the principal and satisfy the promise-keeping and incentive constraints. Since the agent is risk-averse, for a given sequence of promise-keeping and incentive constraints, there is a unique optimal consumption sequence. Thus, there must be differences in investment and enforcement. I use the assumptions above to construct a third allocation that delivers higher utility level at the same profit to the principal, which is a contradiction.

Using the same idea, I can show that the principal's value function is strictly concave on A.

Lemma 6 *$P^*(v)$ is strictly concave on A. Moreover, if $v_1, v_2 \in \mathcal{V}, v_1 \neq v_2$ and $\lambda \in (0,1)$, then there exists some $v_3 \geq \lambda v_1 + (1-\lambda)v_2$, such that $P^*(v_3) > \lambda P^*(v_1) + (1-\lambda)P^*(v_2)$.*

Proof. In the appendix. ■

It is easy to use the Benveniste-Scheinkman theorem and establish differentiability of the value function on the subset of $\mathcal{V}$ where P^* is concave.

Lemma 7 *P^* is differentiable on the interior of A.*

Proof. In the appendix. ■

The fact that the principal's value function is concave on the relevant range allows us to characterize the contract for given k and e. Similar to the static example, the principal tries to keep the consumption profile of the agent as smooth as possible.

Proposition 4 *There is a threshold state j such that the incentive constraint binds for states $j, j+1, \cdots, N$ and is loose for states $1, 2 \cdots, j-1$. If the incentive constraint is loose in states i and i', then $c_i = c_{i'}, v'_i = v'_{i'}$.*

Proof. Assume that the incentive constraint is binding in state i and loose in state $i' > i$. Then we can increase consumption in state i and decrease consumption in state i' keeping the incentive and promise-keeping constraints and lowering average consumption. If consumption is the same in the two states, we can decrease $v'_{i'}$ and increase v'_i again keeping all the constraints satisfied and increase the principal's value. This also proves that consumption and continuation utility are the same in states where the incentive constraints are not binding. ■

Consumption and continuation utility in state i always appear jointly in the incentive and promise-keeping constraints. Define $w_i \equiv u(c_i) + \beta v'_i$. Then the proposition above simply states that

$$w_i = \begin{cases} \tilde{w} & \text{if } i \in \{1, 2, \cdots, j-1\} \\ O(F(k, s_i) - e) & \text{if } i \in \{j, j+1, \cdots, N\} \end{cases}.$$

$\tilde{w}$ is determined by the promise-keeping constraint:

$$\tilde{w} = \frac{v - \sum_{i=j}^{N} \pi_i w_i}{1 - \sum_{i=j}^{N} \pi_i}.$$

If enforcement is not possible, the investment that the principal can make is limited by the inequality $v \geq EO(F(k,s))$. The principal will tend to underinvest because high capital violates the incentive constraints and it requires inefficient spreading of consumption and continuation utility. Enforcement makes investing more capital incentive compatible and it also allows more smooth consumption profile, lowering average consumption for a given promised utility v.

Now I turn to optimal investment and enforcement. First, the Inada properties of the production and enforcement cost functions ensure that we will have an interior solution.

Proposition 5 *Investment $k(v) > 0$ for all v and $k(v) < k^*$ if $v < v^*$, $k(v) = k$ if $v \geq v^*$. Enforcement $e(v) > 0$ if $v < v^*$ and $e(v) = 0$ if $v \geq v^*$.*

Proof. If $v \geq v^*$, then the first-best contract $(k(v) = k^*, c_i(v) = u{-}1((1-\beta)v), v_i'(v) = v, e(v) = 0)$ is incentive-compatible. Now assume that $v < v^*$. If $k(v) = 0$, at the optimum $e(v) = 0$, since positive enforcement provides no incentives and it is costly. Then setting investment to k and enforcement to $e = F(k, s_N)$ keeps all the incentive constraints. The new allocation changes the principal's value by $EF(k,s) - Rk - R\gamma(F(k, s_N))$. The Inada properties of F and γ ensure that if k is small enough this increases the principal's profit. Now assume that $k(v) = k^*$. If $e > 0$, then reduce k and reduce e to keep the incentive constraints satisfied. If $e = 0$, reduce k and reduce the spread of consumption and v'. Since $EF'(k^*, s) - R = 0$, in both cases the modification of the contract increases profits. Finally, assume that $e = 0$. Increase k $(k < k^*)$ and set $e = F(k, s_N)$. The Inada conditions ensure that this modification is profitable. ■

From the characterization of the allocation given by proposition 4, it follows that if $O(F(k, s_n)) \geq v$ there are no incentive problems. Then if $h(k) = O(F(k, s_N))$, then a lower bound for investment would be $\underline{k} = h^{-1}(v)$. The lower bound is monotonically increasing in v. Then clearly, as in the static case, increasing v reduces the distortions due to lower investment. This implies that the principal will have an incentive to raise utility over time.

Lemma 8 *If* $u(c_i(v)) + \beta v_i'(v) \geq v^*$*, then* $v_i'(v) = u(c_i(v))/(1-\beta)$ *If* $u(c_i(v)) + \beta v_i'(v) < v^*$*, then* $v_i'(v) > u(c_i(v)) + \beta v_i'(v)$*.*

Proof. In the appendix. ■

The intuition is that increasing promised utility is costly (due to inefficient pattern of consumption), but allows the principal to increase investment closer to k^*. Taking expectations over the states implies that $E(v') > v$ if $v < v^*$. Then promised utility is a submartingale. The submartingale convergence theorem implies that promised utility converges to v^* almost surely, until the incentive constraints cease to bind.

Proposition 6 v_t *converges to* $\max\{v^*, v_0\}$ *almost surely.*

Proof. In the appendix. ■

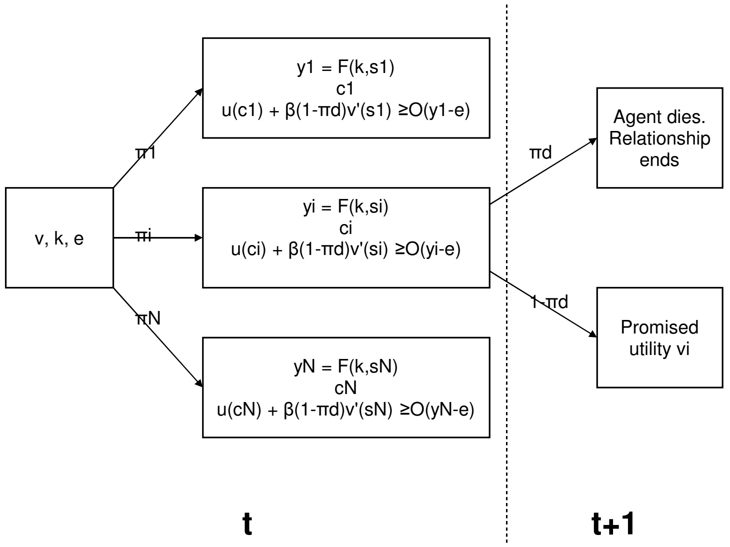
Since $k(v)$ is a continuous and bounded function of v ($0 \leq k(v) \leq k^*$) the proposition above implies that over time average investment converges to the first-best level of investment k^*.

Proposition 7 k_t *converges to* k^* *in* $\mathcal{L}_1$*.*

Proof. Since $k_t \leq k^*$, we have that $E|k_t - k^*| = k^* - Ek_t$. $k_t = k(v_t)$ and therefore $|k_t| \leq k^*$. Therefore by he Lebesgue Dominated Convergence theorem, $\lim_{t\to\infty} Ek_t = E \lim_{t\to\infty} k_t$. But $k_t = k(v_t)$ and k is continuous, therefore $\lim_{t\to\infty} k_t = k^*$ a.s. Therefore $\lim_{t\to\infty} Ek_t = k^*$ and $k_t \longrightarrow^{\mathcal{L}_1} k^*$. ■

3 Applications

In this section I show that the model developed in section 2, can be applied directly to study two applications: firm dynamics and economic development. My focus is on stationary equilibria. As I have demonstrated, in the long run all firms produce at the efficient scale and the incentive frictions cease to bind. The cost of enforcement affects the speed at which the constraints are relaxed and the scale of operation of entrepreneurs of different ages. To analyze the two applications, I modify the basic model minimally. I assume that entrepreneurs die at the end of the period at an exogenous probability π_d, which translates to an exogenous firm destruction probability.



The agent maximizes expected utility:

$$\sum_{t=0}^{\infty} \beta^{t-1}(1-\pi_d)^{t-1}\left[\sum_{s^t} \pi(s^t)u(c_t(s^t))\right] \tag{31}$$

The principal is assumed to be immortal. If entrepreneurs are alive at the beginning of the period, they will produce and consume for certain, therefore the cash flow to the the principal, conditional on survival at date t and history s^t is $F(k_t(s^{t-1}, s_t) - c_t(s^t) - R(k_t(s^{t-1}) + \gamma(e_t(s^{t-1}))$. After the agent dies there are no subsequent payments. There are no additional assets the principal is entitled to, since $F(k, s)$ is gross output and includes undepreciated capital. Then the value for the principal of a contract is:

$$\sum_{t=0}^{\infty} \beta^{t-1}(1-\pi_d)^{t-1} \sum_{s^t} \pi(s^t)\left[F(k_t(s^{t-1}, s_t) - c_t(s^t) - R(k_t(s^{t-1}) + \gamma(e_t(s^{t-1})))\right] \tag{32}$$

Inspecting the value of the principal and entrepreneur, we see that they are identical to the basic model with discount factor $\hat{\beta} = \beta(1 - \pi_d)$. The gross interest rate within a period (when survival probability is 1) is R and between periods is $R/(1 - \pi_d)$.

In the basic model I did not impose the restriction that $R\beta = 1$, only that the principal and the agent have the same discount factors. Thus all the results from section 2.4 carry over without any modification.

In order to look at applications, I need to specify initial utility v_0. Once it is pinned down, the policy function give a unique sequence of observable variables: investment, consumption and enforcement. I assume that financial intermediaries are competitive and at the beginning of the relationship offer a contract maximizing the agent's utility, subject to break-even constraints:

$$v_0 = \sup\{v \in \mathcal{V} : P^*(v) \geq c\}, \tag{33}$$

where $c \geq 0$ is a set-up cost. If c is sufficiently low, the principal will break even for some v and v_0 is well-defined. Also $v_0 \in A$ by definition. If the outside option for the agent is very bad, that is $v^* = O(F(k^*, s_N))$ is very low, incentive problems will not exist in equilibrium. In other words if $P^{FB}(v^*) \geq c$, the incentive constraints will not bind in equilibrium. In what follows, I assume that this is not the case.

The policy functions and initial promised utility v_0, induce a distribution of promised utilities at date t (conditional on survival). Let μ_t be the measure of promised utilities. Then for a set $B \subseteq \mathcal{V}$, $\mu_0(A) = 1$ if $v_0 \in A$ and zero otherwise. Then μ_t are defined recursively by

$$\mu_t(B) = \int_{\mathcal{V}} \left[\sum_{i=1}^{N} \pi_i \mathbf{1}_B(v_i'(v)) \right] d\mu_{t-1}(v). \tag{34}$$

If we assume that there is a continuum of agents and that the productivity shocks are idiosyncratic, the law of large numbers implies that μ_t is not only a probability measure, but it also describes the mass of firms of age t.

I concentrate on stationary equilibria in the following applications. I assume that a mass of π_d entrepreneurs are born at the beginning of every period. They have no resources when they are born so they are entitled to v_0. Therefore the following is the time-invariant measure of agents' utilities:

$$\mu(B) = \sum_{t=0}^{\infty} \pi_d (1 - \pi_d)^t \mu_t(B) \tag{35}$$

With that measure and the policy function for investment, we can find the aggregate capital invested in the economy:

$$K = \int_{\mathcal{V}} k(v) d\mu_t(v) \tag{36}$$

There is no aggregate resource constraint in the model, since the principal can borrow or lend at the risk-free rate with the rest of the world.

3.1 Firm Dynamics

The role of borrowing constraints for the organization of production is a question of huge interest. For example, Cabral and Mata (2003) and Cooley and Quadrini (2001) document that young firms are financially constrained. Moreover, the distribution of firms sizes varies significantly by cohort. For young firms, the distribution is skewed to the right and it evolves towards a more symmetric distribution over time. Cabral and Mata (2003) argue that this change over time is not due to selection effects, but to the relaxation of financing constraints.

An interesting question that this model can address is how differences in enforcement cost affect the dynamics of investment and the cross-section of firms in equilibrium.

To explore this question, I consider a simple numerical example. The numerical values are not calibrated, but are chosen to be consistent with commonly used values. I assume log preferences and technology $F(k, s) = sk^{\theta} + (1 - \delta)k$. s is a 7-state approximation to a log-normal stochastic process. $\gamma(e) = \alpha_i e^2$.

		Source
β	0.96	Real interest rate = 0.04
π_d	0.045	Cooley and Quadrini
θ	0.36	
δ	0.05	Standard value
$\overline{s}$	1	Standardization
$\sigma(s)/\overline{s}$	0.75	
α_1	0.01	10% upper bound on enforcement cost
α_2	0.03	

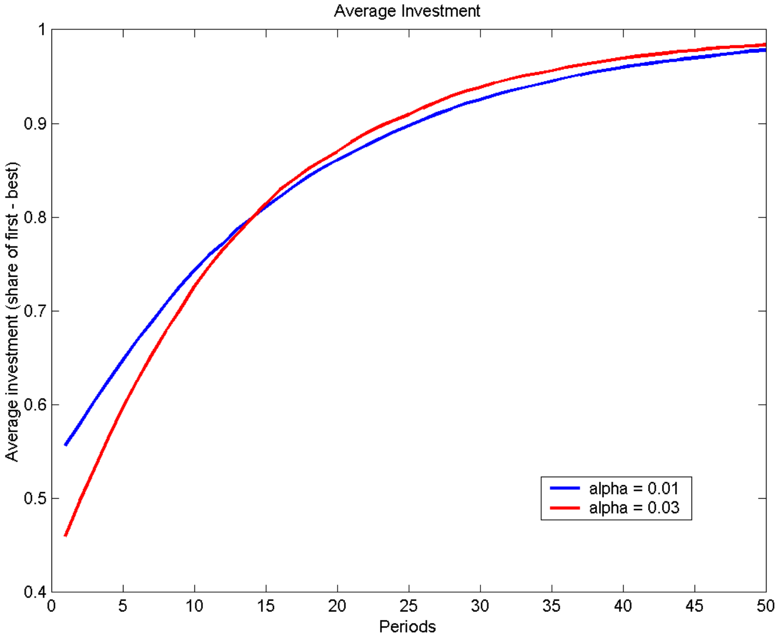


Figure 1: Average investment in firms as a function of age.

$\theta = 0.36$ comes from the assumption that the share of capital in proprietor's income is the same as in other sectors.

The first result of the numerical example is about average capital and capital-output ratios:

	$\alpha = 0.01$	$\alpha = 0.03$
k/k^*	0.79	0.75
k/y	3.2746	3.1622

As expected, lowering the enforcement cost allows the principal to invest at a more efficient scale and average capital is closer to the first-best if enforcement is cheaper.

Next, I plot average investment for a surviving firm of a particular cohort.

Lower enforcement cost allows the principal to offer higher initial utility. So younger firms in economies with good enforcement are less constrained. However, dynamic incentives are more important in economies with high enforcement costs, so these firms grow faster and get to the first-best investment level faster - eventually

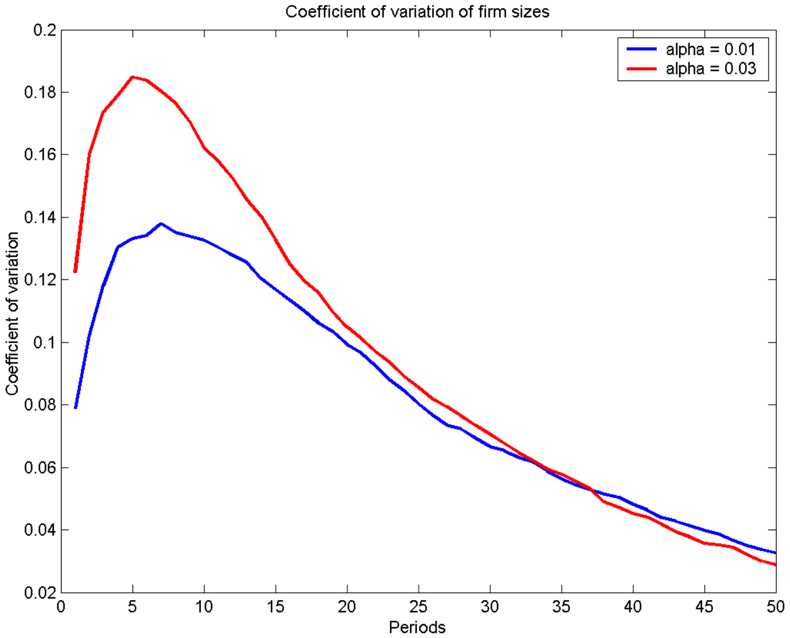


Figure 2: Coefficient of variation of firms sizes.

their average size overcomes firms in economies with lower enforcement costs.

A similar pattern can be observed in the variation of firm sizes. In figure 2, I plot the coefficient of variation of firm sizes of firms of a particular cohort. The history of productivity shocks matters more for firms in economies with high enforcement costs. Increased variation of firm sizes is inefficient if the production function is strictly concave.

3.2 Economic Development

In this section, I show how a basic modification of the model can be used to address the role of enforcement frictions in economic development. This paper relates to two connected fields of research. First, there is a vast literature linking economic development and the degree of the financial intermediation. For example, King and Levine (1993) show that the development of financial intermediation, measured by the ratio of outside financing to GDP, is strongly correlated to GDP per capita.[7]

Second, the degree of property rights protection and the quality of institutions enforcing contracts have been advanced as important causal factors for economic development. In two important papers Djankov et al (2003) and Djankov et al (2008) show that an index of enforcement quality varies very widely across countries. Moreover, they find a very strong correlation between economic development and their measure of enforcement quality. Figure 3 summarizes that evidence.

In my model, enforcement is a decision variable by the principal and its use is affected by its cost. Thus I show how differences in enforcement cost affects the efficiency of financial intermediation and, through it, incomes per capita.

As in the previous application, I will consider the stationary equilibria of two small open economies which are identical in all respects, except for enforcement cost: $\gamma_i(e) = \alpha_i \gamma(e)$, where i is the country index and $\alpha_1 > \alpha_2$. In this situation, we think of country 2 as the relatively less developed. (Enforcement frictions are more severe.)

Enforcement costs affect economic development through two mechanisms. The first is through inefficient firm size. The distribution of firm sizes in developing countries is markedly different from developed ones. Tybout (2000) demonstrates that firms sizes (measured by labor) are much smaller in developing countries. The negative correlation between average firm size and and income per capita persists both in cross section and across time. Moreover, since there is a substantial informal small-scale sector in developing countries, small firms are probably undercounted in estimates of firm size distributions in developing countries. Binding financial constraints, lead to inefficiently small firms (the entrepreneurial input is not used efficiently).

As shown above, increasing enforcement costs leads to firms that are smaller on average, and the variance of firm sizes is higher. Assuming concave production function, this reduces aggregate output. I have demonstrated these effects above.

The second mechanism is through endogenous choice of technology. Hall and Jones (1999) and Klenow and Rodriguez-Clare (1997) show that less than 50% of income per capita differences can be explained by differences in capital-output ratios and human capital. The remainder is attributed to differences in TFP.

---

[7] A very inexhaustive list of quantitative papers that explore this issue would include Quintin(2008), Townsend and Ueda (2006), Castro, Clementi and MacDonald (2004).

The model can easily accommodate technology choice and deliver a theory of TFP. Assume that at the moment of birth, the agent has access to a finite set of stochastic technologies $\mathbf{S}$. After the agent has specialized in a technology $\mathbf{s} \in \mathbf{S}$, she cannot learn a new technology. Since the intermediaries are competitive, there is a function relating technological choice and enforcement cost in the country and initial utility.

$$v_0(\mathbf{s}, \alpha) = \sup\{v \in \mathcal{V} : P^*(v; \mathbf{s}, \alpha) \geq c\}, \tag{37}$$

where $P^*(v; \mathbf{s}, \alpha)$ is the principal's value of delivering v, if the agent uses technology $\mathbf{s}$ and her enforcement cost is $\alpha\gamma(e)$.

Then the choice of technology will maximize date zero utility: $\mathbf{s}(\alpha) = \arg\max_{\mathbf{s}\in\mathbf{S}} v_0(\mathbf{s}, \alpha)$.

Let's assume that $F(k, s) = sf(k) + (1 - \delta)k$ for some strictly concave and differentiable function $f(k)$. Then in the absence of enforcement constraints, the optimal choice of $\mathbf{s}$ depends only on the mean of the process. (All productivity shocks are idiosyncratic.) However, in the presence of incentive problems, increasing the variance may affect the incentive constraints, and thus reduce the agent's initial utility.

I will consider a particular type of stochastic process for concreteness. Let the stochastic process take two values $s_1 = \overline{s} - \Delta$, $s_2 = \overline{s} + \Delta$ and $\pi_1 = \pi_2 = 1/2$. $\sigma(\mathbf{s}) = \Delta$. Increasing Δ is a mean-preserving spread of $\mathbf{s}$. Increasing Δ to Δ' makes the binding incentive constraints more severe, increasing the resources after a default by $(\Delta' - \Delta)f(k)$. However, the outside option may itself be affected by a change in $\mathbf{s}$. (For example, the outside option may be derived from autarchy after default.) Increasing Δ would raise the amount of resources after a default, but on the other hand would make the autarchy production more risky and decrease continuation utility. For Δ small enough, the former effect dominates the latter.

For example if $u(c) = \log(c)$, $F(k, s) = sk^\theta$, then for any k, and e the outside option autarchy would be:

$$O((\overline{s} + \Delta)k^\theta - e) = \frac{\log((\overline{s} + \Delta)k^\theta - e)}{1 - \alpha\beta} + \frac{\beta[\log(\overline{s} - \Delta) + \log(\overline{s} + \Delta)]/2}{(1 - \alpha\beta)(1 - \beta)} + T,$$

where T is a constant that does not depend on Δ. Increasing Δ from 0 raises the outside option, creating incentives for default. Since the incentive problems are less severe if enforcement costs are lower, I will show that for some parameter values, high mean - high variance stochastic process will be chosen if enforcement costs are low and vice-versa if they are high.

Proposition 8 *There exist two productivity processes* $\mathbf{s}_1$ *and* $\mathbf{s}_2$ *such that* $E\mathbf{s}_1 < \mathbf{s}_2$*,* $\sigma(\mathbf{s}_1) < \sigma(\mathbf{s}_2)$ *and* $\alpha_1 > \alpha_2$ *such that* $\mathbf{s}(\alpha_1) = \mathbf{s}_1$ *and* $\mathbf{s}(\alpha_2) = \mathbf{s}_2$*.*

Proof. In the appendix. ■

This proposition implies that the economy with the low enforcement cost (developed economy) would choose the more productive process and an economy with high enforcement costs would choose the less productive, but less volatile stochastic process.

4 Conclusions

In this paper, I develop a tractable model of capital accumulation and investment in the presence of costly enforcement. The principal can enforce compliance with the contract, but that is costly. Allowing the principal to commit to enforcement (alternatively, purchasing the enforcement capacity in advance) simplifies the incentive constraints and allows an efficient computation of the model using recursive techniques.

The model implies that enforcement is never the sole method of providing incentives and that investment will generally be distorted. Over time, the agent's utility (interpreted as equity) rises and the incentive constraints are relaxed.

The model described above is very abstract. In particular, a principal performs a variety of functions (financing investment, providing consumption insurance, allocating consumption over time) that would be performed by many financial institution in a real economy. Therefore the model implicitly assumes that all the financial intermediaries in a relationship with the agent can coordinate their actions costlessly and that the agent cannot enter into unobserved relationships.

If there are no frictions and financial intermediaries are competitive, this assumption does not affect the equilibrium outcomes. However, we know that in private information environments, private saving can constrain the allocations the principal can offer.[8] This is relevant in my model as well: if the agent can save privately, she

---

[8] For example in the model of income insurance with hidden saving by Cole and Kocherlakota (2001), after each history the agent receives zero net expected discounted transfers from the prin-

may build up a stock of saving which would increase the value of the outside option and create incentives for default. In this paper I assume that private saving is not possible. I show that if the outside option is derived from access to storage technology (case 2), this assumption does not bind.

A second issue that the paper abstracts from is the presence of frictions in the relationship between firms. If enforcement frictions constrain the kinds of deals firms can make, then there will be a benefit of vertical integration: one owner would internalize the frictions and improve efficiency. I have excluded this possibility by assuming a one-sector model. There are also benefits from horizontal expansion. The concave production function pins down an efficient production scale. If an agent is financially unconstrained, it would be efficient if she operates more than one establishments. In my model there is no distinction between firms, establishments and entrepreneurs. This is explained by the fact that each establishment requires managerial input and there may be enforcement frictions within a multi-establishment firm itself.

The model provides a feasible framework to study a variety of problems. In particular, I show that differences in enforcement cost have implications for firm dynamics. If enforcement is cheaper, firms start off bigger (in capital) and grow slower. In the case of economic development, the model can allow for endogenous technological choice. For some parameter values, the economies with higher enforcement cost will choose technologies with low mean and low variance.

---

cipal, which restricts the scope of possible insurance. Similar problems exist in the unemployment insurance literature.

References

[1] Albuquerque, R. and H. Hopenhayn (2004), "Optimal Lending Contracts and Firm Dynamics " *Review of Economic Studies 71 (2)*: 285-315

[2] Alfaro, L., A. Charlton and F. Kanczuk (2008), "Firm Size Distribution and Cross-Country Income Differences" *NBER Working Paper No. 14060*

[3] Buera, F., J.P. Kaboskiy and Y. Shin (2008), "Finance and Development: A Tale of Two Sectors " *mimeo*

[4] Cabral, L.M.B. and J. Mata (2003), "On the Evolution of Firm Size Distribution: Facts and Theory" *American Economic Review 93 (4)*: 1075-1090

[5] Castro, R., Clementi, G.L. and G. MacDonald (2004), "Investor Protection, Optimal Incentives, And Economic Growth" *Quarterly Journal of Economics 119 (3)*: 1131-1175

[6] Castro, R., Clementi, G.L. and G. MacDonald (2008), "Legal Institutions, Sectoral Heterogeneity, and Economic Development," *Review of Economic Studies*: forthcoming

[7] Cole, H.L. and N.Kocherlakota (2001), "Efficient Allocations with Hidden Income and Hidden Storage" *Review of Economic Studies 68 (3)*: 523-542

[8] Cooley, T.F. and V. Quadrini (2001), "Financial Markets and Firm Dynamics" *American Economic Review 91 (5)*: 1286-1310

[9] S. Djankov, R. La Porta, F. Lopez-de-Silanes and A. Shleifer (2003), "Courts," *Quarterly Journal of Economics 118*: 453-517

[10] S. Djankov, R. La Porta, F. Lopez-de-Silanes and A. Shleifer (2008), "Debt Enforcement Around the World," *Journal of Political Economy*: forthcoming

[11] Hall, R.E. and C.I. Jones (1999), "Why Do Some Countries Produce So Much More Output Per Worker Than Others?" *Quarterly Journal of Economics 114 (1)*: 83-116

[12] Hopenhayn, H. and I. Werning (2008), “Equilibrium Default” *mimeo*

[13] King , R. and R. Levine (1993), “Finance and Growth: Schumpeter Might be Right” *Quarterly Journal of Economics 108* : 717-737

[14] Klenow, P.J. and A. Rodriguez-Clare (1997), “The Neo-Classical Revival in Growth Economics: Has it Gone Too Far?” *NBER Macroeconomics Manual*

[15] Koeppl, T. (2007), “Optimal Dynamic Risk Sharing when Enforcement is a Decision Variable” *Journal of Economic Theory 134*: 34-60

[16] Krasa, S. and A. Villamil (2000), “Optimal Contracts when Enforcement is a Decision Variable” *Econometrica 68 (1)*: 119-134

[17] Luenberger, D.G., (1969), *Optimization by Vector Space Methods* Wiley, New York

[18] Lucas, R., (1978), “On the Size Distribution of Business Firms,” *Bell Journal of Economics 9 (2)*: 508-523

[19] Koeppl, T., C. Monnet and E. Quintin (2008), “The Poor, the Rich and the Enforcer: Institutional Choice and Growth” *Federal Reserve Bank of Dallas Working Paper 0801.*

[20] Popov, L. (2008), “Costly State Verification and Dynamic Contracts” *University of Iowa manuscript*

[21] Quintin, E. (2008), “Limited Enforcement and the Organization of Production” *Journal of Macroeconomics 30*: 1222-1245

[22] Shorack, G.R., (2000), *Probability for Statisticians* Springer Verlag, New York

[23] Stokey, N.L. and R.E. Lucas with E.C. Prescott, (1989), *Recursive Methods in Economic Dynamics* Harvard University Press, Cambridge

[24] Thomas, J. and T. Worrall (1994), “Foreign Direct Investment and the Risk of Expropriation” *Review of Economic Studies 61 (1)*: 81-108

[25] Tybout, J.R. (2000), “Manufacturing Firms in Developing Countries: How Well Do They Do and Why?” *Journal of Economic Literature 38*: 11-44

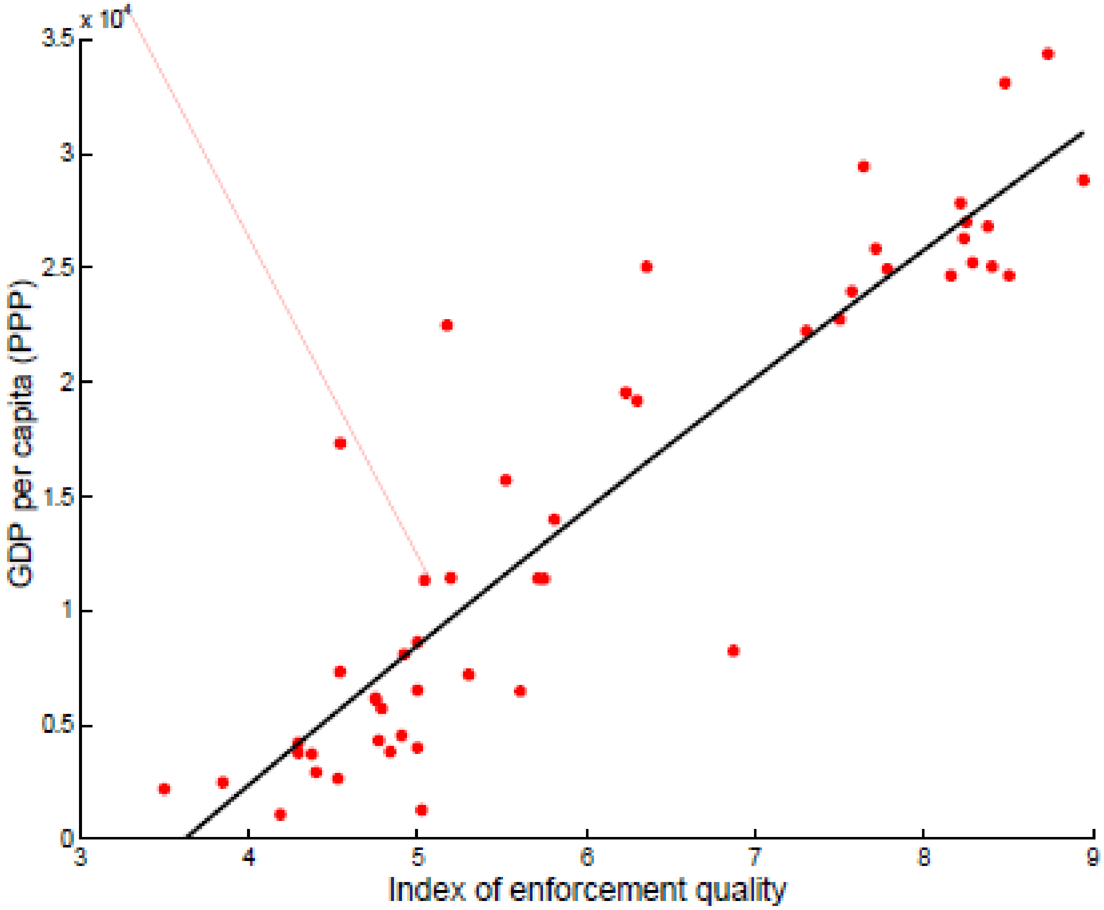


Figure 3: Index of enforcement quality, plotted against per capita PPP income. Sources: Doing Business Database, World Bank; PWT 6.2.

A Appendix

A.1 Static Example

Proposition 1

Proof. The problem can be rewritten as:

$$\max_{u(s)} \int_0^{\overline{s}} F(k,s)dG(s) - R(k+\gamma(e)) - \int_0^{\overline{s}} u^{-1}(u(s))dG(s) \tag{38}$$

$$\text{subject to} \int_0^{\overline{s}} u(s)dG(s) = v$$

$$u(s) \geq u(\max\{F(k,s)-e,0\}) \quad \forall s$$

That's a strictly concave objective function with a convex constraint set that has a nonempty interior. Then by theorem 8.3.1 in Luenberger (1969) there exist Lagrangian multipliers $\lambda, \eta(s)$ such that the optimum $c(s)$ solves the Lagrangian. Then by taking the Gateaux derivative, we establish the result. ■

Proposition 2

Proof. Assume $k > k^*$. Then from 38 we know that decreasing k to k^* increases the principal's profit without affecting the incentives. Now assume that $k = k^*$ for some $v < v^*$. Decreasing k marginally has second order effects on output net of interest costs, but relaxes the constraint on a set of positive measure and allows either reduction of enforcement or more efficient consumption profile. ■

A.2 Value function

Denote the feasibility correspondence by $\Gamma(v)$. If we have a contract (k,c,e,v') and $f \in D$, then define the value of that contract with continuation function f as:

$$\tilde{P}(k,c,e,v';f) = E[F(k,s_i) - c_i - R(k+e) + \beta f(v_i')]$$

Thus:

$$Tf = \sup_{(k,c,e,v') \in \Gamma(v)} \tilde{P}(k,c,e,v';f)$$

Lemma A.1 *If $f_1(v) \leq f_2(v)$ for all v, then $Tf_1(v) \leq Tf_2(v)$ for all v.*

Proof. For any $(k, c, e, v') \in \Gamma(v)$, $\tilde{P}(k, c, e, v'; f_1) \leq \tilde{P}(k, c, e, v'; f_2)$. Then

$$Tf_1(v) = \sup_{(k,c,e,v') \in \Gamma(v)} \tilde{P}(k, c, e, v'; f_1) \leq \sup_{(k,c,e,v') \in \Gamma(v)} \tilde{P}(k, c, e, v'; f_1) = Tf_2(v)$$

■

Lemma A.2 *For any scalar $a \in \mathbb{R}$, $T(f + a)(v) \leq Tf(v) + \beta a$*

Proof. For any $(k, b, e, v') \in \Gamma(v)$, $\tilde{P}(k, b, e, v'; f + a) = \tilde{P}(k, b, e, v'; f) + \beta a$. Then

$$T(f+a)(v) = \sup_{(k,b,e,v') \in \Gamma(v)} \tilde{P}(k, b, e, v'; f+a) \leq \sup_{(k,b,e,v') \in \Gamma(v)} \tilde{P}(k, b, e, v'; f) + \beta a = Tf_2(v) + \beta a$$

■

Lemma A.3 *If $f \in D$, then $Tf \in D$.*

Proof. Fix some v. It is feasible to set $k = 0$, $b_i = -u^{-1}((1 - \beta)v)$, $v'_i = v$, $e = 0$. Then

$$\begin{aligned} Tf(v) &\geq -u^{-1}((1-\beta)v) + \beta f(v) \\ &\geq -u^{-1}((1-\beta)v) + \beta P_1(v) \\ &= P_1(v) \end{aligned}$$

For every $f \in D$, $f(v) \leq P_2(v)$. Then by lemma A.1, $Tf \leq TP_2 \, leqP_2$. Then $Tf \in D$. ■

Then the Blackwell's theorem implies that the operator T is a contraction on D. I showed that $P^* \in D$. So, the final step is to show that $TP^* = P^*$.

Lemma A.4 *P^* solves the Bellman equation.*

Proof.

Consider an arbitrary feasible allocation that satisfies the incentive constraints. Then define the contract $k = k_0$, $c_i = c_0(s_i)$, $e = e_0$, $v'_i = v(s_i)$. This contract is incentive-compatible and feasible.

$$\begin{aligned} TP^*(v) &\geq E[F(k, s_i) - c_i + \beta P^*(v'_i)] - R(k + e) \\ &\geq E[F(k_0, s_i) - c_i + \beta \sum_{t=1}^{\infty} \beta^{t-1} E[F(y_t(s^t) - c_t(s^t) - R(k_t(s^{t-1}) + e_t(s^{t-1}))|s_0 = s_i] - R(k + e \\ &= \sum_{t=0}^{\infty} \beta^t E[y_t(s^t) - c_t(s^t) - R(k_t(s^{t-1}) + e_t(s^{t-1}))] \end{aligned}$$

This implies that

$$\begin{aligned} TP^*(v) &\geq \sup_{(k,b,e)\in\hat{\Gamma}(v)} \sum_{t=0}^{\infty} \beta^t E[y_t(s^t) - c_t(s^t) - R(k_t(s^{t-1}) + e_t(s^{t-1}))] \\ &= P^*(v) \end{aligned}$$

Now I need to show that $P^*(v) \geq TP^*(v)$ for all v. To reach a contradiction, suppose that $TP^*(v) - P^*(v) > \epsilon > 0$ for some v. By definition, there exists some (k, c_i, v_i', e) such that

$$E[F(k, s_i) - c_i + \beta^* P(v_i')] - R(k+e) > TP^*(v) - \epsilon/3$$

and (k, b_i, v_i', e) satisfy the promise-keeping and incentive-compatibility constraints. By the definition of P, for each i there is a contract (k^i, b^i, e^i) that is incentive compatible, delivers promised utility v_i' and satisfies:

$$\sum_{t=0}^{\infty} \beta^t E[F(y_t^i(s^t) - c_t^i(s^t) - R(k_t^i(s^{t-1}) + e_t^i(s^{t-1})) > P^*(v_i') - \epsilon/[3\beta\pi_i]$$

Then construct the sequence contract by: $k_0 = k$, $k_t(s_i, s^{t-2}) = k_{t-1}^i(s^{t-2})$,$e_0 = e$, $e_t(s_i, s^{t-2}) = e_{t-1}^i(s^{t-2})$, $c_0(s_i) = c_i$, $c_t(s_i, s^{t-1}) = c_{t-1}^i(s^{t-1})$. That sequence contract satisfies all the constraints and its value would be greater than $P^*(v) - \epsilon$ which is a contradiction. ■

Proposition 3

Proof. By lemmas A.1, A.2 and A.3 and the Blackwell theorem, we know that T is a contraction on D. D is a complete metric space, therefore the operator T has a unique fixed point in D. Finally, lemma A.4 shows that P^* is the unique solution of the Bellman equation on D. ■

Lemma A.5 *Without loss of generality,* $k(v) \leq k^*$.

Proof. Assume $k > k^*$. Then decreasing investment to k^* relaxes the incentive constraints and increases $EF(k,s) - Rk$, thus strictly increasing the principal's profit. A contradiction. ■

Lemma A.6 *Without loss of generality,* $e(v) \leq F(k^*, s_N)$.

Proof. Assume $e > e^* \equiv F(k^*, s_N)$. From lemma A.5, $k \leq k^*$. Then the resources available to the agent after a default is $0 \leq \max\{F(k, s_i) - e, 0\} \leq \max\{F(k^*, s_i) - e^*, 0\} = 0$. Thus decreasing enforcement to e^* increases the principal's profit. ■

Lemma A.7 *For any the optimal policy v_i' is bounded from below and from above by continuous functions $g_1(v)$ and $g_2(v)$ respectively.*

Proof.

Let $p^* = EF(k^*, s) - Rk^*$. The principal's value from a contract is:

$$\begin{aligned}
\tilde{P}(k, c, e, v'; f) &= EF(k, s) - R(k + \gamma(e)) - E(c_j) + \beta E f(v_j) \\
&\leq p^* + \beta(1 - \pi_i)E[f(v_j')|j \neq i] + \beta\pi_i f(v_i') \\
&\leq p^* + \beta\frac{(1 - \pi_i)p^*}{1 - \beta} + \beta\pi_i \left[\frac{p^*}{1 - \beta} - \frac{u^{-1}((1 - \beta)v_i')}{1 - \beta}\right] \\
&= \frac{p^*}{1 - \beta} - \frac{\beta\pi_i}{1 - \beta}u^{-1}((1 - \beta)v_i')
\end{aligned}$$

I showed that there exists a feasible contract that gives the principal value at least $-u^{-1}((1 - \beta)v)/(1 - \beta)$. Therefore, without loss of generality, I can impose the following constraint:

$$\tilde{P}(k, c, e, v'; f) \geq -u^{-1}((1 - \beta)v)/(1 - \beta)$$

Combining these inequalities, we get:

$$\begin{aligned}
\frac{p^*}{1 - \beta} - \frac{\beta\pi_i}{1 - \beta}u^{-1}((1 - \beta)v_i') &\geq -\frac{u^{-1}((1 - \beta)v)}{1 - \beta} \\
v_i' &\leq \frac{1}{1 - \beta}u\left[\frac{p^* + u^{-1}((1 - \beta)v)}{\beta\pi_i}\right]
\end{aligned}$$

Therefore a lower bound for v' would be $g_2(v) \equiv \frac{1}{1-\beta}u\left[\frac{p^*+u^{-1}((1-\beta)v)}{\beta\min_j\{\pi_j\}}\right]$

Now I derive a lower bound for v'

First, assume that $u(c)$ is bounded from below. Without loss of generality we can assume that $u(0) = 0$. Then $v_i' \geq 0$. So set $g_1(v) \equiv 0$. Now assume that $\lim_{c\to 0} u(c) = -\infty$. From the promise-keeping constraint:

$$\begin{aligned}
Eu(c_i) &= v - \beta E v_i' \\
&\geq v - \beta(1 - \pi_i)g_2(v) - \beta\pi_i v_i'
\end{aligned}$$

Since u is concave, $E(c_i) \geq u^{-1}(Eu(c_i)) \geq u^{-1}[v - \beta(1-\pi_i)g_2(v) - \beta\pi_i v_i']$. Thus:

$$\begin{aligned}
\tilde{P}(k, c, e, v'; f) &= EF(k, s) - R(k + \gamma(e)) - E(c_j) + \beta Ef(v_j) \\
&\leq p^* - u^{-1}[v - \beta(1-\pi_i)g_2(v) - \beta\pi_i v_i'] + \beta Ef(v_j) \\
&\leq -u^{-1}[v - \beta(1-\pi_i)g_2(v) - \beta\pi_i v_i'] + \frac{p^*}{1-\beta}
\end{aligned}$$

Proceeding as before and combining the two inequalities, we get:

$$\begin{aligned}
-u^{-1}[v - \beta(1-\pi_i)g_2(v) - \beta\pi_i v_i'] + \frac{p^*}{1-\beta} &\geq -\frac{u^{-1}((1-\beta)v)}{1-\beta} \\
v_i' &\geq \frac{1}{\beta\pi_i}\left\{v - \beta(1-\pi_i)g_2(v) - u\left[\frac{p^* + u^{-1}((1-\beta)v)}{1-\beta}\right]\right\}
\end{aligned}$$

To get an uniform lower bound for v_i' then set $g_1(v)$ to be the minimum (over i) of the expression above. By construction g_1 and g_2 are continuous. ■

Lemma A.8 *Without loss of generality, consumption c_i is bounded from below and above by continuous functions g_3 and g_4 respectively.*

Proof. First, I derive an upper bound for c.

$$\begin{aligned}
\tilde{P}(k, c, e, v'; f) &= EF(k, s) - R(k + \gamma(e)) - E(c_j) + \beta Ef(v_j) \\
&\leq p^* - \pi_i c_i + \beta Ef(v_j') \\
&\leq \frac{p^*}{1-\beta} - \pi_i c_i
\end{aligned}$$

Using the lower bound for the principal's value, we get:

$$c_i \leq \frac{1}{\min_j \pi_j} \frac{p^* + u^{-1}((1-\beta)v)}{1-\beta} \equiv g_4(v)$$

Now, I consider the lower bound for c. If $u(0) = 0$, then set $g_3(v) \equiv 0$. Assume not. From the promise-keeping constraint:

$$Ev' = \frac{v - Eu(c_j)}{\beta} \leq \frac{v - (1-\pi_i)u(g_4(v)) - \pi_i u(c_i)}{\beta}$$

$$\begin{aligned}
Ef(v') &\leq \frac{p^*}{1-\beta} - \frac{Eu^{-1}((1-\beta)v')}{1-\beta} \\
&\leq \frac{p^*}{1-\beta} - \frac{u^{-1}(E(1-\beta)v')}{1-\beta}
\end{aligned}$$

The principal's value is

$$\begin{aligned}\tilde{P}(k,c,e,v';f) &= EF(k,s)-R(k+\gamma(e))-E(c_j)+\beta Ef(v_j)\\ &\leq p^*+\beta Ef(v_j')\\ &\leq \frac{p^*}{1-\beta}-\frac{\beta}{1-\beta}u^{-1}\left[\frac{1-\beta}{\beta}(v-(1-\pi_i)u(g_4(v))-\pi_i u(c_i))\right]\end{aligned}$$

Using the lower bound of the principal's value, we get:

$$c_i \geq u^{-1}\left\{\frac{1}{\pi_i}\left(v-(1-\pi_i)u(g_4(v))-\frac{\beta}{1-\beta}u\left[\frac{p^*+u^{-1}((1-\beta)v)}{\beta}\right]\right)\right\}$$

Note that, by construction, this lower bound is well-defined and strictly positive. Take the minimum over i to find a uniform bound and set that to be $g_3(v)$. ■

Lemma 1

Proof. The operator T is a contraction and the set of continuous functions in D is complete, therefore I need to show that the operator T maps continuous function to continuous functions. Let $\Gamma(v)$ denote the feasibility correspondence. Define $A(v) \equiv [0,k^*]\times[g_3(v),g_4(v)]^N\times[0,F(k^*,s_N)]\times[g_1(v),g_2(v)]^N$. Define $\tilde{\Gamma}(v)=\Gamma(v)\bigcap A(v)$ to be the correspondence of feasible contracts that satisfy this additional constraints. Lemmas A.5, A.8, A.6 and A.7 state that without loss of generality we can take the supremum in $\tilde{\Gamma}(v)$.

$\tilde{P}(k,c,e,v';f)$ is continuous on $Gr(\tilde{\Gamma})$, and the theorem of the maximum implies that it is sufficient to show that $\tilde{\Gamma}$ is compact-valued and continuous.

Let the sequence $(k^i,c^i,e^i,v'^i)\in\tilde{\Gamma}(v)$ for some fixed v and converge to (k,c,e,v'). $A(v)$ is compact-valued by construction, therefore $(k,c,e,v')\in A(v)$. The constraints are continuous on $Gr(\tilde{\Gamma})$, therefore $(k,c,e,v')\in\Gamma(v)$. Then $(k,c,e,v')\in\tilde{\Gamma}(v)$, which shows that $\tilde{\Gamma}(v)$ is closed. However, it is bounded since $A(v)$ is bounded, therefore $\tilde{\Gamma(v)}$ is compact.

First I show that the correspondence is upper-hemicontinuous. Let $v_i'\to v$ and $(k^i,c^i,e^i,v'^i)\in\tilde{\Gamma}(v)$. Since g_i are continuous and $v_i'\to v$, then for some N if $i>N$, $(k^i,c^i,e^i,v'^i)\in B$, where $B=[0,k^*]\times[g_3(v)-\epsilon,g_4(v)+\epsilon]^N\times[0,F(k^*,s_N)]\times[g_1(v)-\epsilon,g_2(v)+\epsilon]^N$, where $\epsilon>0$. B is compact, therefore there exists a subsequence converging to (k,c,e,v'). Since all the constraints are continuous on $Gr(\tilde{\Gamma})$, $(k,c,e,v')\in\tilde{\Gamma}$.

Second, I prove that the correspondence is lower-hemicontinuous. Let $v_i \rightarrow v$ and $(k, c, e, v') \in \tilde{\Gamma}(v)$. I must show that there exists a sequence $(k^i, c^i, e^i, v'^i) \in \tilde{\Gamma}(v_i')$ such that $(k^i, c^i, e^i, v'^i) \rightarrow (k, c, e, v')$. Without loss of generality, we can assume that $v_1 \leq v_i' \leq v$ or $v \leq v_i' \leq v_1$. Let $\lambda_i \equiv (v_i - v_1)/(v - v_1)$. By assumption $\lambda_i \in [0, 1]$. Let $k^i = k$, $c_j^1 = u^{-1}((1-\beta)v_1)$, $v_j^1 = v_1$. Let $c_j^i = u^{-1}(\lambda_i u(c_j) + (1-\lambda_i)u(c_j^1))$. If this value violates one of the bounds, set c_j^i equal to the respective bound. Similarly, set $v_j^i = \lambda_i v_j + (1-\lambda_i)v_j^1$. If this value violates one of the bounds, set v_j^i equal to the respective bound. If the promised utility of the allocation is too high (low), decrease (increase) c_1^i until the promised utility constraint is satisfied, or one of the bounds is hit. If necessary, do the same adjustment in order to c_h^i and v_h^i. Note that c_j^i and v_j^i are continuous in λ. Finally, set

$$e^i = \max\{F(k^i, s_1) - O^{-1}(u(c_1^i) + \beta v_1^i), \cdots, F(k^i, s_N) - O^{-1}(u(c_N^i) + \beta v_N^i), e\}$$

By construction the contract $(k^i, c^i, e^i, v'^i) \in \tilde{\Gamma}(v_i)$. Since it is continuous in λ it converges to (k, c, e, v'). ■

Theorem 1 Let (k, c, e, v') be a maximizer in the Bellman equation. Then the allocation generated recursively satisfies the promise-keeping and incentive constraints and attains the supremum.

Proof. First I prove that the constraints are satisfied. It is sufficient to show that if $(\tilde{c}_t(s^t)$ is the allocation generated recursively from the policy functions (k, c, e, v') for some started v_0, then it delivers promised utility v_0. By induction on the recursive promise-keeping constraint we get:

$$E\sum_{j=0}^{t} \beta^j u(\tilde{c}_j(s^j)) + \beta^{t+1} E v_{t+1}(s^t) = v_0 \tag{39}$$

Then I need to show that $\lim_{t\rightarrow\infty} \beta^{t+1} E v_{t+1}(s^t) = 0$. Since (k, c, e, v') solves the Bellman equation, we know that

$$P(v_0) \leq \frac{p^*}{1-\beta} - E\sum_{j=0}^{t} \beta^j c_j(s^j) - \beta^{t+1} E \frac{u^{-1}((1-\beta)v_{t+1}(s^t))}{1-\beta} \tag{40}$$

In the inequality I have used the fact that $EF(k_j(s^{j-1}), s_j) - Rk_j(s^{j-1}) - R\gamma(e_j(s^{j-1})) \leq p^*$ and the upper bound of P. If u is bounded from above, then $\limsup_{t\rightarrow\infty} \beta^{t+1} E_0 v_{t+1} \leq$

0. Assume u is not bounded from above and $\limsup_{t\to\infty}\beta^{t+1}E_0v_{t+1} > 0$. Then there exists $A > 0$ such that for every t there exists $t' \geq t$ and $E_0v_{t'+1} \geq A\beta^{-t'-1}$.

Let $\eta(v) = \frac{1}{1-\beta}u^{-1}((1-\beta)v)$. It is strictly convex and $\lim_{v\to\infty}\eta'(v) = \infty$. Clearly $E\eta(v_{t+1}(s^t)) \geq \eta(Ev_{t+1}(s^t))$. Then for any $B > 0$, there exists $\tilde{v}$ such that if $v > \tilde{v}$, $\eta(v) > Bw$. By assumption there exists t' such that $E_0v_{t'+1} \geq A\beta^{-t'-1} > \tilde{v}$. Then:

$$\beta^{t'+1}\eta(E_0v_{t'+1}(s^{t'})) \geq \beta^{t'+1}BE_0v_{t'+1} \geq AB$$

Then $P(v_0) \leq p^*/(1-\beta) - AB$ for any $B > 0$, which is a contradiction, since $P(v_0) \geq -\eta(v_0)$. Therefore $\limsup_{t\to\infty}\beta^{t+1}E_0v_{t+1} \leq 0$.

If u is bounded from below, then $\liminf_{t\to\infty}\beta^{t+1}E_0v_{t+1} \geq 0$. Assume that u is not bounded from below and $\liminf_{t\to\infty}\beta^{t+1}E_0v_{t+1} < 0$.

$$P(v_0) \leq \frac{p^*}{1-\beta} - \frac{1-\beta^{t+1}}{1-\beta}u^{-1}\left(\frac{1-\beta}{1-\beta^{t+1}}[v_0 - \beta^{t+1}Ev_{t+1}(s^t)]\right) - E\beta^{t+1}\eta(v_{t+1}) \quad (41)$$

Using a similar argument as above, we get:

$$P(v_0) \leq \frac{p^*}{1-\beta} - \frac{1-\beta^{t+1}}{1-\beta}u^{-1}(A) \quad (42)$$

for any $A > 0$. This is either infeasible (if u is bounded from above) or arbitrarily negative, which gives a contradiction. ■

Lemma 5

Proof. Assume that the maximizer is not unique for some v. Then there are two different allocations (k^i, c^j, e^j), $j = 1, 2$ that give the principal the same profit. If capital and enforcement are the same (history by history) then we can increase the principal's profit by setting $\tilde{c}_t(s^t) = u{-}1[0.5u(c_t^1(s^t)) + 0.5u(c_t^2(s^t))]$. The new allocation lowers average consumption, but keeps utility, conditional on a state s^t, constant, thus keeping all the incentive constraints satisfied.

Assume that capital or enforcement is not the same history by history. Then for every s^{t-1} define $k_t^3(s^{t-1})$ by $f(k_t^3(s^{t-1})) = [f(k_t^1(s^{t-1})) + f(k_t^2(s^{t-1}))]/2$ and $e_t^3(s^{t-1}) = [e_t^1(s^{t-1}) + e_t^2(s^{t-1})]/2$. Then $y_t^3(s^t) = [y_t^1(s^t) + y_t^2(s^t)]/2$. Also set $c_t^3(s^t) = [c_t^1(s^t) + c_t^2(s^t)]/2$. Since f^{-1} and γ are both convex, this change would increase the principal's value. From the assumption on O it follows that

$$O(y_t^3(s^t) - e_t^3(s^{t-1})) \leq \frac{1}{1-\beta}u\left(\frac{O(y_t^1(s^t) - e_t^1(s^{t-1})) + O(y_t^2(s^t) - e_t^2(s^{t-1}))}{2}\right)$$

Fix some s^t. Consider the expected utility at date $t+j$, conditional on history s^t.

$$\sum_{s^j} \pi(s^j) u(0.5c^1_{t+j}(s^t, s^j) + 0.5c^2_{t+j}(s^t, s^j)) \geq u(0.5ce^1_j(s^t) + 0.5ce^2_j(s^t))$$

where $ce^i_j(s^t)$ is the certainty equivalent of the lottery $(c_{t+j}(s^t, s^j))$.

Now consider the consumption sequence (ce^i_j). By construction, $\sum_{j=0}^{\infty} \beta^j u(ce^i_j) = u(c^i_t(s^t)) + \beta v^i_{t+1}(s^t) \equiv w^i_t(s^t)$. We can think of this as a lottery with probabilities $\beta^j(1-\beta)$. Then:

$$w^3_t(s^t) = \sum_{j=0}^{\infty} \beta^j u\left(\frac{ce^1_j + ce^2_j}{2}\right) \geq \frac{1}{1-\beta} u\left(\frac{u^{-1}((1-\beta)w^1_t(s^t)) + u^{-1}((1-\beta)w^2_t(s^t))}{2}\right)$$

Then combining the three inequalities and using the fact that the original allocation was incentive compatible, we get that

$$u(c^3_t(s^t)) + \beta v^3_t(s^t) \geq O(F(k^3_t(s^{t-1}, s_t) - e^3_t(s^{t-1}))$$

The new allocation is incentive compatible, delivers strictly higher profit to the principal and increases promised utility to the agent. We have reached a contradiction. ■

Lemma 6

Proof. Let $v_1, v_2 \in A$, $\lambda \in (0,1)$ and $v_\lambda = \lambda v_1 + (1-\lambda)v_2 \in A$. Take the optimal allocations and find a new allocation, as described in the proof of lemma 5. The new allocation gives value to the principal at least $\lambda P^*(v_1) + (1-\lambda)P^*(v_2)$ and gives the agent utility strictly higher than v_λ. Since $v_\lambda \in A$, it follows that $P^*(v_\lambda) > \lambda P^*(v_1) + (1-\lambda)P^*(v_2)$. The second statement follows immediately from the earlier discussion. ■

Lemma 7

Proof. Let $v_0 \in int(A)$. The Inada condition for u implies that for some i, $c_i > 0$. Define $\psi(v) = P^*(v_0) + \pi_i c_i - \pi_i u^{-1}(u(c_i) + (v - v_0)/\pi_i)$. $\psi(v)$ is differentiable, strictly concave, defined on some neighborhood of v_0 and $\psi(v) \leq P^*(v)$. Then by the Benveniste-Scheinkman theorem (SLP 4.10), P^* is differentiable at v_0. ■

A.3 Optimal Allocations

Lemma 8 If $u(c_i(v))+\beta v_i'(v) \geq v^*$, then $v_i'(v) = u(c_i(v))/(1-\beta)$ If $u(c_i(v))+\beta v_i'(v) < v^*$, then $v_i'(v) > u(c_i(v)) + \beta v_i'(v)$. **Proof.** Consider the consumption allocation generated recursively by the optimal policy functions.

If $u(c_i(v)) + \beta v_i'(v) \geq v^*$, it is feasible and incentive compatible to set $c_t(s^t) = u^{-1}((1-\beta)v)$ for all s^t, which is the first-best consumption allocation. Then $v_i'(v) = E\sum_{t=1}^{\infty} \beta^{t-1} u(c_t(s^t)) = u(c_i(v))/(1-\beta)$.

Now consider the case when $u(c_i(v)) + \beta v_i'(v) < v^*$. First I show that for any history s^∞, consumption never falls: $c_{t+j}((s^t, s^j)) \geq c_t(s^t)$ where (s^t, s^j) is a history of s^t, followed by s^j. Assume not: $c_t(s^t) > c_{t+j}((s^t, s^j))$ for some s^t and s^j. Define $\tilde{c}$ by:

$$u(\tilde{c}) = \frac{u(c_t(s^t) + \beta^j Prob(s^j) u(c_{t+j}((s^t, s^j)))}{1 + \beta^j Prob(s^j)}$$

Change the consumption allocation to $c_t(s^t) = c_{t+j}((s^t, s^j)) = \tilde{c}$. By construction, the utility of an agent experienced a history of shocks s^h, $u(c_h(s^h)) + \beta v_{h+1}(s^h)$ does not decrease, therefore all the incentive constraints are satisfied. Similarly, the utility of that new allocation $u(c_0(s_i)) + \beta v_1(s_i)$ is not changed. Since u is strictly concave:

$$\beta^t Prob(s^t)[c_t(s^t) + \beta^j Prob(s^j) c_{t+j}((s^t, s^j))] - \beta^t Prob(s^t)\tilde{c}[1 + \beta^j Prob(s^j)] > 0$$

Therefore expected discounted consumption is strictly lower, increasing the value of the principal. We have reached a contradiction.

Now assume that $c_t(s^t) = c_0(s_i)$ for all s^t. If $k_1(s_i) = k^*$, $e_1(s_i) > 0$ and solves the equation $u(c_0(s_i))/(1-\beta) = O(F(k^*, s_N) - e)$. Consider decreasing consumption c_0 and increase c_1 in a way to keep promised utility constant. Increasing consumption marginally would reduce required enforcement. From the implicit function theorem, $de/dc_1 = -u'(c_1)/O'(y_N^* - e)$. Thus increasing consumption c_1 marginally would reduce enforcement costs by $R\gamma'(e) de/dc_1 > 0$. However since consumption is constant, there is no first-order effect on average consumption, therefore for small enough change, the principal's value would increase. Second, assume that $k_1 < k^*$. Then increasing k_1 marginally will increase surplus net of interest costs. In the states in which the incentive constraint binds, increase consumption to keep the incentives. Again, this reallocation of consumption does not have first-order costs, therefore for small enough increase in k, the principal's value is increased.

I have shown that for some s^t, $c_0(s_i) < c_t(s^t)$. Then $u(c_0(s_i)) < (1-\beta)v_0(s_i)$. Then $v_1(s_i) > u(c_0(s_i)) + \beta v_1(s_i)$. ■

Lemma A.9 *If $v < v^*$, then $E(v_i'(v)) > v$.*

Proof. From the promise keeping constraint $\sum_{i=1}^N \pi_i[u(c_i(v))+\beta v_i'(v)] = v$, it follows that for some j, $u(c_j(v)) + \beta v_j(v) \leq v < v^*$. Then lemma 8 implies that $v_j(v) > u(c_j(v)) + \beta v_j(v)$. By the same lemma $v_i'(v) \leq u(c_i(v)) + \beta v_i'(v)$, $\forall i$. Then

$$Ev_i'(v) = \pi_j v_j(v) + \sum_{i \neq j} \pi_i v_i'(v) > \sum_{i=1}^N \pi_i[u(c_i(v)) + \beta v_i'(v)] = v$$

■

Lemma A.10 *If $v \leq v^*$, then $v_i'(v)) \leq v^*$, $\forall i$.*

Proof. Assume that for some i, $v_i'(v) > v^*$. Then for some s^j, $c_{j+1}(s_i, s^j) > u^{-1}((1-\beta)v^*)$. Since $v \leq v^*$, for some s^l, $c_l(s^l) < u^{-1}((1-\beta)v^*)$. Then we can decrease $c_{j+1}(s_i, s^j)$ and increase $c_l(s^l)$ in such a way to keep $\beta^{j+1}\pi(s_i, s^j)u(c_{j+1}(s_i, s^j)) + \beta^l \pi(s^l)u(c_l(s^l))$ constant and $v_i'(v) \geq v^*$. Then this new allocation satisfies all the constraints and reduces average consumption, thus increasing the principal's profit. ■

Proposition 6 v_t converges to $\max\{v^*, v_0\}$ almost surely. **Proof.** If $v_0 \geq v^*$, then the first-best solution is incentive-compatible, therefore $v_i'(v_0) = v_0$, thus $v_t = v_0$ for all s^t.

Now assume that $v_0 \leq v^*$. Define $w_t = v_t - v^*$. Lemma A.7 and lemma A.10 show that $g_1^{(t)}(v_0) \leq v_t \leq v^*$. Therefore $E|w_t| \leq |g_1^{(t)}(v_0) - v^*| < \infty$. Lemma A.9 implies that $E[w_{t+1}|w_t] \geq w_t$. Therefore w_t is a submartingale.

Lemma A.10implies $Ew_t^+ = 0$. Thus $Ew_t^+ \nearrow 0$. So, by the submartingale convergence theorem, w_t converges to a r.v. w_∞ a.s. Define $v_\infty = w_\infty + v^*$. I will show that $v_\infty = v^*$ a.s. By lemma A.10, $v_\infty \leq v^*$ a.s. Assume that $v_\infty \neq v^*$ a.s. Then there exists some interval $[a,b]$ such that $Prob(w_\infty \in [a,b]) \equiv P > 0$ and $b < v^*$. Since $h(v) = Ev'(v) - v$ is a continuous function and $h(v) > 0$ if $v < v^*$, then there exists some $\delta > 0$ such that $h(v) > \delta$ for all $v \in [a,b]$. Let μ_t be the probability measure associated with v_t.

$$\begin{aligned} Ev_{t+1} - Ev_t &= \int h(v) d\mu_t(v) \\ &\geq \int_{v\in[a,b]} h(v) d\mu_t(v) \\ &\geq \delta\mu_t([a,b]) \end{aligned}$$

Since $v_t \rightarrow v_\infty$ a.s., for all t sufficiently large $\mu_t([a,b]) > P/2$, then $Ev_{t+1} > Ev_t + \delta P/2$. This is a contradiction since Ev_t converges. ■

A.4 Economic Development

Proposition 8 Proof. For any $\overline{s}$ there exist Δ and Δ' such that increasing Δ to Δ' would violate some incentive constraints and reduce v_0. By continuity, there is some $\overline{s'} > \overline{s}$ such that $v_0(\overline{s'}, \Delta', \alpha_1) < v_0(\overline{s}, \Delta, \alpha_1)$. But since the utility to the agent if there are no incentive constraints would be higher with $(\overline{s'}, \Delta')$ again by continuity, there exists some $0 < \alpha_2 < \alpha_1$ such that $v_0(\overline{s'}, \Delta', \alpha_2) > v_0(\overline{s}, \Delta, \alpha_2)$. ■