The Dynamic and Stochastic Shortest Path Problem with Anticipation

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Abstract

Mobile communication technologies enable truck drivers to keep abreast of changing traffic conditions in real-time. We assume that such communication capability exists for a single vehicle traveling from a known origin to a known destination where certain arcs en route are congested, perhaps as the result of an accident. Further, we know the likelihood, as a function of congestion duration, that congested arcs will become uncongested and thus less costly to traverse. Using a Markov decision process, we then model and analyze the problem of constructing a minimum expected total cost route from an origin to a destination that anticipates and then responds to changes in congestion, if they occur, while the vehicle is en route. We provide structural results and illustrate the behavior of an optimal policy with several numerical examples and demonstrate the superiority of an optimal anticipatory policy, relative to a route design approach that reflects the reactive nature of current routing procedures.
1 Introduction

In the past decade, the combination of real-time traffic information and in-vehicle communication technology has given drivers the ability to route themselves in coordination with changing traffic conditions. The ability to avoid traffic congestion is particularly important for drivers operating in a just-in-time delivery environment. For example, a trucking company can be penalized thousands of dollars per minute if a late shipment causes an assembly line to shut down at an auto plant [1]. Recent research has recognized the importance of these problems through time-dependent or real-time and stochastic shortest path problems. This research typically assumes that the level of congestion changes according to the time of day or that changes are observed, but are not based on the length of time a road segment has been congested. In this paper, we consider the case where the congestion dissipates over time according to some known probability distribution. This case is the result of an accident or some other unique or relatively rare event whose occurrence is hard to predict, but whose end is governed by such a probability distribution. For example, in

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the event of an accident, the more time that has passed since the accident, the more likely it is that emergency crews have arrived and are cleaning up the accident scene, getting traffic flowing again. As another example, consider United States’ border crossings. In this case, traffic backups can result when a particular vehicle requires additional security checks. The more time that has elapsed since this check was begun, the more likely that the check will end.

One way to overcome the congestion caused by one of these unique traffic events would be to re-route around the congested road segments. However, simply re-routing around traffic congestion or security delays could take the vehicle on an out-of-the-way and potentially much more expensive route. If the real-time traffic information is coupled with distributions on the amount of time that the particular road segments spend in each state of congestion or delay, the driver can geographically position the vehicle so as to be able to anticipate changes in the status of road segments. This ability to anticipate motivates the following problem.

Consider a single vehicle that is to travel on a known road network from a fixed origin to a fixed destination. The vehicle accrues a cost for each arc that is crossed en route to the destination, and this cost is arc dependent and stochastic. For a fixed cost, the driver also has the option of waiting at the current location. There exists a subset of arcs whose level of congestion is observable and for which the travel time is also dependent on the condition of the arc. For example, the travel-time distribution changes with regard to whether or not traffic on the arc is free flowing or has been slowed by an accident. We assume that the driver knows instantaneously if the status of one of these observed arcs has changed. In addition, if an arc is congested due to an accident or some other event, there is a known distribution on the likelihood
of the status of an observed arc transitioning from the congested state to an uncongested state. This distribution is parameterized by the amount of time that the observed arc has spent in its congested state and is assumed to follow the increasing failure rate property. The problem objective is to determine a policy for selecting a path that minimizes the expected total cost of the trip from origin to destination.

We emphasize that this paper is not focused on traffic congestion that results from typical “rush hours” or on traffic congesting events whose origins are unidentifiable, but rather on those events whose duration is stochastic and whose distribution depends on the amount of time that has elapsed since the event occurred. Further, we assume that these congesting events, such as accidents or security delays, are relatively rare and hence will assume that we cannot predict the occurrence of these events. Computational experience demonstrates that when these events are highly unlikely, they have no effect on the optimal policy nor on the expected cost [see [2]]. Thus, we will focus on the case where such events have already occurred and where the uncongested case is a trapping state.

As an example of the described problem, consider a vehicle that leaves a known origin for a known destination, where the destination must be reached before some prespecified future time. Suppose that the driver of the vehicle knows that there is an accident along the preferred path from origin and destination. Intuitively, it seems reasonable that, as a result of the congestion, the driver may want to consider an alternate route. However, if the driver knows a probability distribution on the duration of the congestion, the driver may find that it is still better to follow the preferred path in anticipation of the congestion clearing. At the same time, incorporating probabilities on the status of the
observed arc might produce a routing policy such that the route actually goes somewhere in between the extremes of preferring the congested arc and avoiding it altogether. In this way, the vehicle might be better positioned to divert to the particular arc if the arc’s status changes in a cost-improving way. Such a path would anticipate the change in the arc’s status. Hence, we call this problem the dynamic and stochastic shortest path problem with anticipation.

This paper is outlined as follows. In Section 2, we present a review of the relevant literature. Section 3 establishes a Markov decision process model for the problem with Section 4 providing the optimality equations and preliminary results for the model. In Section 5, we explore the structure of the cost function and optimal policy. Section 6 offers visual examples of optimal anticipatory policies. In Section 7, we characterize a non-anticipating policy. Section 8 explains the design of the numerical experiments intended to empirically describe the performance of anticipatory routing and to determine the value of anticipatory information relative to the non-anticipating policy, with Section 9 presenting the results of these experiments. Section 10 provides concluding remarks.

2 Literature Review

As the name of this paper suggests, this research falls under the broad category of dynamic and stochastic shortest path problems. The problems are dynamic in the sense that the cost of traversing arcs can change over the problem horizon. In general, the only stochasticity in shortest path problems results from decision makers having probabilistic information about the time required to traverse an arc. For generality, our model maintains stochastic travel times, but
we focus on another stochastic element associated with the dynamic changes in arc cost. We assume that certain arc-cost distributions themselves change according to a known probability distribution. To distinguish the two types of stochastic elements, we refer to the probabilistic information on the durations of congestion as anticipatory information. In the rest of this section, we present an overview of related literature. To limit the scope of the review, we have focused our attention to dynamic and stochastic shortest path problems, and the vehicle routing analogy, the dynamic and stochastic routing problem.

Initial work in stochastic and dynamic shortest paths was done by Hall [3] who introduced the stochastic, time-dependent shortest path problem (STDSPP). In the STDSPP, travel times on arcs are stochastic and the travel-time distributions on these arcs are known and time-dependent. Hence, the times at which arc travel time distributions transition is known in advance. Hall showed that the optimal policy is “adaptive.” That is, the optimal actions are subject not only to the vehicle’s location, but also to the time at which the vehicle is at the location. In this way, the policy accounts for the time dependency of the travel-time distributions. Bander and White [4] demonstrate the computational effectiveness of the heuristic search algorithm AO* on the STDSPP. Fu and Rilett [5] extend Hall’s work to the case of continuous-time random arc costs. Miller-Hooks and Mahmassani [6] present and compare algorithms for the least possible cost path in the discrete-time STDSPP. Miller-Hooks and Mahmassani [7] provide an algorithm for finding the least expected cost path in the discrete-time STDSPP. Chabini [8] provides an optimal run-time algorithm for a variation of the STDSPP in which arc costs are time dependent, but not stochastic.

Polychronopoulos and Tsitsiklis [9] relax the assumption on time-dependent
arc cost transitions and introduce the dynamic shortest path problem (DSPP).
In the DSPP, arc costs are randomly distributed, but the realization of the
random arc costs becomes known once the vehicle arrives at the arc. Cheung
[10] provides an iterative algorithm for a similar DSPP. Cheung and Muralhid-
priority shipments in the less-than-truckload trucking industry. Psaraftis and
Tsitsiklis [12] discuss another variation of the DSPP in which the distribu-
tions on arc travel times evolve over time according to a Markov process.
However, unlike the work presented in this paper, the changes in the status
of the arc are not observed until the vehicle reaches the arc. We note that, in
their conclusions, Psaraftis and Tsitsiklis and Polychronopoulos and Tsitsiklis
suggest a problem similar to the one being considered in this paper. Davies
and Lingras [13] propose a genetic algorithm for dynamically rerouting in a
situation similar to [12] in which the arc cost distributions are a function of
time, but these distributions become known only as that time arrives. Waller
and Ziliaskopoulos [14] develop algorithms for problems with limited spatial
and temporal arc dependencies.

Kim, Lewis, and White [15] extend [12] to include real-time information. They
model the problem in a manner similar to that presented in this paper (al-
though they do not consider the amount of time that an observed arc has
spent in a particular state) and present results regarding optimal departure
times from a depot and optimal routing policies. Kim, Lewis, and White [16]
present a state-space reduction technique that significantly improves compu-
tation time for the problem introduced in [15]. Azaron and Kianfar [17] extend
the analytical results shown in [12] for the case where the states of the current
arc and immediately adjacent arcs are known. Azaron and Kianfar also show
an example of a problem similar to that presented in this work. However, they offer no analytical results and the example is too small to show how a vehicle benefits from anticipating changes on its route. Rather than congestion clearing as is the case in this research, Ferris and Ruszczyński [18] present a problem in which arcs can fail and become unusable. Ferris and Ruszczyński model the problem as an infinite-horizon Markov decision process similar to stochastic shortest path formulations in [19] and provide an example that illustrates the behavior of the optimal policy.

In this paper, we define anticipation to mean that we have known probability distributions that are dependent on the amount of time spent in a particular state. Anticipation in this context has been considered in only a few works. Powell et al. [20] introduce a truckload dispatching problem, and Powell [21] provides formulations, solution methods, and numerical results. In these papers, future demand forecasts are used to determine which loads should be assigned to what vehicles in a truckload environment in order to account for forecasted capacity needs in the next period. Extending Datar and Ranade [22] in which bus arrival times are exponentially and independently distributed, Boyan and Mitzenmacher [23] provide a polynomial-time algorithm for the problem of traversing a bus network in which bus-arrival times are distributed according to a probability distribution with the increasing failure rate property. The distributions on bus-arrival times are analogous to the distribution on the duration of congestion presented in this research. Thomas and White [24] consider a problem analogous to the problem presented here in which customer service requests are anticipated instead of changes in the status of arcs. In [24], each customer has a known distribution on the time of day that the customer is likely to request service and the optimal policy
geographically positions the vehicle to respond to requests in a manner that minimizes expected total cost.

3 Model Formulation

In this section, we present a Markov decision model for the dynamic and stochastic shortest path problem with anticipation. Let the graph $G = (N, E)$ serve as a model of the road network, where $N$ is the finite set of nodes, modeling intersections, and $E \subseteq N \times N$ is the set of arcs, modeling one-way roads between intersections. The arc $(n, n') \in E$ if and only if there is a road that permits traffic to flow from intersection $n$ to intersection $n'$. Let $SCS(n) = \{n' : (n, n') \in E\}$ be the successor set of $n \in N$. We assume $n \in SCS(n)$. Let $s \in N$ be the origin, or start node, of the trip, and let $\gamma \in N$ be the destination, or goal node, of the trip. We assume $SCS(\gamma) = \gamma$. Let the set $E^I = \{e^1, \ldots, e^L\} \subseteq E$ be the set of observed arcs, where $L = |E^I|$ and $|E^I|$ is the cardinality of $E^I$. Each element of $E^I$ represents an one-way road for which we can observe congestion, and if congestion occurs, we know a probability distribution on the duration of the congestion.

We assume that there exists a path to $\gamma$ from every $n \in N$. A path $p$ from $n^1$ to $n^j$ in $N$ is a sequence of nodes $\{n^1, n^2, \cdots, n^{j-1}, n^j\}$ such that $n^1, \cdots, n^j$ are in $N$ and $(n^r, n^{r+1}) \in E$ for $r = 1, \cdots, j - 1$. Let $P(n)$ be the set of all paths from $n$ to $\gamma$.

A decision epoch occurs when the vehicle arrives at a node. Let $t_q$ be the time of the $q^{th}$ decision epoch, and let $n_q \in N$ be the position of the vehicle at time $t_q$. Then, $t_1 = 0$ and $n_1 = s$. Let $T$ be a finite integer indicating that action
selection terminates at a time no later than $T - 1$. $T$ might represent the time by which the vehicle is supposed to fulfill its pick-up or delivery obligations. For all $q$, we assume $t_q \in \{1, \ldots, T\}$. Let the random variable $Q$ be such that $\tilde{t}_Q < T$ and $\tilde{t}_{Q+1} \geq T$, where $\tilde{t}_Q$ and $\tilde{t}_{Q+1}$ are realizations of the random variables $t_Q$ and $t_{Q+1}$. We remark that $\tilde{t}_{Q+1} - T$ is the amount of time beyond the delivery deadline required for the vehicle to make its pick-up or delivery.

Assume that the information infrastructure is such that we know the status of each arc $e \in E^l$. We let $K^l(t)$ be a random variable representing the state of arc $e^l \in E^l$ at time $t$. Then, the status of the observed arcs is given by the vector $K(t) = \{K^1(t), \ldots, K^L(t)\}$ where

$$K^l(t) = \begin{cases} 
1 & \text{if arc } e^l \text{ is congested} \\
2 & \text{if arc } e^l \text{ is not congested.}
\end{cases}$$

We denote realizations of $K^l(t)$ and $K(t)$ as $k^l$ and $k$, respectively. We assume that the information infrastructure is such that the driver knows instantaneously when the status of arc $e^l \in E^l, l = 1, \ldots, L$, has changed.

In addition, we assume that the information infrastructure tracks the amount of time that each arc in $E^l$ has spent in its current state if the current state is congested. We let $X^l(t)$ be the number of time units that arc $e^l$ has been in its current state $K^l(t)$. We represent a realization of $X^l(t)$ by $x^l$ and assume $0 \leq x^l \leq \mathcal{X}$, where $\mathcal{X} < \infty$, for all $l$. If arc $e^l$ is in an uncongested state, $K^l(t) = 2$, we let $x^l = 0$ because in this the uncongested state $x^l$ has no effect on the cost or probability calculations. Then, $X(t) = \{x^1, \ldots, x^L\} \in \{1, \ldots, \mathcal{X}\}^L$. A realization of $X(t)$ is denoted by $x$. 

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We model the transition of observed arcs from congested to noncongested as discrete-time Markov chains. We recognize that these transitions actually occur in continuous time. However, in our model, decision epochs occur when the vehicle reaches an intersection and not when a transition occurs at an observed arc. To be consistent with our model of arc travel times, we then approximate the time of observed arc transitions using a discrete-time model.

Given the congestion causing events being discussed in this paper are relatively rare, such as accidents, the congestion on one arc is unlikely to imply congestion on another. Formally, we then make the following assumption.

**Assumption 1** \{K^i(t), t = 0, 1, \ldots, T\} and \{K^j(t), t = 0, 1, \ldots, T\} are independent Markov chains for \(i \neq j\).

Further, because congesting causing events are relatively rare and unlikely to occur while the vehicle is en route, we consider the uncongested state a trapping state. We note that, in the event that a congesting causing event occurs during the problem horizon, we can solve the new problem by simply resolving the problem letting our current position be the start node.

Given these assumptions, for each arc \(e^l \in E^l\), we assume that the state dynamics for \(k^l = 1\) and \(x^l\) are described by the one-step transition matrix:

\[
R^l_{(x,x+1)} = \begin{bmatrix}
1 - \alpha^l_x & \alpha^l_x \\
0 & 1
\end{bmatrix},
\]

where \(\alpha^l_x\) is the probability that, having been in state \(k^l = 1\) for \(x\) time units, arc \(e^l\) transitions from \(k^l = 1\) to \((k^l)' = 2\) in one time unit. We also make the following assumption on the nature of \(\alpha^l_x\).
Assumption 2 Let $\alpha^l_x \leq \alpha^l_{x'}$ for all $l$ and for all $x^l$ and $(x^l)'$ such that $x^l \leq (x^l)'$.

This condition is equivalent to assuming that the distribution on congestion duration has an increasing failure rate property (IFR)[25]. Intuitively, the IFR property says that the longer an arc has been congested, the more likely it is that the congestion will clear. This assumption reflects our desire to model congestion resulting from accidents or security delays rather than from the time-dependent congestion resulting from “rush hours.” The IFR property is frequently used in reliability theory to represent the fact that, the more a machine is used, the more likely it is to fail. For additional discussion of the IFR property, see [25] and [26].

Let $P(k' \mid t, k, x, t')$ be the probability of a transition occurring from $k$ at time $t$ with $x$ units of time in state $k$ to $k'$ at time $t' > t$. By our assumptions,

$$P(k' \mid t, k, x, t') = P(K(t') = k' \mid k(t) = k, X(t) = x) = \prod_{l=1}^{L} P(K^l(t') = (k^l)' \mid K^l(t) = k^l),$$

where each term in the product satisfies an extension of Kolmogorov’s equations for the non-stationary case [see Kim, Lewis, and White [15]] such that

$$R^l_{(x,x')} = \begin{bmatrix} 1 - \alpha^l_x \alpha^l_x & 1 - \alpha^l_{x+1} \alpha^l_{x+1} & \cdots & 1 - \alpha^l_{x'} \alpha^l_{x'} \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$
\[ P(t', k' \mid n, t, k, x, n') = P(t' \mid n, t, k, n')P(k' \mid t, k, x, t'), \]

where \( P(t' \mid n, t, k, n') \) is the probability that a vehicle leaving node \( n \) at time \( t \) will arrive at node \( n' \) at time \( t' > t \), assuming \( k(t) = k \) and that the vehicle travels on arc \((n, n') \in E\).

We make two assumptions with regard to \( P(t' \mid n, t, k, n') \).

**Assumption 3** Let \( P(t' \mid n, t, k, n') \) be stationary in \( t \) and remark that \( P(t' \mid n, t, k, n') \) depends on \( k \) only if \((n, n') = e^l \in E^l\), and then only on \( k^l \).

Our second assumption describes the relationship between travel time and congestion.

**Assumption 4** If \((n, n') = e^l \in E^l \) and \( k^l < \bar{k}^l \), then, for all \( \lambda \in \{1, 2, \ldots\} \),

\[
\sum_{t' \leq \lambda} P(t' \mid n, t, \bar{k}, n') \geq \sum_{t' \leq \lambda} P(t' \mid n, t, k, n').
\]

Assumption 4 implies that \((t' - t \mid n, k^l, n')\) stochastically dominates \((t' - t \mid n, \bar{k}^l, n')\), where \((t' - t \mid n, k^l, n')\) and \((t' - t \mid n, \bar{k}^l, n')\) are random variables representing the amount of time required to traverse arc \((n, n') = e^l\) when in states \( k^l \) and \( \bar{k}^l \), respectively. Essentially, stochastic dominance implies that a congested arc is likely to take a longer time to cross than an uncongested arc.

For additional discussion of stochastic dominance, see [27]. We also assume that there exists some \( \beta \) such that, for all \( n, t, k, \) and \( n' \), \( P(t' \mid n, t, k, n') = 0 \) for all \( t' \) such that \( t' - t > \beta \). Finally, we assume \( P(t + 1 \mid n, t, k, n) = 1 \). Thus, if the action chosen when at node \( n \) is to stay at node \( n \), then the driver must wait 1 time unit before making another decision.

The action set is \( A(n) = SCS(n) \). Thus, when at node \( n \) at time \( t \), \( K(t) = k \),
and \( X(t) = x \), the driver chooses the next node to which to travel. We remark that \(|SCS(n)|\) is necessarily finite.

A decision rule at time \( t \) is a function \( \delta(\cdot, t, \cdot, \cdot) : N \times \{1, 2\}^L \times \{1, \ldots, X\}^L \to A(n) \) that selects an available action at each time \( t \). Thus, \( \delta(n, t, k, x) \in A(n) \).

A policy is a sequence of decision rules \( \pi = \{\delta(\cdot, 1, \cdot, \cdot), \delta(\cdot, 2, \cdot, \cdot), \ldots, \delta(\cdot, T - 1, \cdot, \cdot)\} \). We remark that \( \delta(\cdot, t, \cdot, \cdot) \) is implemented only if \( t \) is a decision epoch.

The cost structure is composed of travel costs between nodes. Let \( c(n, t, k, x, a) = c(n, t, k, a) \) be the expected cost to be accrued between decision epochs \( q \) and \( q + 1 \), given the vehicle is currently at node \( n(t_q) = n, t_q = t, K(t_q) = k, X(t_q) = x \), and \( a \in A(n) \). For \( a = n' \) and \( t_{q+1} = t' \) being the time of arrival at node \( n' \),

\[
c(n, t, k, n') = \sum_{t'} P(t' | n, t, k, n') \tilde{c}(t' - t),
\]

where \( \tilde{c}(t) \) is the cost of traveling for \( t \) units of time and where the strictly positive function \( \tilde{c} \) is non-decreasing in \( t \). We assume \( \tilde{c}(t) < \infty \) for all \( t < \infty \).

We define \( k \leq \bar{k} \) if and only if \( k^l \leq \bar{k}^l \) for all \( l \). Then, Assumption 4 implies that \( c(n, t, k, a) \geq c(n, t, \bar{k}, a) \) for all \( k \leq \bar{k} \) [see [27], p. 405]. We assume \( c(\gamma, t, k, \gamma) = 0 \) for all \( t \) and \( k \). We assume the terminal cost, i.e., \( n_{Q+1} \neq \gamma \), accrued is \( \tilde{c}(n_{Q+1}, t_{Q+1}) \). For \( n_{Q+1} \neq \gamma \), we let \( \tilde{c}(n_{Q+1}, t_{Q+1}) = \tilde{c}(n_{Q+1}, t_{Q+1}) + \tilde{c}(n_{Q+1}, t_{Q+1}) \). We also make the following assumption.

**Assumption 5** For every \( t > T \), we define \( \tilde{c}(n, t) \geq \min_{p(n) \in P(n)} \tilde{c}_\beta(p(n)) \), where \( \tilde{c}_\beta(p(n)) \) is the cost of the path \( p(n) \) with all arcs requiring \( \beta \) units of time to traverse.

In reality, the penalty associated with missing service obligations at time \( T \) would likely be much larger than \( \tilde{c} \) [see [1]]. Our definition of \( \tilde{c} \) simply reflects
the minimum requirement of our structural results. We let \( \hat{c}(n, t) \) be the cost associated with the amount of time by which the time horizon was violated. We assume \( t' \leq t'' \) implies \( 0 < \hat{c}(n, t') \leq \hat{c}(n, t'') \), and for all \( t \) such that \( t - T \leq \beta, \hat{c}(n, t) \leq \beta \) for all \( n \). Thus, the earlier we terminate after time \( T \), the better.

Let
\[
    f^\pi(s, 1, k, x) = E^\pi_{k,x} \left\{ \sum_{q=1}^{Q} c(n, t_q, k_q, x_q, a_q) + \hat{c}(n_{Q+1}, t_{Q+1}) \right\}
\]
be the problem criterion, where \( E^\pi_{k,x} \) is the expectation operator, conditioned on the use of policy \( \pi \), \( K(1) = k \), and \( X(1) = x \). The problem objective is to find a policy \( \pi^* \), called an optimal policy, such that \( f^{\pi^*}(s, 1, 1, 0) \leq f^\pi(s, 1, 1, 0) \) for all policies \( \pi \) and for all \( k \) and \( x \).

### 4 Optimality Equations and Preliminary Results

All of the results in this section can be found in [Puterman [28], section 4.3].

The optimality equation is
\[
    f(n, t, k, x) = \min \{ c(n, t, k, x, a) + \sum_{t', k', x'} P(t' | n, t, k, a) \times P(k' | t, k, x) P(x' | t, k, x, t', k') f(a, t', k', x') : a \in A(n) \},
\]
where \( f(n, t, k, x) = \hat{c}(n, t) \) for \( T \leq t \leq T + \beta \). The solution of the optimality equation is unique, and \( f(n, t, k, x) = \min_\pi f^\pi(n, t, k, x) \), for all \( n, t, k \), and \( x \). We refer to \( f(n, t, k, x) \) as the cost or cost-to-go function. A necessary and sufficient condition for \( \pi^* \) to be optimal is that it is composed of decision rules that cause the minimum in the optimality equation to hold.
Notationally, it will be useful to make the dependence of $f(n, t, k, x)$ explicit on $P_x$, where $P_x = \{P(\cdot | t, k, x') : x' \geq x\}$ and $P(\cdot | t, k, x) = \{P(t' - t | k, x') : t' > t\}$. Then, $P_x$ is a vector in which each component is itself a vector of probabilities. This notation will allow us to differentiate between different distributions on amount of time that arcs remain congested. Note, $P_x = \{P(\cdot | t, k, x), P_{x+1}\}$. Thus, we will occasionally refer to $f(n, t, k, x)$ as $f(n, t, k, x, P_x)$.

5 Structural Results on the Cost Function and Optimal Policy

In this section, we present the structural results on the cost function and on the optimal policy for the dynamic and stochastic shortest path problem with anticipation. These results imply that there exist conditions under which the current optimal policy remains optimal when problem data changes. In addition, the results offer the practitioner an improved intuitive understanding of the behavior of the cost function and the optimal policy. We will first present a useful result, proof of which can be found in Puterman [28], p. 106.

**Lemma 1** Let $\{p_j\}$ and $\{p'_j\}$ be real-valued non-negative sequences satisfying

$$\sum_{j=k}^{\infty} p_j \geq \sum_{j=k}^{\infty} p'_j, \quad (1)$$

for all $k$, with equality holding for $k = 0$. Suppose $v_{j+1} \geq v_j$ for $j = 1, \ldots$, then

$$\sum_{j=0}^{\infty} p_j v_j \geq \sum_{j=0}^{\infty} p'_j v_j, \quad (2)$$

where the limits exist but may be infinite.
Equation 1 is equivalent to the statement that $p_j$ stochastically dominates $p'_j$.

Thus, the consequence of the lemma, Equation 2, is the expected value of $v$ is greater under the distribution given by $\{p_j\}$ than by $\{p'_j\}$.

We now present inequalities on the cost function. Our first result implies that the sooner we reach any intersection, the less the expected future cost. This result is important for future results related to the value function and the arc status vector $k$.

**Theorem 1** Given Assumptions 3 and 5, the cost function $f(n, t, k, x)$ is non-decreasing in $t$ for all $n, k,$ and $x$.

**Proof:** The result holds by Assumption 5 for all $n, k, x$ and for any $t$ and all $t'$ such $t < t'$ and $t' \geq T$. We assume that the result holds for all $n, k, x$ and for all $t''$ and $t'$ such that $t'' < t'$ and $t'' > t$. We note that Assumption 3 implies $P(t + a \mid n, t, k, n') = P(t' + a \mid n, t', k, n')$ for all $n, t, t', k, n'$, and $a$, $P(k' \mid t, k, x, t + a) = P(k' \mid t', k, x, t' + a)$ for all $t, t', k, x, k'$, and $a$, and $P(x' \mid t, k, x, t' + a, k') = P(x' \mid t', k, x, t' + a, k')$ for all $t, t', k, x, k', x'$, and $a$. Then, assuming $t' \geq t$,

$$f(n, t, k, x) = \min_{n'} \{c(n, t, k, n') + \sum_a P(t + a \mid n, t, k, n') \times \sum_{k'} \sum_{x'} P(k' \mid t, k, x, t + a) f(n', t + a, k', x') \leq \min_{n'} \{c(n, t', k, n') + \sum_a P(t' + a \mid n, t', k, n') \times \sum_{k'} \sum_{x'} P(k' \mid t', k, x, t' + a) f(n', t' + a, k', x') \} = f(n, t', k, x),$$

where the inequality follows from the induction hypothesis. □
The next result presents inequalities associated with the status of the observed arcs. In the \( L = 1 \) case, it is intuitive that the more improved the status of the observed arc, the less cost associated with traversing the network. Recall that \( k \leq \bar{k} \) if and only if \( k^l \leq \bar{k}^l \) for all \( l \).

**Theorem 2** Given Assumptions 3, 4, and 5 and assuming \( k \leq \bar{k} \), \( f(n, t, k, x) \geq f(n, t, \bar{k}, x) \), for all \( n, t, \) and \( x \).

**Proof:** It is sufficient to prove the result for the \( L = 1 \) case. By Assumption 5, the result holds for all \( n, k \), and any \( x \) and \( x' \) when \( t \geq T \). We assume that the result holds for all \( t' > t \). The following string of inequalities completes the result:

\[
f(n, t, k = 1, x) = \min_{n'} \{c(n, t, k = 1, n') + \sum_{t'} P(t' | n, t, k = 1, n') \times (P(k' = 2 | t, k = 1, x, t')f(n', t', k' = 2, x' = 0) + (1 - P(k' = 2 | t, k = 1, x, t'))f(n', t', k' = 1, x + t' - t)) \}\]

\( \geq \min_{n'} \{c(n, t, k = 2, n') + \sum_{t'} P(t' | n, t, k = 1, n') \times (P(k' = 2 | t, k = 1, x, t')f(n', t', k' = 2, x + t' - t)) \} \) (3)

\( \geq \min_{n'} \{c(n, t, k = 2, n') + \sum_{t'} P(t' | n, t, k = 2, n') \times (P(k' = 2 | t, k = 2, x, t')f(n', t', k' = 2, x + t' - t)) \} \) (4)

\( = \min_{n'} \{c(n, t, k = 2, n') + \sum_{t'} (P(t' | n, t, k = 2, n')f(n', t', k' = 2, x + t' - t)) \} \) (5)

\( \geq \min_{n'} \{c(n, t, k = 2, n') + \sum_{t'} (P(t' | n, t, k = 2, n')f(n', t', k' = 2, x + t' - t)) \} \) (6)

\( = f(n, t, k = 2, x), \)

where the inequality 3 follows from Assumption 4 and inequality 4 from the induction the hypothesis. The equality in 5 is due to fact that state 2 is a
trapping state and how long we have been in the trapping state has no effect on future costs. The inequality in 6 follows from Lemma 1 and Theorem 1.

We now show that the longer that an arc has been congested, the less the expected total cost-to-go. We say $x \leq \bar{x}$ if and only if $x^l \leq \bar{x}^l$ for all $l$.

**Theorem 3** Given Assumptions 2 and 5, when $x \leq \bar{x}$, $f(n, t, k, \bar{x}) \leq f(n, t, k, x)$ for all $n, t, k$.

**Proof:** It is sufficient to prove the result for the $L = 1$ case. By Assumption 5, the result holds for all $n, k, x$ and $x'$ such that $x \leq x'$ when $t \geq T$. We assume that the result holds for all $t' > t$. To complete the proof, we have the following string of inequalities:

$$f(n, t, k = 1, x) = \min_{n'} \{c(n, t, k = 1, n') + \sum_{n'} P(t' | n, t, k = 1, n') \times (P(k' = 2 | t, k = 1, x, t') f(n', t', k' = 2, x'' = 0) + (1 - (P(k' = 2 | t, k = 1, x, t'))) f(n', t', k' = 1, x + t' - t)) \} \geq \min_{n'} \{c(n, t, k = 1, n') + \sum_{n'} P(t' | n, t, k = 1, n') \times (P(k' = 2 | t, k = 1, \bar{x}, t') f(n', t', k' = 2, 0) + (1 - (P(k' = 2 | t, k = 1, \bar{x}, t'))) f(n', t', k' = 1, x + t' - t)) \} \geq \min_{n'} \{c(n, t, k = 1, n') + \sum_{n'} P(t' | n, t, k = 1, n') \times (P(k' = 2 | t, k = 1, \bar{x}, t') f(n', t', k' = 2, 0) + (1 - (P(k' = 2 | t, k = 1, \bar{x}, t'))) f(n', t', k' = 1, \bar{x} + t' - t)) \} = f(n, t, k = 1, \bar{x}),$$

where the first inequality follows from our Assumption 2 and Theorem 2. Induction completes the proof.

We will now show that the more likely an arc is to become uncongested, the more likely that costs will be reduced. We also show that the cost function is PLNIC. Recall, a real-valued function $h$ on a linear vector space $Y \subseteq \mathbb{R}^L$
is PLNIC if and only if there is a finite set $\Gamma \subseteq \mathbb{R} \times \mathbb{R}^L$ such that $h(y) = \min\{\gamma_m + \gamma^1_m y : (\gamma_m, \gamma^1_m) \in \Gamma\}$, where $\gamma^1_m \leq 0$ for all $m$. For two vectors $y$ and $z$ of the same order, we define $yz$ to be the usual vector dot product.

**Theorem 4** (a) Assume $(P^l)'(t' - t'' | k^l, x^l) \geq P^l(t' - t'' | k^l, x^l)$ and concomitantly $(P^l)'((x^l)' | t, k^l = 1, x^l, t', (k^l)' = 2) \geq P^l((x^l)' | t, k^l = 1, x^l, t', (k^l)' = 2)$ for all $l = 1, \ldots, L$ and for all $t'' \geq t, t' > t''$. Then, given Assumption 5, $f(n, t, k, x, P_x) \geq f(n, t, k, x, P_x')$, for all $n, k, x$.

(b) For $L = 1$, $f(n, t, 1, x, P_x)$ is PLNIC in $\{P(t+1-t | k, x), \ldots, P(T-1-t | k, x)\}$.

**Proof:** (a) Without loss of generality, it is sufficient to prove the result for the $L = 1$ case. By Assumption 5, the result holds for all $n, k, x$, and $t \geq T$. Assume that the result holds for all $t'$ such that $t' > t$. Because the uncongested state is a trapping state, the result holds for $k = 2$. For $k = 1$, the result is due to the following string of inequalities:

$$f(n, t, k = 1, x, P_x) = \min_{n'}\{c(n, t, k = 1, n')$$
$$+ \sum_{t'} P(t' | n, t, k = 1, n')$$
$$\times (P(k' = 2 | t, k, x, t')f(n', t', k' = 2, x' = 0, P_x)$$
$$+ (1 - P(k' = 2 | t, k = 1, x, t'))f(n', t', k' = 1, x + t' - t, P_{x+v-t}))\}$$
$$\geq \min_{n'}\{c(n, t, k = 1, n')$$
$$+ \sum_{t'} P(t' | n, t, k = 1, n')$$
$$\times (P'(k' = 2 | t, k, x, t')f(n', t', k' = 2, x' = 0, P_x')$$
$$+ (1 - P'(k' = 2 | t, k = 1, x, t'))f(n', t', k' = 1, x + t' - t, P_{x+v-t}))\}$$

$$= f(n, t, k, x, P'_x),$$

$$f(n, t, k = 1, x, P_x) = \min_{n'}\{c(n, t, k = 1, n')$$
$$+ \sum_{t'} P(t' | n, t, k = 1, n')$$
$$\times (P(k' = 2 | t, k, x, t')f(n', t', k' = 2, x' = 0, P_x)$$
$$+ (1 - P(k' = 2 | t, k = 1, x, t'))f(n', t', k' = 1, x + t' - t, P_{x+v-t}))\}$$

$$= f(n, t, k, x, P'_x),$$

$$f(n, t, k = 1, x, P_x) = \min_{n'}\{c(n, t, k = 1, n')$$
$$+ \sum_{t'} P(t' | n, t, k = 1, n')$$
$$\times (P(k' = 2 | t, k, x, t')f(n', t', k' = 2, x' = 0, P_x)$$
$$+ (1 - P(k' = 2 | t, k = 1, x, t'))f(n', t', k' = 1, x + t' - t, P_{x+v-t}))\}$$

$$= f(n, t, k, x, P'_x).$$
where inequality 7 follows from assumptions, Theorem 2, and Theorem 3, and the inequality in 8 from the induction hypothesis.

(b) Recursively expanding the optimality equations for $t' = t + 1$ to $t' = T - 1$ and rearranging the terms, we get

\[
\begin{align*}
f(n, t, k = 1, x, P_x) &= \min_{n'} \{ c(n, t, k, n) + \sum_{t' \geq T} \hat{c}(n', t') \\
&+ \sum_{t'=t+1}^{T-1} P(t' \mid n, t, k, n') f(n', t', k' = 1, x + t' - t, P_{x+t'-t}) \\
&+ \sum_{t'=t+1}^{T-1} P(t' \mid n, t, k, n') P(k' = 2 \mid t, k = 1, x, t') \\&\times (\sum_{x'} P(x' \mid t, k = 1, x, t', k' = 2)[f(n', t', k' = 2, x', P_{x'}) \\
&- f(n', t', k' = 1, x + t' - t, P_{x+t'-t})])\}.
\end{align*}
\]

Clearly, the term within the brackets is linear in \{\(P(t' - t \mid k, x), t' = t + 1, \ldots, T - 1\)\} for each \(n' \in SCS(n)\) and each term has a non-positive coefficient, since \(P(t' \mid n, t, n') \geq 0\), and by Theorem 2, \(f(n', t', k = 2, x' P_{x'}) \leq f(n', t', k = 1, x + t' - t, P_{x+t'-t})\) for all \(t' > t\) and for any \(x'\) and \(x + t' - t\). The minimum of a finite number of such functions is PLNIC. Induction completes the proof.

\[\square\]

We now use the inequalities on the value function to explore the structure of the optimal policy. Our result, Corollary 1, indicates that an optimal decision rule is invariant on convex sets of \{\(P(\cdot \mid t, k, x, x'')\) as long as \{\(P(\cdot \mid t, k, x'')\) does not change. For the \(L = 1\) case, this result suggests that if \(\delta^*(n, t, k = 1, x, P_x) = \delta^*(n, t, k = 2, x, P_x)\) and \(\delta^*(n, t, k = 1, x, P_x') = \delta^*(n, t, k = 2, x, P_x')\), then given \(P''(\cdot \mid t, k, x)\) is a convex combination of \(P(\cdot \mid t, k, x)\) and \(P'(\cdot \mid t, k, x)\) and few additional restrictions are met, \(\delta^*(n, t, k = 1, x, P_x') = \delta^*(n, t, k = 1, x, P_x)\). That is, under certain conditions, the optimal
decision remains the same even when the data changes. Thus, optimal policies can be stored and applied despite changes in the data.

**Corollary 1** Given Assumption 5, let $L = 1$, and assume $\delta^*(n, t, k = 1, x, P_x) = \delta^*(n, t, k = 1, x, P'_x)$ where

\[
P_{x+1} = P'_{x+1} \quad \text{and} \quad P(t' - t \mid k = 1, x) \leq P'(t' - t \mid k = 1, x) \quad \text{for} \quad t' = t + 1, \ldots, T - 1.
\]

Let $P''_x$ be such that $P''_{x+1} = P_{x+1}$ and $P''(t' - t \mid k = 1, x) = \lambda P(t' - t \mid k = 1, x) + (1 - \lambda) P'(t' - t \mid k = 1, x)$ for $t' = t + 1, \ldots, T - 1$ and for some $\lambda \in [0, 1]$. Then, $\delta^*(n, t, k = 1, x, P_x) = \delta^*(n, t, k = 1, x, P''_x)$.

**Proof:** Clearly, by Assumption 5, $f(n, t, k = 1, x, P_x) = f(n, t, k = 1, x, P'_x) = f(n, t, k = 1, x, P''_x)$ for all $n, k, x$, and $t \geq T$. We recall from Theorem 4 (b) that $f(n, t, k; x)$ is PLNIC in $P(\cdot \mid x, k)$. More explicitly, it follows from the optimality equations that

\[
f(n, t, k = 1, x, P_x) = \min_{n'} \{x(n') + \sum_{t' = t+1}^{T-1} y(n', t') P(t' - t \mid x, k = 1)\},
\]

where

\[
x(n') = c(n, t, k = 1, n') + \sum_{t' \geq t} P(t' \mid n, t, k = 1, n') \hat{c}(n', t')
\]

\[
+ \sum_{t' = t+1}^{T-1} P(t' \mid n, t, k = 1, n') f(n', t', k' = 1, x + t' - t),
\]

and for $t' = t + 1, \ldots, T - 1,$

\[
y(n', t') = P(t' \mid n, t, k = 1, n') \left[ \sum_{x'} P(x' \mid t, k = 1, x, t', k' = 2) f(n', t', k' = 2, x') \right]
\]

\[
- f(n', t', k' = 1, x + t' - t)].
\]
Note, \( y(n', t') \leq 0 \). We assume that \( f(n, t, k = 1, x, P) \) is the minimum of \(|SCS(n)|\) linear functions and each function is associated with an element in \( SCS(n) \).

Let \( n^* \) be such that for all \( n' \in SCS(n) \),

\[
x(n^*) + \sum_{t'} y(n^*, t') P(t' - t \mid k, x) \\
\leq x(n') + \sum_{t'} y(n', t') P(t' - t \mid k = 1, x).
\]

By assumptions on \( x \) and \( P' \),

\[
x(n^*) + \sum_{t'} y(n^*, t') P'(t' - t \mid k, x) \\
\leq x(n') + \sum_{t'} y(n', t') P'(t' - t \mid k = 1, x).
\]

Multiplying both sides of inequality 9 by \( \lambda \) and both sides of inequality 10 by \( (1 - \lambda) \), adding, and then collecting terms implies the result.

6 Illustrative Examples

The examples presented in this section illustrate the behavior of optimal anticipatory policies. The examples were performed on a network that is a subset of the northeast Ohio highway network that includes Cleveland and is described in Section 8. The origin was arbitrarily chosen to be node 16 and the destination node 13.

We say a path \( p \in \mathcal{P}(n) \) (see Section 3 for the definition of a path) is the shortest uncongested path from \( n \) to the destination \( \gamma \) if, for every arc \( e^r \) implicitly defined by the sequence of nodes \( p, e^r \) is unobserved or \( k^r = 2 \). For a given origin, \( s \), and destination, \( \gamma \), let \( p_a \) be the shortest uncongested path
Fig. 1. Shortest Path between Origin and Destination

from $s$ to $\gamma$. The shortest uncongested path for the origin at node 16 and the destination at node 13 is shown in Figure 1.

To make the example interesting, we chose an observed arc on the shortest uncongested path from node 16 to node 13. In this case, the arc $(22, 28)$ was chosen.

We now consider the case where the vehicle begins its route with the arc $(22, 28)$ in the congested state. For the purposes of the example, we assume that the congestion begins at the same time (time $t = 1$) as the vehicle, and thus $x = 0$ at time $t = 1$. We assume that the random amount of time that arc $(22, 28)$ spends in the congested state is distributed according to a discretized Weibull distribution. The Weibull distribution meets the IFR assumption made in Section 3. While we have not collected statistical data to show that congestion clearance is indeed distributed as a Weibull distribution,
we feel that it is an appropriate choice for numerical tests because, according
to [26], the Weibull distribution is the best known of the IFR distributions. As
a result of its wide-spread application, the Weibull distribution is accessible
to many readers and is then best suited to help build the readers’ intuition re-
garding the performance of anticipatory routing. We consider the four Weibull
distributions, each with the same shape parameter of 1.1, but different scale
parameters (\(\lambda\)) of 0.1, 0.2, 0.5, and 1. We note that the mean of the Weibull
distribution decreases in \(\lambda\).

We assume that travel times were deterministic, and to account for congestion,
we multiplied an arc’s travel time by a scalar. The construction of these travel
times satisfies our stochastic dominance assumption on the time required to
traverse an arc.

In the case where the scale parameter \(\lambda = 0.1\), the case with the longest mean
time of congestion, the optimal anticipatory policy follows the path:

\[
16 \rightarrow 24 \rightarrow 25 \rightarrow 20 \rightarrow 5 \rightarrow 4 \rightarrow 13,
\]

regardless of whether or not the congestion clears at some point during the
trip. As \(\lambda\) increases, and hence the mean time of congestion decreases, the
optimal policy chooses to begin by following the optimal path:

\[
16 \rightarrow 17 \rightarrow 23 \rightarrow 22.
\]

In essence, the optimal policy anticipates that the congestion is likely to clear.

Upon reaching the congested arc, the optimal action varies depending on the
scale parameter. In the case of \(\lambda = 0.2\), if the vehicle reaches node 22 and
the arc (22, 28) is still congested, the policy immediately chooses to follow the
following path to the end:

$$22 \rightarrow 18 \rightarrow 19 \rightarrow 20 \rightarrow 5 \rightarrow 4 \rightarrow 13,$$

regardless of whether or not arc (22, 28) becomes uncongested as the vehicle follows that path. Essentially, the optimal policy recognizes that, if arc (22, 28) has not cleared by the time that the vehicle arrives there, then it is more cost effective to reroute. However, rather than simply routing around the congested arc all together, the anticipatory policy accounts for the high likelihood that the congested arc will have cleared by the time the vehicle arrives to the congested arc.

In the case of $\lambda = 0.5$ and $\lambda = 1$, the optimal policy chooses to wait at node 22 for 3 time units in each case. In the case that the congestion has not cleared at the end of those 3 time units, the remaining path to the end is again:

$$22 \rightarrow 18 \rightarrow 19 \rightarrow 20 \rightarrow 5 \rightarrow 4 \rightarrow 13.$$

Thus, when the expected duration of the congestion is short, the anticipatory policy recognizes the value in waiting rather than incurring the cost of a diversion.

7 Alternate Route Policy

A major drawback of our model of the dynamic and stochastic shortest path problem with anticipation is its computational complexity. For example, for the congested-uncongested status of the observed arcs, there are $2^L$ combinations. This exponential state space growth limits traditional finite-horizon, stochastic dynamic programming solution approaches to small problem sizes.
Consequently, it is important to justify the computational expense relative to simple heuristics for the problem.

To find a reasonable comparison to anticipatory routing, the examples in Section 6 provide us with some direction. While each of the presented anticipatory policies is optimal in the objective of total expected time traveled, it is clear that in the worst case, the case that the arc remains congested over the entire time horizon, that another policy would have performed better for the cases of $\lambda = 0.2$, $\lambda = 0.5$, and $\lambda = 1$. In particular, following the shortest alternative path to $p_u$ would have led to a lower travel time in the worst case.

This observation suggests the following heuristic to which to compare anticipatory routing. If, at the time that the vehicle was leaving, we fix the cost of traversing each arc based on the observed state of the arcs at that time, we could choose our path by simply choosing the least expected cost path under those conditions. With the arc states fixed, the problem remains stochastic, but is now static. In this case, we can use the results in Eiger, Mirchandani, and Soroush [29] and can solve the problem using a Dijkstra-like algorithm.

We now describe the heuristic for the general case. Our heuristic returns a policy $\tilde{\pi} = \{\tilde{\delta}(\cdot, t, \cdot, \cdot) : t = 1, \ldots, T - 1\}$. By assumption, there exists a path $p$ from every node $n$ to $\gamma$, and the set of all paths from $n$ to $\gamma$ is $\mathcal{P}(n)$. Let $\bar{c}(p(n), k)$ be the cost of path $p(n) \in \mathcal{P}(n)$, when the arcs are all fixed in state $k$. We let $\tilde{k} = k(t = 0)$ for the entire problem horizon. Then, for our heuristic, for each $n, t, k$, and $x$, let

$$\delta(n, t, k, x) \in \arg\min_{n'}\{c(n, t, \tilde{k}, n') + \min\{\bar{c}(p(n'), \tilde{k}) : p(n') \in \mathcal{P}(n'), (n, n') \in E\}\}.$$ 

Let $h(s, 1, k, x) = f^{\tilde{\pi}}(s, 1, k, x)$, the expected total cost to be accrued until the
vehicle reaches the destination, assuming the vehicle starts at node \( s \) at time \( t = 1, K(1) = k, X(1) = 0 \), and \( \tilde{\pi} \) is used.

A numerical analysis indicating the quality of \( \tilde{\pi} \), relative to \( \pi^* \), is presented in Section 9.

8 Experimental Design

In the remainder of this paper, we describe and present the results of numerical experiments that were designed to explore the behavior of the optimal anticipatory policy, \( \pi^* \), and the heuristic policy, \( \tilde{\pi} \). Essentially, these comparisons allow us to determine the value of knowing stochastic information about when congestion will clear. We explore this topic by answering two questions:

1. How is the behavior the optimal anticipatory policy, \( \pi^* \), affected by congestion duration and the relative geographic location of the congestion?
2. When is the anticipatory policy, \( \pi^* \), most valuable in comparison to the alternative route heuristic policy, \( \tilde{\pi} \)?

To most accurately answer the questions posed above, we constructed a set of numerical experiments designed to test the performance of \( \pi^* \) and \( \tilde{\pi} \) under a range of conditions. All experiments were performed using a northeast Ohio network that covers the greater Cleveland area (Figure 2). The network was created by the Northeast Ohio Areawide Coordinating Agency during the development of hazardous materials routing strategies for the Cleveland area. The network includes interstates, state routes, U. S. highways, and a few other select roads in the region. The network consists of 131 nodes and 202 arcs. This network was chosen in order to better reflect the performance of anticipatory.
Because we want to focus on the value of knowing stochastic information about when congestion will clear, we assume travel times are deterministic, and we round to 5 minute increments for these experiments. It was assumed that $T = 48$. Thus, this value of $T$ represents a 4 hour time period. The arc cost is measured by the amount of time required to travel the arc. In the case of a congested arc, the arc cost was multiplied by 5 to represent the effect of the congestion. The scalar 5 was chosen as the multiplier so that the cost of the congested arc was greater than the cost of any uncongested arc in the network. Again, the construction of these travel times satisfies our assumptions regarding travel time distributions. For all $n$, the terminal cost was assumed.
Table 1

<table>
<thead>
<tr>
<th>Data Set</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Origin</td>
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<td>8</td>
<td>64</td>
<td>55</td>
<td>96</td>
<td>105</td>
<td>86</td>
<td>99</td>
<td>27</td>
<td>126</td>
</tr>
<tr>
<td>Destination</td>
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<td>93</td>
<td>55</td>
<td>92</td>
<td>13</td>
<td>4</td>
<td>60</td>
<td>105</td>
<td>45</td>
<td>68</td>
</tr>
</tbody>
</table>

Origin and Destination for Each Experiment

to be \( \hat{c}_\beta(p(n)) \), where \( \beta \) is the time required to travel the longest arc in the network. For convenience, \( \hat{c}(t) \) was set to 0 for all \( t > T \).

For the experiments, ten test sets were randomly generated. Each test set contained an origin and a destination. Table 1 gives the origin and destination for each test set. For each data set, we tested the performance of \( \pi^* \) and \( \hat{\pi} \), for three observed-arc positions. Each arc position was chosen from the shortest uncongested path, \( p_u \). For each data set, the first observed-arc position was chosen to be \((n^1 = s, n^2)\) where \( n^1 \) and \( n^2 \) are the first two nodes in the sequence \( p_u \). This first-arc position corresponds to the first arc in the shortest uncongested path. The second-arc position was chosen such that, for the observed arc \((n^r, n^{r+1})\), \( n^r \neq s \), \( n^{r+1} \neq \gamma \), and both \( n^r \) and \( n^{r+1} \) are in the sequence \( p_u \). This second-arc position was chosen to represent an arc arbitrarily in the middle of the shortest uncongested path. Finally, the third-arc position was chosen to be the last arc in \( p_u \). That is, the third-arc position is the arc \((n^r, n^{r+1} = \gamma)\).

It was assumed that each observed arc had two states: congested \((k = 1)\) and uncongested \((k = 2)\). As we did in Section 6, we assume that the random amount of time spent in a congested state is distributed according to a discretized Weibull distribution.
For each data set and each arc position, the scale parameter of the Weibull distribution was iterated from 0.1 to 0.2 to 0.5 to 1. The effect of this iteration was to decrease the expected time that the arc spent in the congested state as the scale parameter was iterated from 0.1 to 1.

For each problem instance, \(\pi^*\) was found using standard stochastic dynamic programming techniques, and \(\tilde{\pi}\) was found using standard dynamic programming methods for deterministic shortest path problems. All algorithms were coded in C++. All experiments were run on a 1.2 ghz Pentium III processor with 512 mb of RAM.

9 Experimental Results

9.1 Effects of Expected Length of Congestion and Geographical Proximity of Congestion on the Expected Value of the Optimal Anticipatory Policy

Table 2 presents the expected values of the optimal anticipatory policy, \(\pi^*\), for each data set, arc-congestion position (first arc, middle arc, or last arc), and scale parameter for the case where the realization of \(X(1)\) is 0. As the table shows, as the scale parameter, \(\lambda\), increases, and hence the expected duration of the congestion decreases, \(f(s, 1, 1, 0)\) decreases, regardless of which arc is congested. The reason for this is that, when, in expectation, the vehicle encounters a shorter period of congestion, the more likely the optimal action is a node on the shortest uncongested path.

At the same time, in general, \(f(s, 1, 1, 0)\) is decreasing in the geographical distance of the congested arc to the origin. For example, all other parameters
equal, the value of \( f(s, 1, 1, 0) \) when the first arc is congested is greater than the value of \( f(s, 1, 1, 0) \) when the middle arc is congested. The reason for this result is that the vehicle cannot gain both time and geographical advantage without having to traverse the congested arc. That is, when the congested arc is the middle or last arc on the uncongested shortest path and when following the uncongested shortest path, the vehicle needs time to reach these arcs. Consequently, by the time that the vehicle reaches these arcs, they are more likely to be uncongested. Additionally, if the vehicle arrives to find these arcs congested, the vehicle has moved relatively closer to the destination and hence can afford to wait longer relative to having followed an alternate route. The same time and geographical advantage cannot be gained when the first arc of the uncongested shortest path is congested. Analogous results exist for the cases where the vehicle leaves the origin after the observed arc has been congested for some number of units of time.

9.2 Comparison of the Value of the Optimal Anticipatory Policy and the Alternative Route Heuristic Policy

This section presents the results of experiments that were run in order to compare the value of anticipatory routing relative to the alternate route heuristic. Table 3 presents the results of the three observed-arc positions for each data set and each scale parameter value for the case where the realization of \( X(1) \) is 0. For each run, the two policies are compared by the percentage difference between \( f(s, 1, 1, 0) \) and \( h(s, 1, 1, 0) \), computed as \( \frac{h(s,1,1,0) - f(s,1,1,0)}{h(s,1,1,0)} \times 100\% \). To begin, we note that anticipatory routing always outperforms alternate route heuristic in terms of expected total cost. In general, the results also show
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Table 2. Comparison of $f(s, 1, 1, 0)$ Values for Each Data Set, Observed-Arc Position, and Scale Parameter
that, for each data set and each experiment, the longer the congestion is expected to last (the larger the scale parameter value, the less time the arc is expected to stay congested), the less value $f(s,1,1,0)$ typically has in relation to $h(s,1,1,0)$. This result occurs because, when the congestion is likely to last long, the anticipatory policy follows the same route as the alternate route policy. When the duration of the congestion is expected to be short, the optimal anticipatory policy anticipates the change in arc status whereas the alternate route policy incurs the cost of an alternate route. It is also important to note the results due to the declining expected duration of the congestion are analogous to the vehicle starting in a situation where the observed arc has been congested for some amount of time.

The value of the optimal anticipatory policy over the alternate route policy with regard to the position of the congested arc is less systematic and depends more on the available paths from origin to destination. For example, in the case where the first arc in the vehicle’s shortest path is congested, consider data set 7. For data set 7, there is only one arc emanating from the origin, node 86. Consequently, the reactive routing policy is forced to cross the congested arc. On the other hand, the anticipatory policy, chooses to wait for some period of time in anticipation of the congestion clearing. Thus, the anticipatory policy gains a large advantage over the reactive policy. Likewise, an advantage occurs when we consider the last arc in data set 3 to be congested. In this case, there exists two arcs that terminate at node 55. However, the preferred arc when there is no congestion is arc (58, 55). The alternate arc (56, 55) requires an alternate route at great additional cost. Thus, because the anticipatory policy can anticipate the change in arc status, it can accomplish large cost savings over the alternate route policy. For a contrast, consider the last arc for data set
1, arc (39, 53). No other arc terminates at node 53. Hence, the alternate route policy must choose arc (39, 53) regardless of its level of congestion. However, if the scale parameter is large, the expected duration of the congestion is short. Hence, by the time a vehicle using the alternate route policy reaches arc (39, 53), it is typically cleared, and hence costs are the same as with the anticipatory policy.

10 Conclusions

In this paper, we introduced a routing problem in which future changes in arc congestion are anticipated. We modeled the problem as a Markov decision process and presented structural results on the cost function and optimal policy. Our structural results showed that the cost function is PLNIC in a probability vector related to the likelihood of the congested arc becoming uncongested. Having identified this structure for the cost function, we were able to show that an optimal policy satisfies a convexity property for the probability vector. As a direct corollary of this convexity property, we identified conditions under which the optimal policy is invariant to changes in the problem data. Given the computational cost of determining the optimal policy, the corollary is important as it identifies the conditions under which we do not need to compute a new optimal policy when data changes.

In addition, we presented the results of a series numerical experiments designed to determine the value of knowing probabilistic information about the duration of congestion. To achieve this goal, we compared the expected cost of the optimal anticipatory policy to the expected cost returned by a reactive routing policy. The experiments test the policies under various geographic positions of
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Table 3. The Percentage Difference between $f(s, 1, 1, 0)$ and $h(s, 1, 1, 0)$ for Each Data Set, Observed-Arc Position, and Scale Parameter.
the congested arc in relation to the shortest uncongested path as well as under various expected lengths of the arc congestion. The results showed that, for a given origin and destination, the anticipatory policy has the least expected value when the expected duration of the observed arc is shortest and when the congested arc is geographically farther away from the origin. At the same time, due to its construction, the alternative route heuristic generally performed the same under all conditions. A comparison of the two policies showed that the optimal anticipatory policy had its best results relative to the alternative route policy in situations where the expected duration of the congestion was shortest and when their did not exist low-cost alternatives to the congested arc.

Three areas stand out for future research. For one, in this paper, we made a reasonable IFR assumption with regard to how long an arc remains congested. To satisfy this property in our computational experiments, we implemented the Weibull distribution. While the Weibull distribution enjoys widespread application, future research should attempt to gather and apply real-world data to the problem. As a second area of future research, we can relax our assumption that there are only two arc states, congested and uncongested, and that the uncongested state is a trapping state. In the case that congesting causing events are rare, it may be necessary to simply react to the change in arc status by re-optimizing the problem with the current vehicle location as the start node and including the new information. Such reoptimization approaches are considered for the dynamic routing and dispatching problem (see [30] and [31] for overviews). Finally, increasing the number of congestion states will lead to exponential state-space growth. Consequently, methods for improving computation speed should be explored. A good starting point would be the state-space reduction techniques presented in [16].
References


